Partial Differential Equations
Exercise Sheet 2

Exercise 5: Scalar equation of first order.

(a) Find the solution of the semilinear PDE $u_t + u_y = -u^2$, which satisfies the initial condition $u(0, y) = 1/(1+y^2)$. For which (t, y) is the solution defined?. What is the range of t, such that the solution exists for all $y \in \mathbb{R}$?

(b) For arbitrary, differentiable $f : \mathbb{R} \to \mathbb{R}$ give the solution of the Cauchy problem

$$u_t + yu_y = y^2, \quad u(0, y) = f(y).$$

Exercise 6: Global versus local solutions. For $R > 0$ consider the strip $\Omega_R = \{ (x_1, x_2) \in \mathbb{R}^2 \mid |x_2| < R \} = \mathbb{R} \times \,]-R, R[$ and the linear PDE

$$x_2 \partial_{x_1} u(x_1, x_2) - x_1 \partial_{x_2} u(x_1, x_2) = 0 \quad \text{for } x \in \Omega_R.$$

(a) Calculate all characteristic curves.

(b) For $\mathcal{C} = \mathbb{R} \times \{0\}$ and $\varphi \in \mathrm{C}^1(\mathbb{R})$ we want to prescribe $u(\xi, 0) = \varphi(\xi)$. Show that for all $\xi \neq 0$ there exists a local solution.

(c) Which additional conditions does φ in (b) have to satisfy to obtain a global C^1 solution $u : \Omega_R \to \mathbb{R}$?

Exercise 7: Linear transport problem on the half axis.

Let Ω be the half axis $]0, \infty[\, \subset \mathbb{R}$. For constants $a, b \in \mathbb{R}$ we consider the scalar, linear equation

$$u_t + au_x = bu \quad \text{for } t > 0 \text{ and } x \in \Omega$$

together with the Cauchy data $u(0, x) = f(x)$ on Ω, where $f \in \mathrm{C}^1(\mathbb{R})$.

(a) Show that for $a \leq 0$ there is a unique solution for this Cauchy problem (recall $t > 0$).

(b) Provide all solutions of the above Cauchy problem for the case $a > 0$. Confirm *local uniqueness* near $(t, x) = (0, x_*)$ and show *global non-uniqueness*.

(c) Under what conditions does the above Cauchy problem (with $t > 0$) has a solution that satisfies the additional *boundary condition* $u(t, 0) = h(t)$, where $h \in \mathrm{C}^1(\mathbb{R})$?

(please turn over)

Exercise 8: The continuity equation. Given a globally Lipschitz-continuous vector field $V \in \mathrm{C}^1(\mathbb{R}^d;\mathbb{R}^d)$ the ***continuity equation*** for a mass density $\rho(t,x) \geq 0$ is given by

$$\partial_t \rho(t,x) + \mathrm{div}_x(\rho(t,x)V(t,x)) = 0.$$

Throughout we assume that $\rho \in \mathrm{C}^1(\mathbb{R}\times\mathbb{R}^d)$ and decays sufficiently fast in x, i.e. $0 \leq \rho(t,x) \leq C(1{+}|x|^2)^{-d}$.

(a) Show that the total mass $M(t) = \int_{\mathbb{R}^d} \rho(t,x)\,\mathrm{d}x$ is constant in time.

(b) Calculate the characteristic curves for the Cauchy problem with

$$\mathcal{C} = \{\, (0,\xi) \in \mathbb{R}\times\mathbb{R}^d \mid \xi \in \mathbb{R}^d \,\} \quad \text{and} \quad \rho(0,\xi) = \varphi(\xi)$$

in terms of the flow mapping $x(\,t\,;r,\xi) = \Phi_{t,r}(\xi)$ given by solving

$$\dot{x}(t) = V(t,x(t)) \quad \text{and} \quad x(r) = \xi.$$

(Hint: Use that all $\Phi_{r,t}(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ are diffeomorphisms and satisfy $\Phi_{r,s}\circ\Phi_{s,t} = \Phi_{r,t}$.)

(c) Show that ρ takes the form

$$\rho(t,x) = \varphi(\Phi_{t,0}(x))\det\big(\mathrm{D}_x\Phi_{t,0}(x)\big).$$

(Hint: Use that for matrices $A(t), X(t) \in \mathbb{R}^{d\times d}$ with $\dot{X}(t) = A(t)X(t)$ the determinant $\mu(t) = \det X(t)$ satisfies $\dot{\mu}(t) = \mathrm{trace}(A(t))\mu(t)$.)

(d) Relate the formula in (c) to the transformation rule applied to $\Phi_{t,0} : \mathbb{R}^d \to \mathbb{R}^d$ restricted to a domain Ω in order to compare the mass $\int_\Omega \rho(0,\xi)\,\mathrm{d}\xi$ and $\int_{\Phi_{t,0}(\Omega)} \rho(t,y)\,\mathrm{d}y$.