Partial Differential Equations
Exercise Sheet 11

Exercise 38. Gronwall's lemma [1919]
(after Thomas Hakon Grönwall (1877-1932)).

(a) Differential form: Let $y \in \mathrm{W}^{1,1}([0,T])$ with $y(t) \geq 0$ and $\alpha \in \mathrm{L}^1([0,T])$ be such that $\dot{y}(t) \leq \alpha(t)y(t)$. Show the Gronwall estimate $y(t) \leq y(0)\mathrm{e}^{\int_0^t \alpha(s)\,\mathrm{d}s}$.

(b) Generalize (a) to the case $\dot{y}(t) \leq \alpha(t)y(t) + \beta(t)$ for $\beta \in \mathrm{L}^1([0,T])$ with $\beta \geq 0$.

(c) Integrated form:
Assume that $z \in \mathrm{C}^0([0,T])$ with $z(t) \geq 0$ satisfies $z(t) \leq z_0 + \int_0^t \big(\alpha_0 z(s)+\beta(s)\big)\,\mathrm{d}s$ with $\alpha_0 \geq 0$ and β as in (b). Conclude $z(t) \leq z_0\mathrm{e}^{\alpha_0 t} + \int_0^t \mathrm{e}^{\alpha_0(t-s)}\beta(s)\,\mathrm{d}s$.
Hint: Define $w(t) = z_0 + \int_0^t \big(\alpha_0 z(s) + \beta(s)\big)\,\mathrm{d}s$ and compare it with $\dot{w}$.

(d) Assume that $y \in \mathrm{W}^{1,1}([0,\infty[)$ and $\phi \in \mathrm{L}^1([0,\infty[)$ satisfy $\phi(t) \geq 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}y^2 \leq 2\,\phi\, y$ for all $t \geq 0$. Show that y remains bounded by $|y(0)| + \int_0^\infty \phi(t)\,\mathrm{d}t$.

Exercise 39. A priori estimates for parabolic equations.

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ we consider the general parabolic equation

$$\begin{aligned} &\rho(x)u_t = \operatorname{div}\big(A(x)\nabla u(x)\big) + b(x)\cdot\nabla u(x) + c(x)u(x) + f(t,x), \quad (t,x) \in \left]0,\infty\right[\times\Omega,\\ &u(0,x) = u_0(x) \text{ in } \Omega, \qquad \big(A(x)\nabla u(t,x)\big)\cdot\nu(x) + \beta(x)u(x) = 0 \text{ for } t>0,\ x \in \partial\Omega, \end{aligned}$$

where the divergence is understood in the weak sense. For the coefficients assume $A \in \mathrm{L}^\infty(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$, $b \in \mathrm{L}^\infty(\Omega,\mathbb{R}^d)$, $\rho,\ c \in \mathrm{L}^\infty(\Omega)$, $\beta \in \mathrm{L}^\infty(\partial\Omega)$ with $\beta \geq 0$. Moreover, assume $\xi\cdot A(x)\xi \geq \alpha_{\min}|\xi|^2$ and $\rho(x) \geq \rho_{\min}$ with $\alpha_{\min},\ \rho_{\min} > 0$ and $f \in \mathrm{BC}([0,\infty[\,,\mathrm{L}^2(\Omega))$.

(a) Let $E_1(t) = \int_\Omega \rho(x)u(t,x)^2\,\mathrm{d}x$ and assume that u is a sufficiently smooth solution. Derive the estimate

$$\dot{E}_1(t) + c_1\|\nabla u\|^2_{\mathrm{L}^2(\Omega)} \leq C_2 E_1(t) + C_3\|f(t)\|^2_{\mathrm{L}^2(\Omega)},$$

where the constants c_1, C_2, C_3 may only depend on the coefficients.

(b) Use the Gronwall lemma to show the a priori estimates

$$\|u(t)\|^2_{\mathrm{L}^2(\Omega)} + \int_0^t \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \ \leq\ C_4\big(\|u(0)\|^2_{\mathrm{L}^2(\Omega)} + F^2\big)\mathrm{e}^{C_5 t},$$

where $F = \sup\{\, \|f(t)\|_{\mathrm{L}^2(\Omega)} \mid t \geq 0 \,\}$ for suitable constants C_4 and C_5.

Exercise 40. An explicit solution for the three-dimensional wave equation. On $\Omega = \mathbb{R}^3$ consider the wave equation

$$u_{tt} = \Delta u, \qquad u(0,x) = 0, \quad u_t(0,x) = g(x).$$

which has the explicit solution

$$u(t,x) = t\, M_g(t,x) = \frac{1}{4\pi t} \int_{|y-x|=t} g(y)\,\mathrm{d}a(y) \qquad \text{(spherical mean)}$$

We consider the case with $g(x) = 1$ for $|x| \le R$ and $g(x) = 0$ for $|x| > R$.

(a) Interprete the solution formula geometrically by intersecting a ball and a sphere. Determine the support of $u(t,\cdot)$ explicitly.

(b) Show that $\|u(t)\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \to 0$.

(c) Calculate the solution explicitly and show that u has a point of discontinuity.
Hints: (i) The initial condition is radially symmetric.
(ii) The area of a spherical cap of the unit sphere is $2\pi(1-\cos\varphi)$, where $\varphi \in]0,\pi[$ is the opening angle.

(d) Calculate explicitly the kinetic energy $E_{\text{kin}}(t) = \int_{\mathbb{R}^3} \frac{1}{2} u_t^2\,\mathrm{d}x$ and show that $E_{\text{kin}}(t) \to \frac{1}{2} E_{\text{total}}$, i.e. the energy distributes equally into kinetic and potential energy for $t \to \infty$. ("Equipartition of energy" is an important principle in physics.)

Dates for the oral exams:
July 22 – 24, 2019 and September 30 – October 2, 2019.