Exercise Sheet 2

Exercise 5. Example without minimizer: Consider the functional

$$I : \ell^2 \to \mathbb{R},\ I(u) = (1 - \|u\|_2^2)^2 + \sum_{n=1}^{\infty} \frac{1}{n} u_n^2 \quad \text{with } \ell^2 = \Big\{ (u_n)_{n\in\mathbb{N}} \,\Big|\, \sum_{n=1}^{\infty} u_n^2 < \infty \Big\}. \quad \text{(Eq.1)}$$

(a) Show that I is continuous and that $I(u) > 0$ for all $u \in \ell^2$.

(b) Construct infimizing sequences to show $\inf I = 0$.

Exercise 6. Local minimizers. Construct $I \in \mathrm{C}^1(\mathbb{R}^2;\mathbb{R})$ with the following properties:
(i) For all straight lines $\gamma_v : \mathbb{R} \ni t \mapsto tv \in \mathbb{R}^2$ the restriction of I has a strict local minimum at $t = 0$.
(ii) $x = 0$ is not a local minimizers of I, i.e. $\forall\, \varepsilon > 0\ \exists\, y \in B_\varepsilon(0) :\ I(y) < I(0)$.
Hint: Look for I which is negative between two parabolas and positive outside.

Exercise 7. Lemma of du Bois–Reymond.
Consider $T = \{ v \in \mathrm{C}^1([\alpha,\beta];\mathbb{R}^m) \mid v(\alpha) = v(\beta) = 0 \}$ and $f,\ g \in \mathrm{C}^0([\alpha,\beta];\mathbb{R}^m)$.

(a) Show that $\int_\alpha^\beta g(x) \cdot v'(x)\,\mathrm{d}x = 0$ for all $v \in T$ implies that g is constant on $[\alpha,\beta]$. (Hint: Construct a $v \in T$ with $v'(x) = g(x) - \gamma$.)

(b) Now assume $\int_\alpha^\beta [f(x) \cdot v(x) + g(x) \cdot v'(x)]\,\mathrm{d}x = 0$ for all $v \in T$. Conclude $g \in \mathrm{C}^1([\alpha,\beta];\mathbb{R}^m)$ and $g'(x) = f(x)$ for all $x \in [\alpha,\beta]$.
(Note that we gain smoothness of g without imposing it.)

Exercise 8. Variations and local extrema. Reconsider $I : \ell^2 \to \mathbb{R}$ from (Eq.1).

(a) Show that the Gâteaux derivative exists and that the first variation takes the form

$$\mathrm{D}I(u)[v] = 2(\|u\|^2 - 1)\langle u, v\rangle + \sum\nolimits_{n\in\mathbb{N}} \frac{1}{n} \langle u, e_n\rangle \langle v, e_n\rangle \quad \text{(where } e_n = (0, ..., 0, 1, 0, ...) \text{)}$$

and derive a formula for the second variation $\mathrm{D}^2 I(u)[v, w]$.

(b) Show that for all $k \in \mathbb{N}$ there exist three critical points of the form $u = \beta e_k$.

(c) Derive definiteness properties of the quadratic form $v \to \mathrm{D}^2 I(\beta e_k)[v, v]$ and try to determine extremal properties.

Exercise 9. Quadratic forms on $\mathrm{L}^2(\Omega)$. For functions $f \in \mathrm{L}^2(\Omega;\mathbb{R}^m)$ and $A \in \mathrm{L}^1(\Omega;\mathbb{R}^{m\times m})$ with $A(x) = A(x)^\top \geq 0$ a.e. in Ω we define

$$I : \mathrm{L}^2(\Omega;\mathbb{R}^m) \to \mathbb{R}_\infty;\ I(u) = \int_\Omega \Big(\frac{1}{2} \langle A(x)u(x), u(x)\rangle - \langle f(x), u(x)\rangle \Big)\,\mathrm{d}x.$$

(a) Explain why $I(u) \in \mathbb{R}_\infty$ is always well-defined.

(b) Discuss necessary and sufficient conditions on A and f for coercivity.

(c) Give necessary and sufficient conditions on A and f for existence of a global minimizer $u_* \in X$. Given examples (i) without a minimizer and (ii) with minimizer but no coercivity.

(d) Argue in favor or against lower semi-continuity for general A.