Exercise Sheet 11

Exercise 32. Minimizers and Euler-Lagrange equations. Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain and $p \in \,]1, \infty[$. For $h \in \mathrm{L}^\infty(\Omega)$ and integers $n, m \in \mathbb{N}$ consider the functional

$$I(u) = \int_\Omega \Big\{ \frac{1}{p}|\nabla u|^p + \frac{1}{2n}u^{2n} + \cos(u^m) - uh \Big\} \,\mathrm{d}x$$

on the space $X = \mathrm{W}^{1,p}(\Omega)$.

(a) Discuss the existence of global minimizers u_*.

(b) Give sufficient conditions such that I is Gateaux differentiable on all of X. Give conditions such that $\mathrm{D}I(u)[\varphi]$ exists for all $\varphi \in \mathrm{C}^1(\overline{\Omega})$.

(c) Using extra conditions for u_* give conditions such that $\mathrm{D}I(u_*)[v] = 0$ holds for $v \in X$ and give conditions such that $\mathrm{I}(u_*)[\varphi] = 0$.

Exercise 33. Continuity of Gateaux derivative implies Fréchet derivative. Consider a functional $I : X \to R$ that is continuously Gateaux differentiable, i.e. $u \mapsto \mathrm{D}^{\mathrm{G}}I(u)$ is a norm-norm continuous mapping from X to X^*. Conclude that I is also Fréchet differentiable with $\mathrm{D}^{\mathrm{F}}I(u) = \mathrm{D}^{\mathrm{G}}I(u)$.
(*Hint: For Gateaux differentiable functionals J show first (*) $|J(u_1){-}J(u_0)| \le \ell\|u_1{-}u_0\|$ with $\ell = \sup\{\, \|\mathrm{D}^{\mathrm{G}}J((1{-}\theta)u_0 + \theta u_1)\|_{X^*} \mid \theta \in [0,1]\,\}$. Then, for fixed u consider the functional $R(h) = I(u{+}h) - I(u) - \mathrm{D}^{\mathrm{G}}I(u)[h]$.*)

Exercise 34. Continuity of Gateaux derivative. Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain and $r, p, q \in \,]1, \infty[$ such that $\mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}^q(\Omega)$.

(a) Consider a function $g \in \mathrm{C}(\overline{\Omega}{\times}\mathbb{R}^m; \mathbb{R}^k)$ satisfying

$$\exists\, C > 0 \;\forall\, (x, u) \in \overline{\Omega} \times \mathbb{R}^m : \quad |g(x, u)| \le C\big(1{+}|u|\big)^{r/q}.$$

We define the Nemitskii operator $\mathcal{G} : \mathrm{L}^r(\Omega; \mathbb{R}^m) \to \mathrm{L}^q(\Omega; \mathbb{R}^k)$ via $\mathcal{G}(u)(x) = g(x, u(x))$. Show that $\mathcal{G}$ is norm-norm continuous.

(b) Consider the functional $I(u) = \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x$ where $f \in \mathrm{C}^1(\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d})$ satisfies

$$|f(x, u, A)| + \big|\partial_u f(x, u, A)\big|^{q'} + \big|\partial_A f(x, u, A)\big|^{p'} \le C\big(1 + |u|^q + |A|^p\big).$$

Show that the Gateaux derivate $u \mapsto \mathrm{D}^{\mathrm{G}}I(u)$ given by

$$\mathrm{D}^{\mathrm{G}}I(u)[v] = \int_\Omega \big\{ \partial_u f(x, u(x), \nabla u(x)){\cdot}v(x) + \partial_A f(x, u(x), \nabla u(x)){:}\nabla v(x) \big\} \,\mathrm{d}x$$

is norm-norm continuous from $\mathrm{W}^{1,p}(\Omega; \mathbb{R}^m)$ to $\mathrm{W}^{1,p}(\Omega; \mathbb{R}^m)^*$.

(c) Conclude that I in (b) is even continuously Fréchet differentiable.