Partial Differential Equations
Exercise Sheet 5

Exercise 17. Analytic solutions for the heat equation. We consider several Cauchy problems for the equation $u_t = u_{xx}$ for $(t, x) \in \mathbb{R}\times\mathbb{R}$.

(a) Transform the equation onto a system of first order equations $A_1\partial_t w + A_2\partial_x w = Bw$ with constant matrices $A_1, A_2, B \in \mathbb{R}^{2\times2}$ and discuss which straight lines $t\nu_1 + x\nu_2 =$const are non-characteristic.

(b) Construct by power series expansion the unique solution of the (characteristic(!)) Cauchy problem with $u(0, x) = \sin(kx)$. (Hint: Work directly on $u_t = u_{xx}$ not using (a).) [Remark: There is no abstract theorem guaranteeing existence, so we have check by hand that there is a solution.]

(c) Show that the non-characteristic Cauchy problem with $u(0, x) = \frac{1}{1+x^2} = \mathrm{Re}\left(\frac{\mathrm{i}}{x+\mathrm{i}}\right)$ cannot have an analytic solutions near any point $(0, x_*)$.
(Hint: Calculate all derivatives $\partial_t^m u(0, x_*)$ and calculate the radius of convergence.)

(d) Find the unique analytic solution for the (non-characteristic) Cauchy problem with $u(t, 0) = \mathrm{Re}(\mathrm{e}^{\mathrm{i}\omega t})$ and $u_x(t, 0) = 0$.

Exercise 18. One-dimensional Dirichlet problem. On $\Omega =]0, \ell[\subset \mathbb{R}^1$ we consider the Dirichlet problem

$$-u''(x) = f(x) \text{ in } \Omega, \quad u(0) = g_0, \quad u(\ell) = g_\ell,$$

where $f \in \mathrm{C}^0([0, \ell])$ and $g_0, g_\ell \in \mathbb{R}$.

(a) Show that $w(x) = -\frac{1}{2}\int_0^\ell |x-y| f(y)\,\mathrm{d}y$ solves the ODE $-w'' = f$.
(Hint: Decompose $[0, \ell] = [0, x[\cup \{x\} \cup]x, \ell]$.)

(b) Give the unique solution for the Dirichlet problem for $f \equiv 0$.

(c) Now take the boundary conditions into account as well and show that there exist functions $G : [0, \ell]^2 \to \mathbb{R}$ and $\gamma_0, \gamma_\ell : [0, \ell] \to \mathbb{R}$ such that the unique solutions has the form $u(x) = \int_0^\ell G(x, y)f(y)\,\mathrm{d}y + \gamma_0(x)g_0 + \gamma_\ell(y)g_\ell$. Show that $G(x, y) = G(y, x)$, $G(0, y) = 0 = G(\ell, y)$, and $G(x, y) \geq 0$.

(please turn over)

Exercise 19. Harmonic functions. A function $u : \Omega \to \mathbb{R}$ is called ***harmonic*** on the open domain Ω, if it satisfies the following ***mean-value property***:

$$\forall\, x \in \Omega\ \forall\, r > 0 \text{ with } B_r(x) \Subset \Omega : \quad u(x) = \frac{1}{\omega_d r^{d-1}} \int_{y \in \partial B_r(x)} u(y)\,\mathrm{d}a(y).$$

Note that the right-hand side is an average over the sphere $\partial B_r(x)$ as ω_d denotes the $(d{-}1)$-dimensional surface measure of the sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. We want to show that a function $u \in \mathrm{C}^2(\Omega)$ is harmonic if and only if $\Delta u(x) = 0$ for all $x \in \Omega$.

(a) For fixed $x \in \Omega$ let $R(x) = \mathrm{dist}(x, \mathbb{R}^d \setminus \Omega)$ and define

$$\Phi(r) = \frac{1}{\omega_d r^{d-1}} \int_{y \in \partial B_r(x)} u(y)\,\mathrm{d}a(y).$$

Transform the integral via $y = x + rz$ with $z \in \mathbb{S}^{d-1}$, and show that $u \in \mathrm{C}(\Omega)$ implies that $\Phi : [0, R(x)[\ \to \mathbb{R}$ is continuous with $\Phi(0) = u(x)$.

(b) Show that a harmonic function $u \in \mathrm{C}^2(\Omega)$ satisfies $\Delta u(x) = 0$ for all $x \in \Omega$.
(Hint: Expand u in a Taylor series at x via $u(x{+}rz) = u(x) + r\nabla u(x)\cdot z + \frac{1}{2} r^2 z \cdot \nabla^2 u(x) z + o(r^2)$ for $r \to 0$.)

(c) For $u \in \mathrm{C}^2(\Omega)$ satisfying $\Delta u(x) = 0$ for all $x \in \Omega$ show that u is harmonic.
(Hint: Show that $\Phi'(r) = 0$ by differentiating the integral over $\mathbb{S}^{d-1}$ obtained in (a) and show $\omega_d r^{d-1} \Phi'(r) = \int_{\partial B_r(x)} \nabla u(y) \cdot \nu(y)\,\mathrm{d}a(y)$.)

Exercise 20. Theorem of Cauchy (1789-1857) & Kovalevskaya (1850-1891)

Construct the solution of the following Cauchy problems via power-series expansion.

$$u_{xx} + \sigma u_{yy} = 0, \quad u(x,0) = u_0(x) = \frac{x^2 - 1}{(x^2+1)^2},\ u_y(x,0) = u_1(x) = \frac{2 - 6x^2}{(x^2+1)^3},$$

where $\sigma \in \{-1, 1\}$.

(a) Check that the Theorem of Cauchy–Kovalevskaya is applicable in both cases.

(b) Find the power-series expansion explicitly. (Hint: Write u_0 as real part and u_1 as imaginary part of simple complex-valued functions.) Discuss the radius of convergence and the singularities of the explicit solutions.