Exercise Sheet 13

Exercise 37. Constraint sets. Consider a closed subset C of $\mathbb{R}^m$ and a bounded domain $\Omega \subset \mathbb{R}^d$. For $p \in [1, \infty]$ define $\mathcal{C}_p = \{ u \in \mathrm{L}^p(\Omega; \mathbb{R}^m) \mid u(x) \in C \text{ a.e. in } \Omega \}$. (In the case $p = \infty$ we mean weak* convergence.)

(a) Show that $\mathcal{C}_p$ is strongly closed in $\mathrm{L}^p(\Omega; \mathbb{R}^m)$.

(b) Show that $\mathcal{C}_p$ is weakly closed if C is convex.

(c) Show that weak closedness of $\mathcal{C}_p$ implies convexity of C.

Exercise 38. Variational inequalities generalize the Lax-Milgram lemma. On a Hilbert space H, we consider a symmetric, bounded, and coercive bilinear form $B : H \times H \to \mathbb{R}$. Moreover, let M be a closed convex subset of H. For $\xi \in H^*$ we consider the following variational inequality:

$$\text{Find } u \in M \text{ such that } B(u, w-u) \geq \langle \xi, w-u \rangle \text{for all } w \in M.$$

(a) Construct for each ξ a solution u exists and show that it is unique, thus defining $U : H^* \to M \subset H; \xi \mapsto u = U(\xi)$.

(b) Show that U is Lipschitz continuous.

(c) Give a case where U is explicit and nonlinear.

Exercise 39. Quadratic densities. Assume that $f : \mathbb{R}^{m \times d} \to \mathbb{R}$ is quadratic, i.e. $f(A) = (MA){:}A$ for some $M \in \mathrm{Lin}(\mathbb{R}^{m \times d}; \mathbb{R}^{m \times d})$. The following equivalences hold, where (i) and (iii) have been established in earlier excercises.

(i) f is convex $\iff$ $f(A) \geq 0$ for all A.

(ii) f is polyconvex $\iff$ $\exists\, \boldsymbol{\beta} \in \mathbb{R}^{\tau_2(m,d)}\ \forall\, A \in \mathbb{R}^{m \times d} :\ f(A) \geq \boldsymbol{\beta}_* \cdot T_2(A)$.

(iii) f is rank-one convex and quasiconvex $\iff$ $\forall\, \xi \in \mathbb{R}^m,\ \eta \in \mathbb{R}^d :\ f(\xi \otimes \eta) \geq 0$.

(a) Show the direction $\Longleftarrow$ in (ii).
(*Hint: Construct a convex function* $g(A, \alpha) = h(A) + \gamma \cdot \alpha$.)

(b) Establish the direction $\Longrightarrow$ in (ii). (*Hint: Use* $f(\lambda A) = \lambda^2 f(A)$.)

(c) Show that for $\min\{m, d\} \leq 2$ for quadratic densities f we have that polyconvexity is equivalent to rank-one convexity.