Partial Differential Equations
Exercise Sheet 8

Exercise 27. Fourier series and Hilbert spaces.
Let $\Omega =]0, 2\pi[$ and $k \in \mathbb{N}_0$ and consider the Hilbert spaces

$$\begin{aligned}
\mathrm{H}^k(\Omega) &= \{ f \in \mathrm{L}^2(\Omega) \mid f, f', ..., f^{(k)} \in \mathrm{L}^2(\Omega) \}, \\
\mathrm{H}^k_{\mathrm{per}}(\Omega) &= \{ f \in \mathrm{H}^k(\Omega) \mid f^{(j)}(0) = f^{(j)}(2\pi) \text{ for } j = 0, ..., k{-}1 \}, \\
\mathrm{H}^k_0(\Omega) &= \text{closure of } \mathrm{C}^\infty_\mathrm{c}(\Omega) \text{ in } \mathrm{H}^k(\Omega).
\end{aligned}$$

Further let $S_n(t) = s_n \sin(nt)$ and $C_n(t) = c_n \cos(nt)$. Then, from previous courses we know that $\mathrm{L}^2(\Omega)$ has the following three complete orthonormal systems (cONS)

$$O_1 = \{ C_n, S_m \mid m \in \mathbb{N},\ n \in \mathbb{N}_0 \}, \quad O_2 = \{ C_{m/2} \mid m \in \mathbb{N}_0 \}, \quad O_3 = \{ S_{m/2} \mid m \in \mathbb{N} \}.$$

(a) For O_1 show that $f = \sum_1^\infty a_m S_m + \sum_0^\infty b_n C_n \in \mathrm{L}^2(\Omega)$ lies in $\mathrm{H}^1_{\mathrm{per}}(\Omega)$ if and only if $\sum_1^\infty l^2(|a_l|^2{+}|b_l|^2)$ is finite and that in this case we may differentiate the series representation of f term by term.

(b) For O_2 show that $f = \sum_0^\infty b_m C_{m/2}$ lies in $\mathrm{H}^1(\Omega)$ if and only if $\sum_0^\infty m^2 b_m^2 < \infty$.

(c) Show that O_3 lies in $\mathrm{H}^1_0(\Omega)$ and that $f = \sum_1^\infty a_m S_{m/2}$ lies in $\mathrm{H}^1_0(\Omega)$ if and only if $\sum_1^\infty m^2 a_m^2 < \infty$.

(General hints: (i) Use, without proof, that $\mathrm{H}^{k+1}(]0, 2\pi[) \subset \mathrm{C}^k([0, 2\pi])$ for $k = 0, 1, \ldots$. (ii) Compare the series differentiated term by terms with a suitable new expansion of the derivative. (iii) Take care of boundary terms in integrations by parts.)

Exercise 28. Lax-Milgram lemma in unbounded domains. Let $\Omega \subset \mathbb{R}^d$ be any domain (i.e., open and connected). On $H = \mathrm{H}^1_0(\Omega)$ define the bilinear form $B : H \times H \to \mathbb{R}$ via $B(u, v) = \int_\Omega \nabla u(x) \cdot A(x) \nabla v(x) + c(x)u(x)v(x)\, \mathrm{d}x$, with $A \in \mathrm{L}^\infty(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $c \in \mathrm{L}^\infty(\Omega)$. Further there is an $\alpha > 0$ with $\xi \cdot A(x)\xi \geq \alpha|\xi|^2$ for all $x \in \Omega$ and all $\xi \in \mathbb{R}^d$.

(a) Show that B is a symmetric and continuous bilinear form. Give sufficient conditions on c that guarantee that B is also coercive.

(b) Consider the Schrödinger operator $L_\lambda u = -\Delta u + V u + \lambda u$ in $\mathrm{H}^1_0(\Omega)$ with $\Omega = \mathbb{R}\times]0, \pi[$, where $V(x) = 1/(1{+}x_1^2)$. Show that the bilinear form B_λ (of Lax-Milgraqm type) associated with L_λ is coercive for $\lambda > -1$ but not for $\lambda \leq -1$.

(please turn over)

Exercise 29. Friedrichs' inequality. A domain $\Omega \subset \mathbb{R}^d$ is said to satisfy a FRIEDRICHS' INEQUALITY, if there exists a constant $C > 0$ such that

$$(*) \qquad C \int_\Omega u(x)^2 \, \mathrm{d}x \leq \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x \text{ for all } u \in \mathrm{C}_\mathrm{c}^\infty(\Omega).$$

The largest such C is called the Friedrichs constant $C_{\text{Fried}}(\Omega)$ of the domain. $C_{\text{Fried}}(\Omega) = 0$ means that no such positive constant exists.

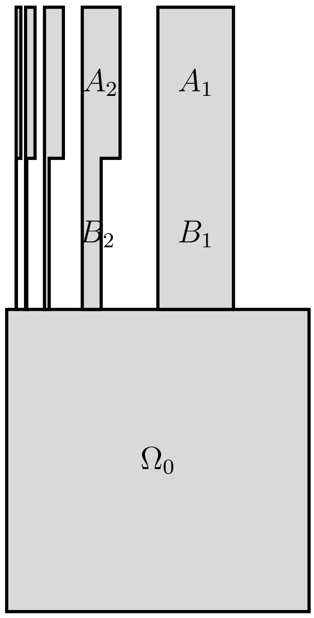
(a) Show that (*) holds if and only if the same inequality holds for all $u \in \mathrm{H}_0^1(\Omega)$.

(b) For $\Omega =]a, b[\subset \mathbb{R}^1$ calculate $C_{\text{Fried}}(\Omega)$ explicitly in terms of $b-a$ via a suitable Fourier expansion.

(c) For $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^m$ set $\Omega = X \times Y \subset \mathbb{R}^{k+m}$ and show $C_{\text{Fried}}(\Omega) = C_{\text{Fried}}(X) + C_{\text{Fried}}(Y)$. With this calculate the Friedrichs' constant for $]0, \ell_1[\times \cdots \times]0, \ell_d[$.

(d) For $\Omega_1 \subset \Omega_2$ show $C_{\text{Fried}}(\Omega_1) \geq C_{\text{Fried}}(\Omega_2)$ and conclude that every domain Ω fitting between two parallel hyperplanes with distance d satisfies $C_{\text{Fried}}(\Omega) \geq (\pi/d)^2$.

Exercise 30. (Counterexample to Friedrichs and Poincaré inequality).



We construct a bounded domain Ω (open and connected) that does not have a Lipschitz boundary. We set

$$\Omega = \Omega_0 \cup \bigcup_{n=0}^{\infty} (A_n \cup B_n) \qquad \text{with } \Omega_0 =]0,1[\times]-1,0[,$$
$$A_n =]1/2^n, 3/2^{n+1}[\times]1/2, 1[\quad \text{and}$$
$$B_n =]1/2^n, 1/2^n + 1/4^n[\times [0, 1/2].$$

Show that there exists a sequence of functions $u_n \in \mathrm{H}^1(\Omega)$ with the following properties:

(a) $u_n|_{\Omega_0} = 0$, (b) $\int_\Omega u_n \, \mathrm{d}x = 0$,

(c) $\text{support}(u_n) \subset \overline{A_n \cup B_n \cup A_{n+1} \cup B_{n+1}}$,

(d) $\int_\Omega u_n^2 \, \mathrm{d}x = 1$, (e) $\int_\Omega |\nabla u_n|^2 \, \mathrm{d}x \to 0$.

Dates for the teaching evaluation: June 17–28, 2019.

Dates for the oral exams:
July 22–25, 2019 and September 30 – October 2, 2019.