Exercise Sheet 1

Exercise 1. Linear Algebra: Let $I(u) = \frac{1}{2}\langle u, Au\rangle - \langle b, u\rangle$ with $A \in \mathbb{R}^{n\times n}$, $b, u \in \mathbb{R}^n$, and $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product.

(a) Show that u is a critical point of I if and only if $\frac{1}{2}(A + A^\top)u = b$ holds.

(b) Assume that A is symmetric and positive definite. Does I attain its infimum over $\mathbb{R}^d$?

(c) Assume now that A is symmetric and positive semidefinite. Under which conditition on b do we have $\inf_{\mathbb{R}^n} I > -\infty$? Do we then have $\inf I = \min I$?

Exercise 2. Extrema without differentiablity. Consider $f_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ given by

$$f_{\alpha,\beta}(x) = \begin{cases} \alpha & \text{for} \quad x = 0, \\ \frac{\beta x^2}{1+x^2} & \text{for} \quad x \neq 0, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$. Determine all extrema (local and global minima and maxima) of $f_{\alpha,\beta}$ and as well as the infimum and the supremum in dependence of α and β.

Exercise 3. Example of *Weierstraß*: Consider the functional $I : M \to \mathbb{R}$ with

$$I(u) = \int_{-1}^{1} [xu'(x)]^2 \,\mathrm{d}x \quad \text{and} \quad M = \{\, u \in \mathrm{C}^1([-1,1]) \mid u(-1) = -1,\ u(1) = 1 \,\}.$$

Show that any sequence $(u_k)_{k\in\mathbb{N}}$ with $I(u_k) \to 0$ converges uniformly on compact subsets of $[-1,0) \cup (0,1]$ to the limit $u_* : x \mapsto \mathrm{sign}(x)$.

For $x > 0$ estimate $|1-u(x)| = |\int_x^1 u'(s)\,\mathrm{d}s|$ by $\sqrt{\alpha}C(x)$, where $\alpha = I(u)$ and C is a continuous function on $]0,1]$. (*Hint: Let* $a(x) = [xu'(x)]^2$ *and use* $\int_{-1}^1 a(s)\,\mathrm{d}s = \alpha$*.*)

Exercise 4. Semicontinuity: For functions $f : \mathbb{R}^n \to \mathbb{R}$ we define sequential upper semicontinuity (usc) and lower semicontinuity (lsc) as follows:

f is usc at x_*, if $x_k \to x_*$ implies $f(x_*) \geq \limsup_{k\to\infty} f(x_k)$,

f is lsc at x_*, if $x_k \to x_*$ implies $f(x_*) \leq \liminf_{k\to\infty} f(x_k)$

(a) Give an (ε, δ) definition for usc and lsc.

(b) For $\alpha, \beta \in \mathbb{R}$, consider the functions $g_{\alpha,\beta} : [0,1] \to \mathbb{R}$ with $g_{\alpha,\beta}(x) = \begin{cases} \alpha & \text{for } x = 0, \\ \mathrm{e}^x & \text{for } x \in]0,1[, \\ \beta & \text{for } x = 1. \end{cases}$

For which parameters do we have usc and for which lsc?

(c) For $\Omega \subset \mathbb{R}^n$ consider $f_k \in \mathrm{C}(\Omega)$ such that $f_k(x) \nearrow f(x)$ (convergence from below). Do we have usc or lsc? Given an example of f_k, where f is not continuous.

Exercise 5. Example without minimizer: Consider the functional

$$I : \ell^2 \to \mathbb{R},\ \ I(u) = (1 - \|u\|_2^2)^2 + \sum_{n=1}^{\infty} \frac{1}{n} u_n^2, \quad \text{where } \ell^2 = \{\, (u_n)_{n\in\mathbb{N}} \mid \sum_{n=1}^{\infty} u_n^2 < \infty \,\}.$$

(a) Show that I is continuous and that $I(u) > 0$ for all $u \in \ell^2$.

(b) Construct infimizing sequences to show $\inf I = 0$.