Partial Differential Equations
Exercise Sheet 3

Exercise 9: Transformation of quasilinear equations.

We consider the quasilinear problem

$$a(x, u(x)) \cdot \nabla u(x) = g(x, u(x)) \text{ for } x \in \Omega \subset \mathbb{R}^d, \tag{Q}$$

where the function $u : \Omega \to \mathbb{R}$ is to be determined.

(a) Consider a coordinate change $x = \Phi(y)$ with a bijective mapping $\Phi : \widetilde{\Omega} \to \Omega$, where Φ and Φ^{-1} are in C^1. We let $v(y) = u(\Phi(y)) = (u\circ\Phi)(y)$. Which equation holds for v, if u is a solution of (Q)?

(b) Moreover, consider a bijection $\psi : \mathbb{R} \to \mathbb{R}$, such that ψ and ψ^{-1} are in C^1. Which equation holds for $w : \Omega \to \mathbb{R}; x \mapsto \psi(u(x))$, if u is a solution of (Q)?

(c) For the special case $\partial_t u + \widetilde{a}(t, x, u) \cdot \nabla_x u = b(t, x, u)$ give the equations for v and w from (a) und (b) in the form $\partial_t v + \dots$ and $\partial_t w + \dots$, respectively. Here, the coordinate change Φ in (a) should not depend on t.

(d) Which bijections Φ and ψ transform the equation $\partial_t u + \partial_{x_1} u + x_1 \partial_{x_2} u = u$ into the equation $\partial_t w + \partial_{x_1} w = 1$? Construct the general solution w and provide a formula for the general solution u.

Exercise 10: The Fundamental Lemma of the Calculus of Variations.

Consider an open domain $\Omega \subset \mathbb{R}^d$, a scalar function $b \in \mathrm{C}(\Omega)$ and a vector field $v \in \mathrm{C}^1(\Omega; \mathbb{R}^d)$.

(a) Assume that for all closed balls $B_r(x) \subset \Omega$ (as test volumes) we have $\int_{B_r(x)} b(y)\,\mathrm{d}y = 0$. Conclude that $b \equiv 0$.

(b) Assume that for all closed balls $B_r(x) \subset \Omega$ we have $\int_{B_r(x)} b(y)\,\mathrm{d}y = \int_{\partial B_r(x)} v(\eta) \cdot \nu(\eta)\,\mathrm{d}a(\eta)$. Conclude $b = \operatorname{div} v$ in Ω.

(c) Assume that $\int_\Omega b(x)\psi(x)\,\mathrm{d}y = 0$ for all $\psi \in \mathrm{C}^\infty_\mathrm{c}(\Omega)$. Show that $b \equiv 0$. (Here $\mathrm{C}^\infty_\mathrm{c}(\Omega)$ denotes the space of all infinitely often differentiable functions $\psi : \Omega \to \mathbb{R}$ such that the support $\operatorname{sppt}(\psi) = \operatorname{closure}\big(\{\, x \in \Omega \mid \psi(x) \neq 0 \,\}\big)$ is compact and contained in Ω.)

(please turn over)

Exercise 11: The Lemma of Du Bois–Reymond. This is a variant of the fundamental lemma of the calculus of variations. (See there for the definition of $\mathrm{C}_\mathrm{c}^\infty(\Omega)$.)

(a) (Classical version) Consider an open interval $I \subset \mathbb{R}$ and a function $a \in \mathrm{C}(I)$ satisfying

$$\forall\, \phi \in \mathrm{C}_\mathrm{c}^\infty(I): \quad \int_I \dot{\phi}(t)\, a(t)\, \mathrm{d}t = 0.$$

Show that there exists $a_* \in R$ such that $a(t) = a_*$ for all $t \in \mathbb{R}$.
(Hint: Choose $h \in \mathrm{C}_\mathrm{c}^\infty(I)$ with $\int_I h(t)\,\mathrm{d}t = 1$ and show that every $\psi \in \mathrm{C}_\mathrm{c}^\infty(I)$ can be written in the form $\psi = \dot{\phi} + ch$ for suitable $c \in \mathbb{R}$ and $\phi \in \mathrm{C}_\mathrm{c}^\infty(I)$.)

(b) Consider now a function $a \in \mathrm{C}(\mathbb{R}{\times}\mathbb{R}^d)$ satisfying

$$\int_{(t,x)\in\mathbb{R}\times\mathbb{R}^d} \partial_t\phi(t,x)a(t,x)\,\mathrm{d}(t,x) = 0 \quad \text{for all } \phi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}{\times}\mathbb{R}^d).$$

Construct a function $a_* \in \mathrm{C}(\mathbb{R}^d)$ such that $a(t,x) = a_*(x)$ for all $(t,x) \in \mathbb{R}{\times}\mathbb{R}^d$.
(Hint: Maybe test functions in product form are useful.)

Exercise 12: Weak solutions of a linear transport equation.
We define the ***weak solutions*** for the equation $\partial_t u(t,x) + \mathbf{v}\cdot\nabla_x u(t,x) = bu(t,x)$ to be any function $u \in \mathrm{C}^0(\mathbb{R}{\times}\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}\times\mathbb{R}} \big(\partial_t\phi(t,x) + \mathbf{v}\cdot\nabla_x\phi(t,x) + b\phi(t,x)\big)\, u(t,x)\,\mathrm{d}(t,x) = 0 \quad \text{for all } \phi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}{\times}\mathbb{R}^d).$$

(a) For a function $U \in \mathrm{C}^1(\mathbb{R}^d)$ find the unique (classical) solution of the Cauchy problem $\partial_t u + \mathbf{v}\cdot\nabla_x u = bu$ and $u(0,x) = U(x)$.

(b) Consider $W \in \mathrm{C}^0(\mathbb{R}^d)$ and set $\widetilde{u}(t,x) = \mathrm{e}^{bt}W(x{-}t\mathbf{v})$. Show that $\widetilde{u}$ is a weak solution.
(Hint: It may be helpful to introduce the coordinate $\xi = x{-}t\mathbf{v}$ and $\psi(t,\xi) = \mathrm{e}^{\pm bt}\phi(t,x)$.)

(c) Show that the Cauchy problem with $\widetilde{u}(0,x) = W(x)$ has a unique solution $\widetilde{u} \in \mathrm{C}^0(\mathbb{R}{\times}\mathbb{R}^d)$. (Hint: The lemma of Du Bois-Reymond can be useful.)