Partial Differential Equations
Exercise Sheet 9

Exercise 31. Rellich's embedding theorem for a square. Consider the domain $\Omega = \,]0,1[^2 \subset \mathbb{R}^2$. We want to show that $\mathrm{H}^1(\Omega)$ (no boundary conditions) embeds compactly into $\mathrm{L}^2(\Omega)$.

(a) Set $\widetilde{\Omega} = \,]-1,2[^2$ and define the mapping $R : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\widetilde{\Omega})$ via

$$(Ru)(x_1,x_2) = u(\varrho(x_1),\varrho(x_2)), \quad \text{where } \varrho(y) = \begin{cases} -y & \text{for } y \in [-1,0], \\ y & \text{for } y \in [0,1], \\ 2-y & \text{for } y \in [1,2]. \end{cases}$$

Show that R restricted to $\mathrm{H}^1(\Omega)$ defines a bounded linear operator from $\mathrm{H}^1(\Omega)$ to $\mathrm{H}^1(\widetilde{\Omega})$ and satisfies $\|Ru\|_{\mathrm{H}^1(\widetilde{\Omega})} = 3\|u\|_{\mathrm{H}^1(\Omega)}$.

(b) Construct a cut-off function $\chi : \mathbb{R}^2 \to \mathbb{R}$ such that the mapping $S : v \mapsto\in \chi v$ is a bounded linear operator from $\mathrm{H}^1(\widetilde{\Omega})$ to $\mathrm{H}^1_0(\widetilde{\Omega})$ and that $Sv|_\Omega = v|_\Omega$.

(c) Combine (a) and (b) with Rellich's embedding theorem for $\mathrm{H}^1_0(\widetilde{\Omega})$ to show that the embedding of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^2(\Omega)$ is compact.

Exercise 32. Eigenvalue problem on a rectangle with Dirichlet and Neumann boundary conditions. Consider the domain Ω and Dirichlet boundary Γ_{Dir} with

$$\Omega = \,]0,a[\times]0,b[\;\text{ and }\; \Gamma_{\mathrm{Dir}} = \Big(]0,a[\times \{0\}\Big) \cup \Big(]0,a[\times \{b\}\Big).$$

Let $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega) := \big\{ u \in \mathrm{H}^1(\Omega) \,\big|\, u|_{\Gamma_{\mathrm{Dir}}} = 0 \big\}$ and consider the following eigenvalue problem for $(\lambda,\varphi) \in \mathbb{R}\times\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$:

$$\int_\Omega \nabla\varphi \cdot \nabla v \,\mathrm{d}x = \lambda \int_\Omega \varphi v \,\mathrm{d}x \quad \text{for all } v \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega). \tag{1}$$

(a) Show that sufficiently smooth eigenpairs satisfy the PDE

$$-\Delta\varphi = \lambda\varphi \;\text{ in } \Omega, \quad \varphi = 0 \;\text{ on } \Gamma_{\mathrm{Dir}}, \quad \nabla\varphi\cdot\nu = 0 \;\text{ on } \partial\Omega \setminus \Gamma_{\mathrm{Dir}}.$$

(b) Use the separation ansatz $\varphi(x_1,x_2) = V(x_1)W(x_2)$ to find eigenpairs (λ,φ) for all $\lambda \in \big\{ (\frac{\pi n_1}{a})^2 + (\frac{\pi n_2}{b})^2 \,\big|\, n_1 \in \mathbb{N}_0,\; n_2 \in \mathbb{N} \big\}$.

(c) Show that the eigenfunctions constructed in (b) form a complete ONS.

(please turn over)

Exercise 33. A one-dimensional eigenvalue problem. For the domain $\Omega =]0,1[\subset \mathbb{R}$ and $\alpha \in \mathbb{R}$ consider the following eigenvalue problem: Find $(\lambda, \varphi) \in \mathbb{R} \times \mathrm{H}^1(\Omega)$ such that

$$\int_\Omega \varphi' \, v' \, \mathrm{d}x + \alpha \varphi(1) v(1) = \lambda \int_\Omega \varphi \, v \, \mathrm{d}x \quad \text{for all } v \in \mathrm{H}^1(\Omega).$$

(a) Show that all eigenpairs solve the equations

$$-\varphi'' = \lambda \varphi \ \text{ in } \Omega, \qquad \varphi'(0) = 0, \qquad \varphi'(1) + \alpha \varphi(1) = 0.$$

(b) Show that $\alpha > 0$ implies that all eigenvalues are positive as well and give all eigenvalues $\lambda_k^{(0)}$ for the case $\alpha = 0$.

(c) Find the "characteristic equation" for the eigenvalue problem, i.e. a function $F : \mathbb{R} \to \mathbb{R}$ such that $F(\lambda) = 0$ holds if and only if λ is an eigenvalue.

(d) Derive from (c) in the case $\alpha < 0$ there is at most one negative eigenvalue and that the ordered eigenvalues $\lambda_k^{(\alpha)}$ satisfy

$$\lambda_1^{(\alpha)} < 0 = \lambda_1^{(0)} < \lambda_2^{(\alpha)} < \lambda_2^{(0)} < \cdots \lambda_{k-1}^{(0)} < \lambda_k^{(\alpha)} < \lambda_k^{(0)} < \cdots .$$

Exercise 34. Poincaré's inequality. Using Rellich's embedding theorem we want to derive the following Poincaré's inequality: *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Then,*

$$\exists \, C_{\mathrm{Poin}} > 0 \ \forall \, u \in \mathrm{H}^1(\Omega) \text{ with } \int_\Omega u \, \mathrm{d}x = 0 : \quad C_{\mathrm{Poin}} \int_\Omega u^2 \, \mathrm{d}x \le \int_\Omega |\nabla u|^2 \, \mathrm{d}x. \tag{2}$$

(a) Assume that (2) is false, and conclude using Rellich's embedding theorem that there must be a function $u \in \mathrm{H}^1(\Omega)$ such that

$$\nabla u = 0 \ \text{ a.e. in } \Omega, \qquad \int_\Omega u^2 \, \mathrm{d}x = 1, \qquad \text{and} \qquad \int_\Omega u \, \mathrm{d}x = 0.$$

(b) Prove that (a) produces a contradiction by showing that $\nabla u = 0$ a.e. implies that u must be equal to a constant a.e. in Ω. For functions in $\mathrm{H}^1(\Omega)$ this result is not immediate, because we do not even know that u is continuous.

Hint: Show that mollification $u_\varepsilon = u * \psi_\varepsilon$ satisfy $\nabla u_\varepsilon = 0$ away from $\partial\Omega$ and use connectedness of Ω.

Dates for the teaching evaluation: June 17–28, 2019
(token = name-year of Abel Prize winner).

Dates for the oral exams:
July 22–24, 2019 and September 30 – October 2, 2019.