Partial Differential Equations
Exercise Sheet 6

Exercise 21. Representation of second derivative for the Poisson kernel. Consider a radius $R > 0$ and a function $f \in \mathrm{C}^2_\mathrm{c}(\mathbb{R}^d)$ with $\mathrm{sppt}(f) \subset B_{R/3}(0)$. For $x \in B_{R/3}(0)$ we start from the representation of the first partial derivatives:

$$\partial_j u(x) = \int_{y \in B_R(x)} \partial_j K_d(x-y) f(y)\,\mathrm{d}y.$$

(a) Show that the second derivative has the representation

$$\partial_i \partial_j u(x) = \int_{y \in B_R(x)} \partial_j K_d(x-y) \partial_i f(y)\,\mathrm{d}y.$$

(b) Show that there exists constants M_{ij} such that for all radii $R > 0$ we have the identity

$$\int_{|z|=R} \partial_j K(z)\,\frac{z \cdot e_i}{|z|}\,\mathrm{d}a(z) = M_{ij}, \quad \text{where } e_i \text{ is the } i\text{th unit vector.}$$

(c) Establish the representation

$$\partial_i \partial_j u(x) = \int_{y \in B_R(x)} \partial_i \partial_j K_d(x-y)\big(f(y)-f(x)\big)\,\mathrm{d}y - M_{ij} f(x).$$

Hint: Subtract a ball $B_\varepsilon(x)$ and use $\partial_i f(y) = \partial_{y_i}\big(f(y)-f(x)\big)$ in (a).

Exercise 22. Approximation of Hölder continuous functions
(Otto L. Hölder, 1859 – 1937, Leipzig, studied in Berlin)
For $R > 0$ consider $\Omega_R = \overline{B_R(0)} \subset \mathbb{R}^d$ and the Hölder spaces with $\alpha \in \,]0,1[$:

$$\mathrm{C}^\alpha(\Omega_R) = \{\, u \in \mathrm{C}^0(\overline{\Omega}_R) \mid \|u\|_{\mathrm{C}^\alpha} := \|u\|_{\mathrm{L}^\infty} + [u]_\alpha < \infty \,\} \text{ with } [u]_\alpha := \sup_{x \neq y} \tfrac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

It is well-known that $(\mathrm{C}^\alpha(\Omega_R), \|\cdot\|_{\mathrm{C}^\alpha})$ is a Banach space.

(a) Show that for $\beta \in \,]\alpha, 1[$ we have the inclusion $\mathrm{C}^\beta(\Omega_R) \subset \mathrm{C}^\alpha(\Omega_R)$ with a strict inclusion. Hint: Consider $u(x) = |x|^\gamma$.

(b) Consider a sequence $(u_n)_{n\in\mathbb{N}}$ with $\|u_n\|_{\mathrm{C}^\alpha} \le C < \infty$ and $u_n \to u$ uniformly in Ω_R. Show that $u \in \mathrm{C}^\alpha(\Omega_R)$ with $\|u\|_{\mathrm{C}^\alpha} \le C$.

(c) Show that for any $u \in \mathrm{C}^\alpha(\Omega_R)$ there is a sequence $(\phi_n) \in \mathrm{C}^2(\Omega_R)$ with $\phi_n \to u$ uniformly in Ω_R and $\|\phi_n\|_{\mathrm{C}^\alpha} \le C < \infty$.
Hint: Consider smoothening convolutions $\phi_n = \Psi_n * u$ for suitable $\Psi_n \in \mathbb{R}^d$ with $\Psi_n \ge 0$.

(please turn over)

Exercise 23. Poisson's formula for a disc. (Siméon D. Poisson, 1781 – 1840, Paris)
Let $\Omega = B_R(0) \subset \mathbb{R}^2$, $g \in \mathrm{C}^0(\partial\Omega)$, and

$$u(x) = \int_{|y|=R} P(x,y)\, g(y)\, \mathrm{d}a \quad \text{with } P(x,y) = \frac{R^2 - |x|^2}{2\pi R\, |x-y|^2}. \tag{PI}$$

(a) Show that (PI) defines a function $u \in \mathrm{C}^2(\Omega)$ satisfying $\Delta u = 0$ in Ω.

(b) Establish $u \in \mathrm{C}(\overline{\Omega})$ and $u(y) = g(y)$ for $y \in \partial\Omega$.

(Hint: Show $P(x,y) \geq 0$, $\int_{\partial\Omega} P(x,y)\, \mathrm{d}a = 1$ for all $x \in \Omega$, and $P(x,y) \to 0$ for $x \to y_* \in \partial\Omega \setminus \{y\}$. Polar coordinates $x = r(\cos\phi, \sin\phi)$ and $y = R(\cos\psi, \sin\psi)$ may come in handy.)

Prize exercise (not solved in tutorial)
Prize = 50 Euro book coupon.
For some $d \in \mathbb{N}$ find a function $f \in \mathrm{C}^0_c(\mathbb{R}^d)$,
such that the convolution $u = K_d * f$ does not lie in $\mathrm{C}^2(\mathbb{R}^d)$.

Deadline of submission of solutions:
July 7, 2019 at 23:59 h as PDF-file
per email to `alexander.mielke@wias-berlin.de`