Partial Differential Equations
Exercise Sheet 7

Exercise 24. An alternative proof of the Lax-Milgram lemma. Let $B : H\times H \to \mathbb{R}$ be a bounded and coercive bilinear mapping satisfying

$$\exists\, C_1, c_0 > 0\ \forall\, u, v \in H : \quad |B(u, v)| \le C_1\|u\|\,\|v\| \ \text{ and } \ B(u, u) \ge c_0\|u\|^2.$$

Denote by $R : H^* \to H$ the Riesz isometry such that $\langle RF, v\rangle = F(v)$ for all $v \in H$.

(a) Show that $w = RB(u, \cdot)$ satisfies the relation $\langle w, v\rangle = B(u, v)$.

(b) For $\lambda \in \mathbb{R}$ and $F \in H^*$ we define the mapping $\Phi_\lambda : H \to H; u \mapsto u - \lambda R\big(B(u, \cdot) - F\big)$. Show that Φ_λ is affine and continuous and that $\Phi_\lambda(u) = u$ is equivalent to $B(u, \cdot) = F$.

(c) Construct $\lambda \in \mathbb{R}$ such that Φ_λ is a contraction (i.e. $\|\Phi_\lambda(u) - \Phi_\lambda(v)\| \le \rho\|u - v\|$ for some $\rho \in\,]0, 1[$). Conclude that for each $F \in H^*$ there exists a unique u with $B(u, \cdot) = F$.
Hint: For $\lambda > 0$ establish the estimate $\|\Phi_\lambda(u) - \Phi_\lambda(v)\|^2 \le (1 - a\lambda + b\lambda^2)\|u - v\|^2$ for suitable $a, b > 0$.

Exercise 25. Weak derivatives. We consider the domain $\Omega = B_1(0) \subset \mathbb{R}^d$ for $d \in \mathbb{N}$.

(a) For $\alpha \in R$ define the functions $u : x \mapsto |x|^\gamma$. Give necessary and sufficient conditions on (γ, d) such that $u \in \mathrm{L}^p(\Omega)$.

(b) For the functions u in (a) give necessary and sufficient conditions for $u \in \mathrm{W}^{1,p}(\Omega)$.

(c) Let v_k be a sequence with $v_k \rightharpoonup v_*$ in $\mathrm{L}^p(\Omega)$. Moreover, assume that for some $\alpha \in \mathbb{N}_0^d$ we have $\mathrm{D}^\alpha v_k = w_k \in \mathrm{L}^p(\Omega)$ with $w_k \rightharpoonup w_*$. Show that $\mathrm{D}^\alpha v_* = w_*$ in the sense of weak derivatives. (I.e. we can interchange weak convergence and differentiation.)

(please turn over)

Exercise 26. Approximation error in the Lax-Milgram lemma. We consider a bounded and coercive bilinear form $B : H \to H \to \mathbb{R}$ satisfying

$$\exists\, C_1, c_0 > 0\ \forall\, u, v \in H : \quad |B(u,v)| \leq C_1 \|u\|\,\|v\| \ \text{ and } \ B(u,u) \geq c_0 \|u\|^2.$$

Fix $F \in H^*$ let u be the solution of $B(u,v) = F(v)$ for all $v \in H$. Moreover, let $\left\{ \phi_j \in H \,\middle|\, j \in \mathbb{N} \right\}$ be a countable complete orthonormal system (cONS).

(a) Denote by $u^N \in X_N := \operatorname{span}\left\{ \phi_i \in H \,\middle|\, i = 1, \ldots, N \right\}$ the solution of $B(u^N, v^N) = F(v^N)$ for all $v^N \in X_N$ and by $e^N := u - u^N \in H$ the approximation error. Show that $B(e^N, w^N) = 0$ for all $w^N \in X_N$.

(b) For a given $u \in H$ we set $f_N(u) := \min\left\{ \|u - w^N\|_H \,\middle|\, w^N \in X_N \right\}$ which measures how good u can be approximated by elements in X_N. Show that the solution u satisfies

$$\|e^N\| = \|u - u^N\| \leq \frac{C_1}{c_0} f_N(u).$$

Hint: Insert $u - w^N$ in a suitable way.

(c) Now assume that B is also symmetric, i.e. $B(u,v) = B(v,u)$ for all $u, v \in H$. Show the improved estimate

$$\|e^N\| = \|u - u^N\| \leq \Big(\frac{C_1}{c_0}\Big)^{1/2} f_N(u).$$

Hint: Show the Pythagoras-type identity $B(u - w^N, u - w^N) = B(u - u^N, u - u^N) + B(u^N - w^N, u^N - w^N)$.

Prize exercise (not solved in tutorial)
Prize = 50 Euro book coupon.
For some $d \in \mathbb{N}$ find a function $f \in \mathrm{C}^0_c(\mathbb{R}^d)$,
such that the convolution $u = K_d * f$ does not lie in $\mathrm{C}^2(\mathbb{R}^d)$.

Deadline of submission of solutions:
July 7, 2019 at 23:59 h as PDF-file
per email to `alexander.mielke@wias-berlin.de`

Dates for the oral exams:
July 22–25, 2019 and September 30 – October 2, 2019.