

# Damage of nonlinearly elastic materials at small strain – Existence and regularity results –

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Received 2 February 2009, revised 6 September 2009, accepted 5 November 2009

Published online 25 January 2010

**Key words** Damage evolution with spatial regularization, partial damage, rate-independent systems, energetic solutions, convexity of energy functional, temporal Lipschitz- and Hölder-continuity.

**MSC (2000)** 35K85, 49S05, 74C15, 74R20

This paper discusses an existence result for energetic solutions of rate-independent damage processes and the temporal regularity of the solution. We consider a body consisting of a physically nonlinearly elastic material undergoing small deformations and partial damage. The present work is a generalization of [16] concerning the properties of the stored elastic energy density as well as the suitable Sobolev space for the damage variable: While previous work assumes that the damage variable  $z$  satisfies  $z \in W^{1,r}(\Omega)$  with  $r > d$  for  $\Omega \subset \mathbb{R}^d$ , we can handle the case  $r > 1$  by a new technique for the construction of joint recovery sequences. Moreover, this work generalizes the temporal regularity results to physically nonlinearly elastic materials by analyzing Lipschitz- and Hölder-continuity of solutions with respect to time.

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## 1 Introduction

Damage describes the creation and growth of cracks and voids on the micro-level of a solid material. This process can be investigated by means of continuum damage mechanics, which goes back on Kachanov in 1958, see [9]. Within this approach, an inner variable, the damage variable, is incorporated to the constitutive law, where it describes the influence of damage on the elastic behavior of the material. In this paper we treat the case of isotropic damage, which presumes a uniform orientation distribution of the cracks and voids in the material. Hence the damage variable  $z$  is a scalar-valued function of time and space, where  $z(t, x)$  is defined as the volume fraction at time  $t$  of the undamaged material in a neighborhood of a material point  $x$  in the reference configuration  $\Omega \subset \mathbb{R}^d$ . Thus, the values of the function  $z$  range between 0 and 1, where  $z(t, x) = 1$  means no damage and  $z(t, x) = 0$  stands for maximal damage in the neighborhood of the point  $x \in \Omega$  at time  $t$ . Damage is considered as a unidirectional process, so that  $\partial_t z(t, x) \leq 0$  for a.e.  $(t, x) \in [0, T] \times \Omega$ . As in [4, 16, 24] we focus on partial damage, i.e.  $z = 0$  does not mean that all material is disintegrated. Instead we have in mind that the material consists of two constituents, like a matrix and fibers, where only one of them may experience damage. Thus, for  $z = 0$  the material is still able to support arbitrary stresses without further damage. For rate-independent models with complete damage we refer to [1, 18, 24].

The model that is analyzed in this paper was proposed by Frémond and Nedjar to describe the damage of concrete, see [6, Ch. 12] or [7]. It consists of a functional representing the free energy of the body and a dissipation potential accounting for the energy dissipated by the damage process. We restrict our analysis to the rate-independent case and neglect viscous effects. The free energy depends on time  $t \in [0, T]$ , the damage variable  $z \in [0, 1]$  and – in the small strain case – on the linearized Green-St. Venant strain tensor  $e(u) := \frac{1}{2}(\nabla u + \nabla u^\top)$ , where  $u : \Omega \rightarrow \mathbb{R}^d$  is the displacement field. The free energy is defined via three different energy terms:

$$\mathcal{E}(t, u, z) := \int_{\Omega} W(x, e(u), z) dx + \int_{\Omega} \frac{\kappa}{r} |\nabla z|^r dx - \langle l(t), u \rangle, \quad (1.1)$$

where the first term in (1.1) denotes the stored elastic energy, which is determined by the stored elastic energy density  $W : \Omega \times \mathbb{R}_{\text{sym}}^{d \times d} \times [0, 1] \rightarrow \mathbb{R}_{\infty}$ , where  $\mathbb{R}_{\infty} := \mathbb{R} \cup \{\infty\}$ . The properties of  $W$  will be specified in Subsect. 3.1 more

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precisely. The third term in formula (1.1) represents the work of external loadings, which may comprise both volume and surface forces. The second term in (1.1) involves the gradient  $\nabla z$  of the damage variable  $z$  and thus introduces an internal length scale. This term accounts for microscopic interactions, i.e. it considers the influence of damage in a point  $x$  on its neighborhood. Hence,  $\kappa > 0$  is often called the *factor of influence of damage*. The ansatz involving the damage gradient is quite often used in engineering for the exponent  $r = 2$ , see [8, 12]. Since [16] only deals with  $r > d$  it is the aim in this work to cover all  $1 < r < \infty$ .

The dissipation potential is considered to be of the following form:

$$\mathcal{R}(\dot{z}) := \int_{\Omega} R(x, \dot{z}) dx, \quad \text{where } R(x, v) := \begin{cases} \varrho(x)|v| & \text{if } v \in (-\infty, 0] \\ \infty & \text{if } v > 0 \end{cases} \tag{1.2}$$

with  $\varrho \in L^\infty(\Omega)$  satisfying  $0 < \varrho_0 \leq \varrho(x)$  for a.e.  $x \in \Omega$ .

This definition of the dissipation potential accounts for the unidirectionality of the damage process: Only those damage variables, that describe an increase of damage, lead to finite dissipation. Moreover, the dissipation potential defined via (1.2) is rate-independent, since it is homogeneous of degree one, i.e.:  $R(x, \alpha w) = \alpha R(x, w)$  for every  $\alpha > 0$  and every  $w \in \mathbb{R}$ . Hence, the dissipation potential generates a so-called dissipation distance:

$$\mathcal{D}(z_0, z_1) = \mathcal{R}(z_1 - z_0). \tag{1.3}$$

Specifying a suitable state space  $\mathcal{Q}$ , the triple  $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$  is called a rate-independent system and our aim is to construct energetic solutions  $(u, z) : [0, T] \rightarrow \mathcal{Q}$ . They are defined by satisfying the global energy balance (E) and the global stability (S) of Definition 2.1. These main tools for the energetic approach are explained in Sect. 2. Sect. 3 provides the assumptions that are made on the setting of the damage process throughout this paper and it contains the existence result and its proof. In Sect. 4 the temporal regularity of energetic solutions is analyzed under the assumption of additional convexity properties on the free energy. Finally, Sect. 5 discusses classes of free energies known in engineering, which fit into the framework of our setting.

One main difference to previous works on the existence analysis of rate-independent processes [5, 19, 20] is, that we do not claim a growth property on the stored elastic energy density of the form  $c_1|e|^p - C \leq W(x, e, z) \leq c_2|e|^p + \tilde{C}$  for constants  $c_1, c_2, C, \tilde{C} > 0$  and  $1 < p < \infty$ , which would lead under the assumption of convexity to a growth condition on the stresses of the form

$$(H4^*) \quad |\partial_e W(x, e, z)| \leq c(|e|^{p-1} + \tilde{c}) \quad \text{for constants } c, \tilde{c} > 0.$$

This condition is not applicable for our purposes, since we want to allow for stored elastic energy densities used in literature [28] to describe strain hardening, such as, e.g.,

$$W(e, z) := \frac{1+z}{4} (\text{tr } e)^2 + |e^D|^{\tilde{p}}, \quad \text{where } e^D := e - \frac{\text{tr } e}{d} \text{Id} \tag{1.4}$$

for some  $3 < \tilde{p} < \infty$ . Here, a coercivity inequality is only obtained for the exponent  $p = 2$ , i.e.  $\frac{1}{4}(|e|^2 - 1) \leq W(e, z)$ , whereas condition (H4\*) can only be verified for the exponent  $p = \tilde{p}$ , namely  $|\partial_e W(e, z)| = |\frac{1+z}{2} (\text{tr } e) \text{Id} + \tilde{p}|e^D|^{\tilde{p}-2} 2e^D| \leq |\text{tr } e| + 2\tilde{p}|e^D|^{\tilde{p}-1} \leq \max\{2\tilde{p}, 2^{\frac{\tilde{p}-3}{2}}\}(1 + |e|^{p-1})$ . We circumvent this problem by using the weaker stress control

$$(H4) \quad |\partial_e W(x, e, z)| \leq c(W(x, e, z) + \tilde{c}) \quad \text{for constants } c, \tilde{c} > 0.$$

The main challenge of analyzing the damage problem lies in the discontinuity of the dissipation distance  $\mathcal{D}$  arising from the unidirectionality of the damage process. Compared to [5, 19, 20], where the dissipation distance was assumed to be (weakly) continuous, another method is required for proving the stability of limit states, see (C2) in the abstract existence theorem 2.4 and Sect. 3.2.5. The possibly infinite-valued dissipation distance does not allow us to pass to the limit along a stable sequence in stability condition (S), see Definitions 2.1 and 2.2 as well as Sect. 3.2.5. To overcome this problem the so-called *joint recovery condition* was introduced in [16, 23], see here Sect. 3.2.5. It is based on the construction of suitable testfunctions for the global stability inequality to recover global stability of the limit. In the unidirectional damage problem the main difficulty is to find for each stable sequence  $(z_k)_{k \in \mathbb{N}}$  with  $z_k \rightharpoonup z$  in  $W^{1,r}(\Omega)$  and each testfunction  $\hat{z}$  with  $\mathcal{D}(z, \hat{z}) < \infty$  a sequence  $(\hat{z}_k)_{k \in \mathbb{N}}$  such that  $\hat{z}_k \rightharpoonup \hat{z}$  and

$$\limsup_{k \rightarrow \infty} \left( \int_{\Omega} \frac{\kappa}{r} |\nabla \hat{z}_k|^r dx + \mathcal{D}(z_k, \hat{z}_k) - \int_{\Omega} \frac{\kappa}{r} |\nabla z_k|^r dx \right) \leq \int_{\Omega} \frac{\kappa}{r} |\nabla \hat{z}|^r dx + \mathcal{D}(z, \hat{z}) - \int_{\Omega} \frac{\kappa}{r} |\nabla z|^r dx. \tag{1.5}$$

In [16], for  $r > d$  the choice  $\widehat{z}_k = \max\{0, \widehat{z} - \|z - z_k\|_\infty\}$  was possible because of the continuous embedding of  $W^{1,r}(\Omega) \subset C(\overline{\Omega})$ . To handle all  $r > 1$  we use the sequence  $\widehat{z}_k = \min\{z_k, \max\{0, \widehat{z} - \delta_k\}\}$  for a suitable sequence  $(\delta_k)_{k \in \mathbb{N}}$  with  $0 \leq \delta_k \rightarrow 0$ . Since we do not know the strong convergence  $\widehat{z}_k \rightarrow \widehat{z}$  in  $W^{1,r}(\Omega)$  we need to exploit certain cancellations in  $\int_\Omega \frac{\kappa}{r} |\nabla \widehat{z}_k|^r dx - \int_\Omega \frac{\kappa}{r} |\nabla z_k|^r dx$  to obtain (1.5), see the proof of Theorem 3.14.

In Sect. 4, following the ideas in [17] we prove the temporal continuity of energetic solutions under the assumption of uniform convexity of the free energy on sublevels. In [17] it was proven, that an energetic solution is Lipschitz-continuous with respect to time, if the free energy satisfies a uniform convexity inequality of the form

$$\mathcal{E}(t, \theta q_1 + (1-\theta)q_2) \leq \theta \mathcal{E}(t, q_1) + (1-\theta)\mathcal{E}(t, q_2) - c\theta(1-\theta)\|q_1 - q_2\|^\alpha \quad \text{for all } q_1, q_2 \in \mathcal{Q} \quad (1.6)$$

with  $\alpha = 2$ . We only need that (1.6) holds on sublevels of  $\mathcal{E}(t, \cdot)$ , i.e.  $c$  depends on the sublevel, and we allow for  $\alpha \geq 2$ , so that we prove Hölder-continuity of the energetic solution with respect to time. In Sect. 5 we demonstrate for an example that  $\alpha = 2$  is restricted to free energies  $\mathcal{E}$  of (sub-)quadratic growth with respect to the state  $q$ . Free energies having super-quadratic growth in one of the state variables may satisfy more general uniform convexity inequalities with  $\alpha > 2$ .

## 2 Energetic formulation

We analyze the damage problem within its energetic formulation. For this, we fix a state space  $\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$ , which is assumed to be a weakly closed subset of a reflexive Banach space. Our approach is solely based on the free energy functional  $\mathcal{E} : [0, T] \times \mathcal{Q} \rightarrow \mathbb{R}_\infty$  and the dissipation distance  $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \rightarrow \mathbb{R}_\infty$ . We search for an energetic solution  $q : [0, T] \rightarrow \mathcal{Q}$ , which is supposed to satisfy the global stability condition (S) and the global energy balance (E).

**Definition 2.1** (Energetic solution). A function  $q = (u, z) : [0, T] \rightarrow \mathcal{Q}$  is called an *energetic solution* for the rate-independent system  $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ , if  $t \mapsto \partial_t \mathcal{E}(t, q) \in L^1((0, T))$  and if for all  $t \in [0, T]$  we have  $\mathcal{E}(t, q(t)) < \infty$ , stability (S) and energy balance (E):

$$\text{for all } \tilde{q} = (\tilde{u}, \tilde{z}) \in \mathcal{Q} \text{ holds: } \quad \mathcal{E}(t, q(t)) \leq \mathcal{E}(t, \tilde{q}) + \mathcal{D}(z(t), \tilde{z}); \quad (S)$$

$$\mathcal{E}(t, q(t)) + \text{Diss}_{\mathcal{D}}(z, [0, t]) = \mathcal{E}(0, q(0)) + \int_0^t \partial_t \mathcal{E}(\xi, q(\xi)) d\xi \quad (E)$$

with  $\text{Diss}_{\mathcal{D}}(z, [s, t]) := \sup \sum_{j=1}^M \mathcal{D}(z(\xi_{j-1}), z(\xi_j))$ , where the supremum is taken over all  $M \in \mathbb{N}$  and all partitions  $s = \xi_0 < \xi_1 < \dots < \xi_M = t$ .

Stability inequality (S) suggests to introduce sets of stable states.

**Definition 2.2** (Set of stable states, stable sequence). The *set of stable states* at time  $t \in [0, T]$  is defined by

$$\mathcal{S}(t) := \{q \in \mathcal{Q} \mid \mathcal{E}(t, q) < \infty, \forall \tilde{q} \in \mathcal{Q} : \mathcal{E}(t, q) \leq \mathcal{E}(t, \tilde{q}) + \mathcal{D}(z, \tilde{z})\}.$$

A sequence  $(t_k, q_k)_{k \in \mathbb{N}} \subset [0, T] \times \mathcal{Q}$  is called a *stable sequence*, if (i) and (ii) hold:

(i)  $\sup_{k \in \mathbb{N}} \{\mathcal{E}(t_k, q_k)\} < \infty$ , i.e. there is a constant  $E \in \mathbb{R}$  such that

$$q_k \in L_E(t_k) := \{q \in \mathcal{Q} \mid \mathcal{E}(t_k, q) \leq E\}, \quad (2.1)$$

(ii)  $q_k \in \mathcal{S}(t_k)$  for every  $k \in \mathbb{N}$ .

In order to guarantee the existence of an energetic solution, certain general assumptions have to be made on  $\mathcal{E}$  and  $\mathcal{D}$ , see also [13, 23]. The energy  $\mathcal{E} : [0, T] \times \mathcal{Q} \rightarrow \mathbb{R}_\infty$  has to fulfill the following conditions:

$$\begin{aligned} \text{Compactness of energy sublevels: } & \forall t \in [0, T] \forall E \in \mathbb{R} : \\ & L_E(t) := \{q \in \mathcal{Q} \mid \mathcal{E}(t, q) \leq E\} \text{ is weakly seq. compact.} \end{aligned} \quad (E1)$$

$$\begin{aligned} \text{Uniform control of the power: } & \exists c_0 \in \mathbb{R} \exists c_1 > 0 \forall (t, q) \in [0, T] \times \mathcal{Q} \text{ with } \mathcal{E}(t, q) < \infty : \\ & \mathcal{E}(\cdot, q) \in C^1([0, T]) \text{ and } |\partial_t \mathcal{E}(t, q)| \leq c_1(c_0 + \mathcal{E}(t, q)) \text{ for all } t \in [0, T]. \end{aligned} \quad (E2)$$

Condition (E2) enables to apply Gronwall’s lemma in order to derive a Lipschitz-estimate for  $\mathcal{E}$  with respect to time:

$$|\mathcal{E}(t, q) - \mathcal{E}(s, q)| \leq \left( e^{c_1|t-s|} - 1 \right) (\mathcal{E}(t, q) + c_0) \leq e^{c_1 T} (\mathcal{E}(t, q) + c_0) |t - s|. \tag{2.2}$$

Hence, if  $\mathcal{E}(t, q) < E$  for  $E \in \mathbb{R}$ , then, for  $c_E := e^{c_1 T} (E + c_0)$ , estimate (2.2) implies

$$|\mathcal{E}(t, q) - \mathcal{E}(s, q)| \leq c_E |t - s|. \tag{2.3}$$

The abstract existence theory requires the following general assumptions on the dissipation distance  $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \rightarrow [0, \infty]$ :

$$\begin{aligned} \text{Quasi-distance: } \forall z_1, z_2, z_3 \in \mathcal{Z} : \mathcal{D}(z_1, z_2) = 0 &\Leftrightarrow z_1 = z_2 \text{ and} \\ \mathcal{D}(z_1, z_3) &\leq \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3); \end{aligned} \tag{D1}$$

$$\text{Semi-continuity: } \mathcal{D} : \mathcal{Z} \times \mathcal{Z} \rightarrow [0, \infty] \text{ is weakly sequentially lower semi-continuous.} \tag{D2}$$

**Remark 2.3.**  $\mathcal{D}$  is an extended quasi-distance on  $\mathcal{Z}$ , since all metric axioms except symmetry are satisfied and since the value  $\infty$  is allowed.  $\mathcal{D}$  on  $\mathcal{Q}$  is a pseudo-distance or semi-distance, because for  $q_1 = (u_1, z_1)$ ,  $q_2 = (u_2, z_2)$  the property  $\mathcal{D}(z_1, z_2) = 0$  not necessarily implies  $q_1 = q_2$ .

Conditions (E1), (E2) and (D1), (D2) are useful to state an abstract existence result for the energetic formulation of rate-independent problems. This abstract version of the main existence theorem was developed within the works [5, 13, 23].

**Theorem 2.4** (Abstract main existence theorem). *Let  $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$  satisfy conditions (E1), (E2) and (D1), (D2). Moreover, let the following compatibility conditions hold: For every stable sequence  $(t_k, q_k)_{k \in \mathbb{N}}$  with  $t_k \rightarrow t$ ,  $q_k \rightarrow q$  in  $[0, T] \times \mathcal{Q}$  we have*

$$\partial_t \mathcal{E}(t, q_k) \rightarrow \partial_t \mathcal{E}(t, q), \tag{C1}$$

$$q \in \mathcal{S}(t). \tag{C2}$$

Then, for each  $q_0 \in \mathcal{S}(0)$  there exists an energetic solution  $q : [0, T] \rightarrow \mathcal{Q}$  for  $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$  satisfying  $q(0) = q_0$ .

The proof of Theorem 2.4 is based on a time-discretization, where conditions (E1), (D2) ensure the existence of a minimizer for the time-incremental minimization problem at each time-step. For a given partition  $\Pi := \{0 = t_0 < t_1 < \dots < t_M = T\}$ , for every  $k = 1, \dots, M$  we have to

$$\text{find } q_k \in \text{Argmin} \{ \mathcal{E}(t_k, \tilde{q}) + \mathcal{D}(z_{k-1}, \tilde{z}) \mid \tilde{q} = (\tilde{u}, \tilde{z}) \in \mathcal{Q} \}. \tag{IP}$$

One then defines a piecewise constant interpolant  $q^\Pi$  with  $q^\Pi(t) := q_{k-1}$  for  $t \in [t_{k-1}, t_k)$  and  $q^\Pi(T) = q_M$ . Choosing a sequence  $(\Pi_m)_{m \in \mathbb{N}}$  of partitions, where the fineness of  $\Pi_m$  tends to 0 as  $m \rightarrow \infty$ , it is possible to apply Helly’s selection principle to the sequence  $(q^{\Pi_m})_{m \in \mathbb{N}}$ . Then, it is shown that the limit function fulfills the properties (S) and (E) of an energetic solution. See e.g. [23] for a detailed proof.

### 3 Existence analysis for the damage model

The aim in this section is to prove the existence of an energetic solution for the damage problem by applying the abstract existence theorem 2.4 on this setup. Thereto, we introduce general assumptions on the given data like the domain  $\Omega \subset \mathbb{R}^d$ , the external loadings and the stored elastic energy density.

#### 3.1 Assumptions and the existence result

We consider a bounded domain  $\Omega \subset \mathbb{R}^d$  with a Lipschitz-boundary  $\partial\Omega$  modeling the reference configuration of a nonlinearly elastic material. This body undergoes a damage process driven by exterior forces  $l(t)$ , which may change with time. Furthermore, the body is assumed to be fixed at one part  $\Gamma_D$  of its boundary  $\partial\Omega$  with positive  $(d-1)$ -dimensional measure  $\mathcal{L}^{d-1}(\Gamma_D) > 0$ , such that the displacement field  $\tilde{u} : \Omega \rightarrow \mathbb{R}^d$  is prescribed there:  $\tilde{u} = u_D(t)$  on  $\Gamma_D$  for  $t \in [0, T]$ . This means that we allow for time-dependent Dirichlet conditions, where the Dirichlet boundary  $\Gamma_D$  itself is fixed in time. From now on we write  $u_D(t)$  also for the given extension into the domain  $\Omega$  of the function  $u_D$  specifying the Dirichlet condition on the boundary. Hence, using the splitting  $\tilde{u} = u + u_D(t)$ , we define the state  $q = (u, z)$  and the free energy

$$\mathcal{E}(t, u, z) = \int_{\Omega} W(x, e(u) + e_D(t), z) dx + \frac{\kappa}{r} \int_{\Omega} |\nabla z|^r dx - \langle l(t), u + u_D(t) \rangle, \tag{3.1}$$

where  $u = 0$  on  $\Gamma_D$ , such that  $u + u_D(t) = u_D(t)$  on  $\Gamma_D$ . Moreover,  $e(u) := \frac{1}{2}(\nabla u + \nabla u^\top)$  and similarly  $e_D(t) := \frac{1}{2}(\nabla u_D(t) + \nabla u_D(t)^\top)$  denote the linearized strain tensor of  $u$  and  $u_D(t)$ , respectively.

We make the following general assumptions on the domain  $\Omega$  and the given data  $u_D, l$ :

$$\left. \begin{array}{l} \text{(A1)} \quad \Omega \text{ is a bounded Lipschitz-domain, } \Gamma_D \subset \partial\Omega \text{ with } \mathcal{L}^{d-1}(\Gamma_D) > 0, \\ \text{(A2)} \quad u_D \in C^1([0, T], W^{1,\infty}(\Omega, \mathbb{R}^d)) \text{ with } c_D := \max\{1, \mathcal{L}^d(\Omega)\} \|u_D\|_{C^1([0, T], W^{1,\infty}(\Omega, \mathbb{R}^d))}, \\ \text{(A3)} \quad l \in C^1([0, T], W^{-1,p'}(\Omega, \mathbb{R}^d)) \text{ with } c_l := \|l\|_{C^1([0, T], W^{-1,p'}(\Omega, \mathbb{R}^d))}. \end{array} \right\} \quad (3.2)$$

Here  $p' = p/(p-1)$ , where  $p \in (1, \infty)$  will be fixed in (H3) below. Throughout this paper we use  $\mathcal{L}^m(A)$  for the  $m$ -dimensional Lebesgue-measure of a set  $A \subset \mathbb{R}^m$  with  $m \in \{d-1, d\}$ .

Furthermore, we claim the following hypotheses on the stored elastic energy density:

**(H1) Carathéodory function:**  $W(x, \cdot, \cdot) \in C^0(\mathbb{R}_{\text{sym}}^{d \times d} \times [0, 1])$  for a.e.  $x \in \Omega$  and  $W(\cdot, e, z)$  is measurable in  $\Omega$ .

**(H2) Convexity:** For every  $(x, z) \in \Omega \times [0, 1]$  the function  $W(x, \cdot, z)$  is convex.

**(H3) Coercivity:** There are constants  $c_1, C > 0$ , and  $1 < p < \infty$  such that for all  $(x, e, z) \in \Omega \times \mathbb{R}_{\text{sym}}^{d \times d} \times [0, 1]$  we have

$$c_1 |e|^p - C \leq W(x, e, z).$$

**(H4) Stress control:** For all  $(x, z) \in \Omega \times [0, 1]$  we have  $W(x, \cdot, z) \in C^1(\mathbb{R}_{\text{sym}}^{d \times d})$  and there exist constants  $c > 0, \tilde{c} \geq 0$  such that for all  $(x, e, z) \in \Omega \times \mathbb{R}_{\text{sym}}^{d \times d} \times [0, 1]$  we have

$$|\partial_e W(x, e, z)| \leq c(W(x, e, z) + \tilde{c}).$$

**(H5) Monotonicity:** There are constants  $K > 0, \tilde{K} \geq 0$  so that for all  $(x, e, z), (x, e, \tilde{z}) \in \Omega \times \mathbb{R}_{\text{sym}}^{d \times d} \times [0, 1]$  with  $z \leq \tilde{z}$  we have

$$W(x, e, z) \leq W(x, e, \tilde{z}) \leq K(W(x, e, z) + \tilde{K}).$$

Hypotheses (H1)-(H3) will ensure condition (E1). Hypothesis (H4) is the basis to prove Lipschitz-estimate (2.3). The first estimate in assumption (H5) reflects the physical property of damage, that an increase of damage decreases the stored elastic energy. The second estimate in (H5) states that the remaining elastic properties after all damage has occurred are still comparable to the undamaged material. This assumption is reasonable, because we only treat partial damage in our analysis. Total damage would neither allow for the second inequality in (H5) nor for coercivity (H4), since for a completely disintegrated body the displacement field has no meaning any longer.

In view of hypothesis (H3) we choose the space of admissible displacements as

$$\mathcal{U} := \{u \in W^{1,p}(\Omega, \mathbb{R}^d) \mid u = 0 \text{ on } \Gamma_D\}. \quad (3.3)$$

Under consideration of formula (3.1) we put the set of admissible damage variables

$$\mathcal{Z} := \{z \in W^{1,r}(\Omega) \mid 0 \leq z \leq 1 \text{ a.e. in } \Omega\} \quad (3.4)$$

and  $\mathcal{Q} := \mathcal{U} \times \mathcal{Z}$  indicates the set of admissible states. By  $\mathcal{X} := W^{1,p}(\Omega, \mathbb{R}^d) \times W^{1,r}(\Omega)$  we denote the Banach space that specifies the topology for weak convergence. Within the analysis we will consider the convergence of sequences  $(q_k)_{k \in \mathbb{N}} \subset \mathcal{Q}$  to a limit  $q$  with respect to the weak topology of  $\mathcal{X}$  and we will indicate the weak convergence in  $\mathcal{X}$  by  $q_k \rightharpoonup q$  in  $\mathcal{X}$ .

With these tools at hand we state the existence theorem for the damage problem.

**Theorem 3.1** (Existence theorem for the damage problem). *Let  $\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$  be given as above. Let  $\mathcal{E}$  be defined via (3.1) such that (3.2) and (H1)-(H5) hold. Let  $\mathcal{D}$  be given by (1.2) and (1.3). Then, for the rate-independent damage process defined by  $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$  there exists an energetic solution for any initial state  $q_0 \in \mathcal{S}(0)$ .*

The proof of Theorem 3.1 is carried out in Sect. 3.2. The main difficulty lies in the missing weak continuity of the dissipation distance, which especially complicates the proof of the compatibility conditions (C1) and (C2), see Sects. 3.2.4, 3.2.5.

### 3.2 Proof of the existence theorem for the damage problem

In this subsection the assumptions (E1), (E2), (D1), (D2) and (C1), (C2) of the abstract main existence theorem 2.4 are checked. An analysis similar to ours is given in [14, 19, 20]. As our damage model allows for more general assumptions in (H1)-(H5) we repeat all steps for the readers convenience. In particular, previous work (e.g. [19]) assumes (H2) and (H4\*), where (H4\*) ensures that  $\partial_A W(x, A, z) \in L^{p'}(\Omega, \mathbb{R}^{d \times d})$ , which is not guaranteed by (H4).

For a shorter notation in the proofs we introduce the following abbreviations:

$$\left. \begin{aligned} \mathcal{I}(t, u, z) &:= \int_{\Omega} W(x, e(u) + e_D(t), z) dx, \\ \mathcal{C}(z) &:= \frac{\kappa}{r} \int_{\Omega} |\nabla z|^r dx, \\ \mathcal{J}(t, u, z) &:= \mathcal{I}(t, u, z) - \langle l(t), u + u_D(t) \rangle, \end{aligned} \right\} \tag{3.5}$$

such that

$$\mathcal{E}(t, u, z) = \mathcal{I}(t, u, z) + \mathcal{C}(z) - \langle l(t), u + u_D(t) \rangle = \mathcal{J}(t, u, z) + \mathcal{C}(z). \tag{3.6}$$

A basic tool in the proofs is Korn’s inequality, which holds for functions  $u \in \mathcal{U} \subset W^{1,p}(\Omega, \mathbb{R}^d)$  for  $\mathcal{U}$  defined by (3.3).

**Theorem 3.2** (Korn’s inequality [11, 26]). *Let  $\Omega \subset \mathbb{R}^d$  and  $\Gamma_D \subset \partial\Omega$  satisfy (A1) and let  $1 < p < \infty$ . There is a constant  $C_K = C_K(\Omega, p)$  such that for every  $v \in \mathcal{U}$  the following estimate holds:*

$$\|v\|_{W^{1,p}(\Omega, \mathbb{R}^d)} \leq C_K \|e(v)\|_{L^p(\Omega, \mathbb{R}^{d \times d})}. \tag{3.7}$$

#### 3.2.1 Compactness of the energy sublevels (E1)

In the following, the weak sequential compactness of the energy sublevels is established using the standard approach in the direct method of the calculus of variations.

**Lemma 3.3.** *Let the assumptions (3.2) and (H1)-(H5) hold. Then there exist constants  $c_3, C_3 > 0$  such that  $\mathcal{E}(t, \cdot, \cdot) : \mathcal{U} \times \mathcal{Z} \rightarrow \mathbb{R}$  satisfies a growth estimate of the form*

$$\mathcal{E}(t, u, z) \geq c_3 \left( \|u\|_{W^{1,p}(\Omega, \mathbb{R}^d)}^p + \|z\|_{W^{1,r}(\Omega)}^r \right) - C_3 \text{ for all } (u, z) \in \mathcal{U} \times \mathcal{Z}. \tag{3.8}$$

**Proof.** For  $(x, e, z, A) \in \Omega \times \mathbb{R}_{\text{sym}}^{d \times d} \times [0, 1] \times \mathbb{R}^d$  we set

$$\overline{W}(x, e, z, A) := W(x, e, z) + \frac{\kappa}{r} |A|^r.$$

Let  $u \in \mathcal{U}$ . Using hypotheses (A2), (A3), (H3), Young’s and Korn’s inequality we get

$$\begin{aligned} \mathcal{E}(t, u, z) &= \int_{\Omega} \overline{W}(x, e(u) + e_D(t), z, \nabla z) dx - \langle l(t), u + u_D(t) \rangle \\ &\geq c_1 (\|e(u)\|_{L^p} - c_D)^p - (C + \frac{\kappa}{r}) \mathcal{L}^d(\Omega) - c_l (\|u\|_{W^{1,p}} + c_D) + \frac{\kappa}{r} \|z\|_{W^{1,r}}^r \\ &\geq c_1 (2^{1-p} \|e(u)\|_{L^p}^p - c_D^p) - (C + \frac{\kappa}{r}) \mathcal{L}^d(\Omega) - c_l (\|u\|_{W^{1,p}} + c_D) + \frac{\kappa}{r} \|z\|_{W^{1,r}}^r \\ &\geq \frac{2^{1-p} c_1}{C_K^p} \|u\|_{W^{1,p}}^p - (C + \frac{\kappa}{r}) \mathcal{L}^d(\Omega) - c_1 c_D^p - \frac{1}{p'} \left(\frac{c_l}{\varepsilon}\right)^{p'} - \frac{(\varepsilon \|u\|_{W^{1,p}})^p}{p} - c_l c_D + \frac{\kappa}{r} \|z\|_{W^{1,r}}^r \\ &\geq \frac{2^{-p} c_1}{C_K^p} \|u\|_{W^{1,p}}^p + \frac{\kappa}{r} \|z\|_{W^{1,r}}^r - (C + \frac{\kappa}{r}) \mathcal{L}^d(\Omega) - c_1 c_D^p - c_l c_D - \frac{1}{p'} \left(\frac{c_l}{\varepsilon}\right)^{p'}, \end{aligned} \tag{3.9}$$

where Young’s inequality with  $\varepsilon := \left(\frac{2^{-p} c_1 p}{C_K^p}\right)^{\frac{1}{p}}$  leads to the third inequality of (3.9).

This proves (3.8) with suitable  $c_3$  and  $C_3$ . □

**Proposition 3.4.** *Let assumptions (3.2) as well as (H1)-(H5) hold. Then  $\mathcal{E}(t, \cdot, \cdot)$  is sequentially lower semicontinuous with respect to the weak topology of  $\mathcal{X}$  and its sublevels  $L_E(t)$  are weakly sequentially compact in  $\mathcal{X}$ .*

**Proof.** First, we obtain that  $\mathcal{C}(\cdot) : W^{1,r}(\Omega) \rightarrow \mathbb{R}$  is bounded from below by 0 and lower semicontinuous, since every  $L^r$ -converging sequence contains a subsequence that converges pointwise a.e. by Riesz' convergence theorem. Moreover,  $\mathcal{C}(\cdot)$  is convex and hence weakly sequentially lower semicontinuous by [2] p. 49, Th. 1.2.. Furthermore, [2] p. 74 states the weak sequential lower semicontinuity of  $J(\xi, \eta) = \int_{\Omega} W(x, \xi(x), \eta(x)) dx$  for  $\eta = z, \xi = e(u)$  on  $W^{1,p}(\Omega, \mathbb{R}^d) \times L^r(\Omega)$  if hypotheses (H1)-(H3) are satisfied, because the compact embedding of  $W^{1,r}(\Omega) \Subset L^r(\Omega)$  by Rellich's embedding theorem implies the strong  $L^r$ -convergence of a sequence converging weakly in  $W^{1,r}(\Omega)$ . Hence,  $\mathcal{E}$  is weakly sequentially lower semicontinuous on  $\mathcal{X}$ .

Let now  $(u_k, z_k)_{k \in \mathbb{N}} \subset L_E(t) \subset \mathcal{Q}$ . Then estimate (3.8) yields

$$\|e(u_k)\|_{W^{1,p}(\Omega, \mathbb{R}^d)} + \|z_k\|_{W^{1,r}(\Omega)} \leq \left(\frac{E+C_3}{c_3}\right)^{\frac{1}{p}} + \left(\frac{E+C_3}{c_3}\right)^{\frac{1}{r}}. \quad (3.10)$$

Since the spaces  $W^{1,p}(\Omega, \mathbb{R}^d), W^{1,r}(\Omega)$  are real, reflexive Banach spaces for  $1 < p, r < \infty$ , the sequence  $(u_k, z_k)_{k \in \mathbb{N}}$  contains subsequence converging weakly in  $\mathcal{X}$ . In particular, due to the compact embedding of  $\mathcal{X}$  into  $L^p(\Omega, \mathbb{R}^d) \times L^r(\Omega)$  and Riesz' convergence theorem we find a further subsequence converging pointwise a.e. in  $\Omega$  with their limits  $z \in \mathcal{Z}$  and  $u \in W^{1,p}(\Omega, \mathbb{R}^d)$  with  $u=0$  on  $\Gamma_D$ . Due to the weak sequential lower semicontinuity of  $\mathcal{E}(t, \cdot)$  on  $\mathcal{X}$  the limit  $(u, z)$  of the subsequence is an element of  $L_E(t)$ . This proves that the sublevels are weakly sequentially compact, i.e. (E1) and it implies that  $u \in \mathcal{U}$ .  $\square$

**Remark 3.5** (Existence, uniqueness of minimizers). As a direct consequence of Proposition 3.4 one obtains the existence of minimizers for the minimization problems

$$\min_{(\tilde{u}, \tilde{z}) \in \mathcal{Q}} (\mathcal{E}(t, \tilde{u}, \tilde{z}) + \mathcal{D}(z, \tilde{z})), \quad \min_{\tilde{u} \in \mathcal{U}} \mathcal{J}(t, \tilde{u}, z) \quad \text{and} \quad \min_{\tilde{q} \in \mathcal{Q}} \mathcal{J}(t, \tilde{q})$$

for all  $t \in [0, T]$  and all  $z \in \mathcal{Z}$ , as well as for the time-incremental problems (IP) in every time step. This implies that the stable sets  $\mathcal{S}(t)$  are non-empty for every  $t \in [0, T]$ . If strict convexity is claimed in (H2), then the minimizers  $u \in \mathcal{U}$  of  $\mathcal{J}(t, \cdot, z)$  are even unique.

### 3.2.2 Control of the power of the energy (E2)

In this subsection condition (E2) is proven under the assumptions (3.2) and (H1)-(H4).

As a first step we derive a Lipschitz-estimate for the stored elastic energy density.

**Lemma 3.6** (Lipschitz-estimate for  $W$ ). *Let  $W$  satisfy (H2) and (H4). Then for every  $(x, z) \in \Omega \times [0, 1]$  and any  $e, \tilde{e} \in \mathbb{R}_{\text{sym}}^{d \times d}$  it holds:*

$$|W(x, \tilde{e}, z) - W(x, e, z)| \leq \frac{c}{2} (W(x, e, z) + W(x, \tilde{e}, z) + 2\tilde{c}) |\tilde{e} - e|. \quad (3.11)$$

**Proof.** Under consideration of (H2) and (H4) we obtain for  $\alpha \in [0, 1]$ :

$$\begin{aligned} |W(x, \tilde{e}, z) - W(x, e, z)| &= \left| \int_0^1 \partial_e W(x, (e + \alpha(\tilde{e} - e)), z) : (\tilde{e} - e) d\alpha \right| \\ &\leq \int_0^1 c(\alpha(W(x, \tilde{e}, z) + \tilde{c}) + (1 - \alpha)(W(x, e, z) + \tilde{c})) |\tilde{e} - e| d\alpha \\ &= \frac{c}{2} (W(x, \tilde{e}, z) + \tilde{c}) |\tilde{e} - e| + \frac{c}{2} (W(x, e, z) + \tilde{c}) |\tilde{e} - e|, \end{aligned}$$

which gives the result.  $\square$

Now, we are in a position to prove condition (E2).

**Theorem 3.7.** *Let (H2)-(H4) and (3.2) be satisfied. Then there exist constants  $c_0 \geq 0, c_1 > 0$  such that for every  $(t_q, q) \in [0, T] \times \mathcal{Q}$  with  $\mathcal{E}(t_q, q) < \infty$  holds:*

$\mathcal{E}(\cdot, q) \in C^1([0, T])$ , where

$$\partial_t \mathcal{E}(t, q) = \int_{\Omega} \partial_e W(x, e(u) + e_D(t), z) : \dot{e}_D(t) dx - \langle \dot{l}(t), u + u_D(t) \rangle - \langle l(t), \dot{u}_D(t) \rangle \quad (3.12)$$

$$\text{and} \quad |\partial_t \mathcal{E}(t, q)| \leq c_1 (\mathcal{E}(t, q) + c_0) \quad \text{for every } t \in [0, T]. \quad (3.13)$$

**Proof.** Note that the assumption  $\mathcal{E}(t_q, q) =: E_q < \infty$  for some  $t_q \in [0, T]$  together with (A2), (A3) and (H4) yields  $\mathcal{E}(t, q) < \tilde{E}_q < \infty$  for every  $t$  in a sufficiently small neighborhood  $\mathcal{U}(t_q) \subset [0, T]$  of  $t_q$ , since  $\mathcal{E}(\cdot, q)$  as the sum and composition of the continuous functions  $l(\cdot), u_D(\cdot), W(x, \cdot, z), \langle \cdot, \cdot \rangle$  and  $\int_{\Omega} (\cdot) dx$  is a continuous function itself. In a first step, we prove that the time-derivative  $\partial_t \mathcal{E}(\cdot, q)$  exists in  $\mathcal{U}(t_q)$ . In this neighborhood the estimate (3.13) can be derived as a second step. We will obtain that the constants are independent of  $t_q$  and  $\mathcal{U}(t_q)$ . This allows us to apply Gronwall's lemma and Lipschitz-estimate (2.3) uniformly in each neighborhood of any time  $t_q$  with finite energy. Thus,  $\mathcal{E}(\cdot, q) \in C^1([0, T])$  follows.

Now, we prove the existence of  $\partial_t \mathcal{E}(t, q)$  for  $t \in \mathcal{U}(t_q)$ . Thereto we define for  $t \in \mathcal{U}(t_q)$

$$h(x, t, \alpha) := \begin{cases} \frac{1}{\alpha} (W(x, e(u)+e_D(t+\alpha), z) - W(x, e(u)+e_D(t), z)) & \text{if } \alpha \neq 0 \\ \partial_e W(x, e(u)+e_D(t), z) : \dot{e}_D(t) & \text{if } \alpha = 0 \end{cases}$$

and we must show that  $h(x, t, \cdot) \in C^0([-\alpha_t, \alpha_t])$  for  $\alpha_t$  suitably. By the mean value theorem of differentiability, we know the existence of  $\tilde{\alpha} = \tilde{\alpha}(\alpha)$  for every  $\alpha \in [-\alpha_t, \alpha_t]$ , such that

$$\begin{aligned} & \frac{1}{\alpha} (W(x, e(u)+e_D(t+\alpha), z) - W(x, e(u)+e_D(t), z)) \\ &= \partial_e W(x, e(u)+e_D(t+\tilde{\alpha}), z) : \dot{e}_D(t+\tilde{\alpha}) =: F_{\alpha} \rightarrow \partial_e W(x, e(u) + e_D(t), z) : \dot{e}_D(t) \end{aligned} \tag{3.14}$$

as  $\alpha, \tilde{\alpha} \rightarrow 0$  by (H4) and (A2). In order to show that the integrals converge as well, we are going to apply the dominated convergence theorem. Thereto we estimate by (A2) and (H4)

$$|F_{\alpha}| \leq c_{DC} (W(x, e(u) + e_D(t + \tilde{\alpha}), z) + \tilde{c}) \rightarrow c_{DC} (W(x, e(u) + e_D(t), z) + \tilde{c})$$

as  $\alpha, \tilde{\alpha} \rightarrow 0$  due to (A2) and (H4). By Lipschitz-estimate (3.11), (A2) and (A3) we have

$$\begin{aligned} & \left| \int_{\Omega} W(x, e(u) + e_D(t + \tilde{\alpha}), z) - W(x, e(u) + e_D(t), z) dx \right| \\ & \leq \|e_D(t+\tilde{\alpha}) - e_D(t)\|_{L^{\infty}(\Omega, \mathbb{R}^{d \times d})} \left( 2c\tilde{c}\mathcal{L}^d(\Omega) + \mathcal{E}(t, u, z) + \mathcal{E}(t+\tilde{\alpha}, u, z) + 2c_1 c_D \right) \xrightarrow{\tilde{\alpha} \rightarrow 0} 0, \end{aligned} \tag{3.15}$$

since  $\mathcal{E}(t + \tilde{\alpha}, u, z) < \tilde{E}_q$  for every  $t + \tilde{\alpha} \in \mathcal{U}(t_q)$ . The differentiability of  $\langle l(t), u + u_D(t) \rangle$  is ensured by (A2), (A3). Thus we have proven the existence of  $\partial_t \mathcal{E}(\cdot, q)$  in  $\mathcal{U}(t_q)$ .

By (3.8) we find an upper estimate for  $\|e(u) + e_D(t)\|_{L^p(\Omega, \mathbb{R}^{d \times d})}^p$  in terms of  $\mathcal{E}(t, q)$ :

$$\begin{aligned} \|e(u) + e_D(t)\|_{L^p(\Omega, \mathbb{R}^{d \times d})}^p & \leq 2^{p-1} \left( \|e(u)\|_{L^p(\Omega, \mathbb{R}^{d \times d})}^p + c_D^p \right) \\ & \leq 2^{p-1} \left( \frac{\mathcal{E}(t, q) + C_3}{c_3} + c_D^p \right) =: A_1 \mathcal{E}(t, q) + B_1. \end{aligned} \tag{3.16}$$

This estimate will be used in the following to get (3.13). We have

$$|\partial_t \mathcal{E}(t, q)| \leq \left| \int_{\Omega} \partial_e W(x, e(u)+e_D(t), z) : \dot{e}_D(t) dx \right| + |\langle \dot{l}(t), u + u_D(t) \rangle| + |\langle l(t), \dot{u}_D(t) \rangle|,$$

where the loading terms are treated with Korn's and Young's inequality as in the proof of (3.8), such that one obtains an estimate of the form

$$|\langle \dot{l}(t), u + u_D(t) \rangle| + |\langle l(t), \dot{u}_D(t) \rangle| \leq A_2 \mathcal{E}(t, q) + B_2. \tag{3.17}$$

Application of (H4) to the stored elastic energy term yields

$$\begin{aligned} & \left| \int_{\Omega} \partial_e W(x, e(u)+e_D(t), z) : \dot{e}_D(t) dx \right| \leq c_{DC} (\mathcal{I}(t, q) + \tilde{c}\mathcal{L}^d(\Omega)) \\ & \leq c_{DC} \left( \mathcal{E}(t, q) + c_l \|u\|_{W^{1,p}(\Omega, \mathbb{R}^{d \times d})} + c_1 c_D + \tilde{c}\mathcal{L}^d(\Omega) \right) \\ & = A_3 (\mathcal{E}(t, q) + \|u\|_{W^{1,p}(\Omega, \mathbb{R}^{d \times d})}) + B_3 =: G. \end{aligned} \tag{3.18}$$

Applying Korn's inequality (3.7) to  $\|u\|_{W^{1,p}(\Omega, \mathbb{R}^{d \times d})}$  leads to the estimate

$$\begin{aligned}
& \left| \int_{\Omega} \partial_e W(x, e(u) + e_D(t), z) : \dot{e}_D(t) \, dx \right| \leq G \\
& \leq A_3 (\mathcal{E}(t, q) + C_K \|e(u) + e_D(t)\|_{L^p(\Omega, \mathbb{R}^{d \times d})}) + A_3 C_K c_D + B_3 \\
& \leq A_4 (1 + \|e(u) + e_D(t)\|_{L^p(\Omega, \mathbb{R}^{d \times d})}^p) + A_3 \mathcal{E}(t, q) + B_3 \\
& \leq A_4 2^{p-1} (1 + \|e(u) + e_D(t)\|_{L^p(\Omega, \mathbb{R}^{d \times d})}^p) + A_3 \mathcal{E}(t, q) + B_3 \\
& \leq A_4 2^{p-1} (1 + A_1 \mathcal{E}(t, q) + B_1) + A_3 \mathcal{E}(t, q) + B_3 = A_5 \mathcal{E}(t, q) + B_5,
\end{aligned} \tag{3.19}$$

where (3.16) has been applied to obtain the last inequality. Combining (3.17), (3.19) yields the desired estimate (3.13).  $\square$

### 3.2.3 Proof of the abstract assumptions on the dissipation distance

Now, we show that the dissipation distance that refers to a rate-independent damage process satisfies the assumptions (D1) and (D2).

**Theorem 3.8.** *The dissipation distance  $\mathcal{D}$  on  $\mathcal{Z}$  given by (1.2), (1.3) satisfies (D1), (D2).*

**Proof.** **Ad (D1):** By (1.2) we have  $\mathcal{D}(z_1, z_2) \geq \varrho_0 \|z_2 - z_1\|_{L^1(\Omega)}$ . Hence,  $\mathcal{D}(z_1, z_2) = 0$  implies  $z_1 = z_2$ . Let now  $z_1, z_2, z_3 \in \mathcal{Z}$  to show that the triangle-inequality holds. If its right-hand side is infinite, then the inequality is satisfied trivially. For a finite right-hand side  $z_1 \geq z_2 \geq z_3$  is necessary and hence we even obtain equality.

**Ad (D2):** To show sequential lower semicontinuity, let  $z_{0_k} \rightarrow z_0$ ,  $z_{1_k} \rightarrow z_1$  in  $W^{1,r}(\Omega)$  and put  $w_k := z_{1_k} - z_{0_k}$ ,  $w := z_1 - z_0$ . Hence  $0 \geq w_k \rightarrow w$  pointwise a.e. for a subsequence so that also  $w \leq 0$  a.e.. Thus  $\mathcal{D}(z_0, z_1) = \infty$  can be excluded. Assume now that  $\liminf_{k \rightarrow \infty} \mathcal{D}(z_{0_k}, z_{1_k}) < \infty$ , otherwise the inequality trivially holds. For a subsequence that attains the limit inferior, i.e.  $w_k \leq 0$  for all  $k \in \mathbb{N}$ , we obtain that

$$|\mathcal{D}(z_{0_k}, z_{1_k}) - \mathcal{D}(z_0, z_1)| \leq \|\varrho\|_{L^\infty(\Omega)} \|w_k - w\|_{L^1(\Omega)} \rightarrow 0 \text{ as } k \rightarrow \infty$$

due to the compact embedding  $W^{1,r}(\Omega) \Subset L^1(\Omega)$ . Thus  $\mathcal{D}(z_0, z_1) \leq \liminf_{k \rightarrow \infty} \mathcal{D}(z_{0_k}, z_{1_k})$ .  $\square$

### 3.2.4 Convergence of the time-derivative of the energies (C1)

The aim in this subsection is to prove the first compatibility condition.

**Theorem 3.9.** *Let hypotheses (H1)-(H5), (3.2) and (D1), (D2) hold true. Then, for every stable sequence  $(t_k, q_k)_{k \in \mathbb{N}} \subset [0, T] \times \mathcal{Q}$  with  $t_k \rightarrow t$  and  $q_k \rightarrow q$  in  $\mathcal{X}$  we have*

$$\partial_t \mathcal{E}(t, q_k) \rightarrow \partial_t \mathcal{E}(t, q). \tag{C1}$$

**Proof.** Since  $\mathcal{C}(z) := \int_{\Omega} \frac{\kappa}{r} |\nabla z|^r \, dx$  does not depend on time  $t$  we have  $\partial_t \mathcal{E}(t, q) = \partial_t \mathcal{J}(t, q)$ , where  $\mathcal{J}(t, q) = \mathcal{I}(t, q) - \langle l(t), u(t) + u_D(t) \rangle$ . As the last term is linear, it is sufficient to prove Theorem 3.9 for  $\mathcal{I}$ .

The following properties (P1) and (P2), proved in separate lemmas below, are utilized to obtain the convergence result:

**(P1)** It holds  $\mathcal{I}(t, u_k, z_k) \rightarrow \mathcal{I}(t, u, z)$  for every stable sequence  $(t_k, u_k, z_k)_{k \in \mathbb{N}}$ , where  $t_k \rightarrow t$ ,  $(u_k, z_k) \rightarrow (u, z)$  in  $\mathcal{X}$ , see Lemma 3.10.

**(P2)** For  $q \in L_E(0)$  the derivatives  $\partial_t \mathcal{I}(\cdot, q)$  are uniformly continuous, see Lemma 3.11.

Using properties (P1) and (P2) we are able to apply Proposition 3.3 of [5] to  $\mathcal{I}$  and conclude  $\partial_t \mathcal{I}(t, q_k) \rightarrow \partial_t \mathcal{I}(t, q)$ . Thus, (C1) is established.  $\square$

In the following, the two properties (P1) and (P2) from the proof of Theorem 3.9 are verified. Property (P1) is a consequence of

**Lemma 3.10.** *Let  $(t_k, u_k, z_k)_{k \in \mathbb{N}}$  be a stable sequence with  $t_k \rightarrow t$ ,  $(u_k, z_k) \rightarrow (u, z)$  in  $\mathcal{X}$  as  $k \rightarrow \infty$  and let (H1)-(H5) and (3.2) hold. Then*

$$\mathcal{J}(t, u, z_k) \rightarrow \mathcal{J}(t, u, z) \quad \text{and} \quad \mathcal{J}(t, u_k, z_k) \rightarrow \mathcal{J}(t, u, z) \quad \text{as } k \rightarrow \infty.$$

**Proof.** As a first step, we show that  $\mathcal{J}(t, u, z_k) \rightarrow \mathcal{J}(t, u, z)$ .

We have  $W(\cdot, e(u)+e_D(t), z_k) \rightarrow W(\cdot, e(u)+e_D(t), z)$  in measure, since each subsequence  $(W(\cdot, e(u)+e_D(t), z_{k_l}))_{l \in \mathbb{N}}$  contains a further subsequence that converges pointwise a.e.. This is due to the continuity of  $W$  with respect to  $z$  and Riesz' convergence theorem. By (H5) we obtain for every  $k \in \mathbb{N}$  that

$$W(x, e(u)+e_D(t), z_k) \leq K(W(x, e(u)+e_D(t), 0) + \tilde{K}) \leq K(W(x, e(u)+e_D(t), z) + \tilde{K}).$$

Moreover, we have

$$\begin{aligned} \int_{\Omega} (W(x, e(u)+e_D(t), z) + \tilde{K}) dx &\leq \mathcal{E}(t, u, z) + \tilde{K} \mathcal{L}^d(\Omega) + c_l(\|u\|_{W^{1,p}(\Omega, \mathbb{R}^d)} + c_D) \\ &\leq \liminf_{k \rightarrow \infty} (\mathcal{E}(t_k, u_k, z_k) + c_E|t - t_k|) + \tilde{K} \mathcal{L}^d(\Omega) + c_l(\|u\|_{W^{1,p}(\Omega, \mathbb{R}^d)} + c_D) < \infty \end{aligned}$$

by lower semicontinuity, (2.1) and (2.3). The dominated convergence theorem now yields  $\mathcal{J}(t, u, z_k) \rightarrow \mathcal{J}(t, u, z)$ . Since  $u_k$  minimizes  $\mathcal{J}(t_k, \cdot, z_k)$  and since (2.1), (2.3) hold, we infer

$$\mathcal{J}(t, u_k, z_k) - c_E|t_k - t| \leq \mathcal{J}(t_k, u_k, z_k) \leq \mathcal{J}(t, u, z_k) + c_E|t_k - t| \rightarrow \mathcal{J}(t, u, z)$$

and by weak sequential lower semicontinuity we conclude  $\mathcal{J}(t_k, u_k, z_k) \rightarrow \mathcal{J}(t, u, z)$ . □

The next lemma refers to property (P2) from the proof of Theorem 3.9. It is based on the fact that the given data are continuously differentiable on the compact time interval  $[0, T]$  by (A2), (A3) in (3.2), and hence they and their time-derivatives are uniformly continuous.

**Lemma 3.11** (Equi-uniform continuity of the power  $\partial_t \mathcal{I}(\cdot, q)$ ). *Let (H1)-(H5) and (3.2) be satisfied. Then, for each  $E, \varepsilon > 0$  there exists a  $\delta > 0$  such that for every  $q \in \mathcal{Q}$  with  $\mathcal{E}(0, q) < E$  it holds:*

$$\text{If } |t - s| < \delta \text{ then } |\partial_t \mathcal{I}(t, q) - \partial_t \mathcal{I}(s, q)| < \varepsilon.$$

**Proof.** Due to (A2) and (A3) we find for every  $\tilde{\varepsilon} > 0$  a  $\tilde{\delta} > 0$  such that for all  $s, t \in [0, T]$  with  $|s - t| < \tilde{\delta}$  we have  $\|u_D(s) - u_D(t)\|_{W^{1,\infty}(\Omega, \mathbb{R}^d)} + \|\dot{u}_D(s) - \dot{u}_D(t)\|_{W^{1,\infty}(\Omega, \mathbb{R}^d)} < \tilde{\varepsilon}$ . Choose now  $\varepsilon, E > 0$  and let  $(u, z) \in L_E(0)$ . By estimate (3.8) we obtain for  $t = 0$ :

$$\|u\|_{W^{1,p}(\Omega, \mathbb{R}^d)} \leq \left( \frac{\mathcal{E}(0, u, z) + C_3}{c_3} \right)^{\frac{1}{p}} \leq \left( \frac{E + C_3}{c_3} \right)^{\frac{1}{p}} =: \tilde{B}.$$

This shows that  $u + u_D(t)$  with  $(u, z) \in L_E(0)$  are uniformly bounded for every  $t \in [0, T]$ , since  $\|u + u_D(t)\|_{W^{1,p}(\Omega, \mathbb{R}^d)} \leq \|u\|_{W^{1,p}(\Omega, \mathbb{R}^d)} + \|u_D(t)\|_{W^{1,p}(\Omega, \mathbb{R}^d)} \leq \tilde{B} + c_D =: B$ .

Furthermore we estimate

$$\begin{aligned} &|\partial_t \mathcal{I}(t, q) - \partial_t \mathcal{I}(s, q)| \\ &\leq \left| \int_{\Omega} \partial_e W(x, e(u)+e_D(t), z) : (\dot{e}_D(t) - \dot{e}_D(s)) dx \right| \end{aligned} \tag{3.20}$$

$$+ \left| \int_{\Omega} (\partial_e W(x, e(u)+e_D(t), z) - \partial_e W(x, e(u)+e_D(s), z)) : \dot{e}_D(s) dx \right|. \tag{3.21}$$

In view of (H3), (H4) and Lipschitz-estimate (2.3) we see that

$$\begin{aligned} (3.20) &\leq \|\partial_e W(\cdot, e(u)+e_D(t), z)\|_{L^1(\Omega)} \|\dot{e}_D(t) - \dot{e}_D(s)\|_{L^\infty(\Omega, \mathbb{R}^{d \times d})} \\ &\leq (\mathcal{E}(0, q) + C \mathcal{L}^d(\Omega) + c_E T + c_l B) \|\nabla \dot{u}_D(t) - \nabla \dot{u}_D(s)\|_{L^\infty(\Omega, \mathbb{R}^{d \times d})} < \frac{\varepsilon}{2}, \end{aligned}$$

if only  $|t - s| < \tilde{\delta}_1$  is sufficiently small. Moreover we have  $|e(u)+e_D(s) - (e(u)+e_D(t))| \rightarrow 0$   $\mathcal{L}^d$ -a.e.. Keeping in mind the continuity of  $\partial_e W(x, \cdot, z)$  by (H2) we choose  $\tilde{\varepsilon}_2 := \frac{\varepsilon}{2c_D}$  so that

$$(3.21) = \|\partial_e W(\cdot, e(u)+e_D(t), z) - \partial_e W(\cdot, e(u)+e_D(s), z)\|_{L^1(\Omega, \mathbb{R}^{d \times d})} < \tilde{\varepsilon}_2$$

for  $|s - t| < \tilde{\delta}_2$  sufficiently small. Hence we obtain (3.21)  $< \frac{\varepsilon}{2}$  if  $|s - t| < \tilde{\delta}_2$ . Altogether we conclude that  $|\partial_t \mathcal{I}(s, q) - \partial_t \mathcal{I}(t, q)| < \varepsilon$  if  $|s - t| < \delta := \min\{\tilde{\delta}_1, \tilde{\delta}_2\}$ . □

### 3.2.5 Closedness of the stable sets (C2) and joint recovery condition

In the framework of damage we have to cope with a dissipation distance that is not weakly continuous on  $W^{1,r}(\Omega)$ . Hence it is not possible to show (C2) directly as in [5, 20], where weak continuity is essential. Like in [16, 23] we get (C2) via the so-called joint recovery condition.

**Definition 3.12** (Joint recovery condition).

The rate-independent system  $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$  satisfies the *joint recovery condition* if for all stable sequences  $(t_k, q_k)_{k \in \mathbb{N}} = (t_k, u_k, z_k)_{k \in \mathbb{N}} \subset [0, T] \times \mathcal{Q}$  with  $(t_k, q_k) \rightharpoonup (t, q)$  in  $[0, T] \times \mathcal{X}$  and for every  $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$  there is a sequence  $(\hat{q}_k)_{k \in \mathbb{N}} = (\hat{u}_k, \hat{z}_k)_{k \in \mathbb{N}}$  with  $\hat{q}_k \rightharpoonup \hat{q}$  in  $\mathcal{X}$  and

$$\limsup_{k \rightarrow \infty} (\mathcal{E}(t_k, \hat{q}_k) + \mathcal{D}(z_k, \hat{z}_k) - \mathcal{E}(t_k, q_k)) \leq \mathcal{E}(t, \hat{q}) + \mathcal{D}(z, \hat{z}) - \mathcal{E}(t, q). \quad (\text{JRC})$$

Clearly, the estimate (JRC) is equivalent to

$$\mathcal{E}(t, q) - \mathcal{E}(t, \hat{q}) - \mathcal{D}(z, \hat{z}) \leq \liminf_{k \rightarrow \infty} \underbrace{(\mathcal{E}(t_k, q_k) - \mathcal{E}(t_k, \hat{q}_k) - \mathcal{D}(z_k, \hat{z}_k))}_{=: L_k}.$$

Using  $q_k \in \mathcal{S}(t_k)$  we have  $L_k \leq 0$  for all  $k \in \mathbb{N}$ ; hence (JRC) implies the closedness of the sets of stable states (C2).

If  $\mathcal{D}(z, \hat{z}) < \infty$ , the joint recovery sequence has to be constructed in such a manner that also  $\mathcal{D}(z_k, \hat{z}_k) < \infty$  is satisfied for every  $k \in \mathbb{N}$ . Otherwise the left-hand side in (JRC) is too big. In fact, we will enforce  $\mathcal{D}(z_k, \hat{z}_k) \rightarrow \mathcal{D}(z, \hat{z})$ , which follows from  $z_k \rightharpoonup z$  and  $\hat{z}_k \rightharpoonup \hat{z}$  only if the additional constraint  $\hat{z}_k \leq z_k$  holds.

To this end, the case  $1 < r \leq d$  requires substantially new ideas compared to [16]. There, the embedding  $W^{1,r}(\Omega) \Subset C^0(\overline{\Omega})$  permitted to gain the finiteness of  $\mathcal{D}$  by choosing  $\hat{z}_k := (\hat{z} - \|z_k - z\|_\infty)^+$ , where  $(f)^+ := \max\{0, f\}$ .

In the following, the result of [16] is extended to the case of  $1 < r < \infty$  by constructing the joint recovery sequence in such a manner that the compact embedding  $W^{1,r}(\Omega) \Subset C(\overline{\Omega})$  is not needed for the proof of estimate (JRC).

For the construction of a joint recovery sequence we will entirely use that the superposition of a  $W^{1,r}$ -function with a globally Lipschitz-continuous function  $g$  again gives a  $W^{1,r}$ -function:

**Lemma 3.13** (Superposition lemma, [15]). *Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be globally Lipschitz-continuous and  $v \in W^{1,r}(\Omega)$ . Then  $g \circ v \in W^{1,r}(\Omega)$  and*

$$\nabla(g \circ v)(x) = g'(v(x))\nabla v(x) \quad \text{for a.a. } x \in \Omega.$$

The following result establishes the compatibility condition (C2).

**Theorem 3.14** (Joint recovery condition for  $1 < r < \infty$ ). *Let (H1)-(H5) hold. Then, the rate-independent system  $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$  satisfies the joint recovery condition. Hence, if  $(t_k, q_k)_{k \in \mathbb{N}}$  is a stable sequence with  $t_k \rightarrow t$ ,  $q_k \rightharpoonup q$  in  $\mathcal{X}$ , then  $q \in \mathcal{S}(t)$ , i.e. (C2) holds.*

**Proof.** Let  $(u_k, z_k)_{k \in \mathbb{N}} \subset \mathcal{U} \times \mathcal{Z}$  with  $u_k \rightharpoonup u$  in  $W^{1,p}(\Omega, \mathbb{R}^d)$  and  $z_k \rightharpoonup z$  in  $W^{1,r}(\Omega)$ . Choose  $\hat{q} \in \mathcal{Q}$  such that  $\hat{q} \in L_E(t)$  for some  $E \in \mathbb{R}$ , otherwise (JRC) trivially holds. Now we distinguish between the following two cases:

**Case A:** Let  $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$  be such that there exists a  $\mathcal{L}^d$ -measurable set  $B \subset \Omega$  with  $\mathcal{L}^d(B) > 0$  and  $\hat{z} > z$  on  $B$ . Then  $\mathcal{D}(z, \hat{z}) = \infty$  and (JRC) holds.

**Case B:** Let  $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$  be such that  $\hat{z} \leq z$  a.e. in  $\Omega$ . Then,  $\mathcal{D}(z, \hat{z}) = \int_{\Omega} \varrho(z - \hat{z}) \, dx < \infty$ .

To construct a joint recovery sequence we put  $\hat{u}_k := \hat{u}$  for every  $k \in \mathbb{N}$  and

$$\hat{z}_k := \min \{ (\hat{z} - \delta_k)^+, z_k \} := \begin{cases} (\hat{z} - \delta_k)^+ & \text{if } (\hat{z} - \delta_k)^+ \leq z_k \\ z_k & \text{if } (\hat{z} - \delta_k)^+ > z_k \end{cases}, \quad (3.22)$$

where  $0 < \delta_k \xrightarrow{\mathbb{R}} 0$  will be chosen suitably in step 2. Thus,  $\hat{z}_k \leq z_k$  a.e. and therefore  $\mathcal{D}(z_k, \hat{z}_k) < \infty$  for every  $k \in \mathbb{N}$ . Besides, it holds  $\hat{z}_k(x) < \hat{z}(x) \leq z(x)$  for a.e.  $x \in \Omega$  with  $\hat{z}(x) \neq 0$ . Again we have  $\hat{z}_k = z_k - \max\{z_k - (\hat{z} - \delta_k)^+, 0\} \in W^{1,r}(\Omega)$  by Lemma 3.13.

For a joint recovery sequence constructed by (3.22) we can in general only prove weak convergence in  $W^{1,r}(\Omega)$ . This can be seen from Example 3.16 below the proof.

It holds  $\mathcal{E}(t_k, \hat{q}_k) \leq \mathcal{E}(t_k, \hat{q}) + \mathcal{C}(\hat{z}_k) \leq \hat{c}$  due to  $\hat{q} \in L_E(t)$  and estimate (2.3) for  $\hat{q}$ . Furthermore, (2.3) provides a uniform Lipschitz-constant for  $(\hat{q}_k)_{k \in \mathbb{N}}$  such that

$$\mathcal{E}(t_k, \hat{q}_k) + \mathcal{D}(z_k, \hat{z}_k) - \mathcal{E}(t_k, q_k) \leq \mathcal{E}(t, \hat{q}_k) + \mathcal{D}(z_k, \hat{z}_k) - \mathcal{E}(t, q_k) + 2L|t_k - t|, \quad (3.23)$$

where  $L$  is the maximum of the uniform Lipschitz-constants for  $(q_k)_{k \in \mathbb{N}}$  and  $(\hat{q}_k)_{k \in \mathbb{N}}$ . Since  $|t_k - t| \rightarrow 0$ , inequality (JRC) holds if we can prove

$$\limsup_{k \rightarrow \infty} (\mathcal{E}(t, \hat{q}_k) + \mathcal{D}(z_k, \hat{z}_k) - \mathcal{E}(t, q_k)) \leq \mathcal{E}(t, \hat{q}) + \mathcal{D}(z, \hat{z}) - \mathcal{E}(t, q). \tag{JRC*}$$

In order to show (JRC\*) we take into account that

$$\begin{aligned} & \limsup_{k \rightarrow \infty} (\mathcal{E}(t, \hat{q}_k) + \mathcal{D}(z_k, \hat{z}_k) - \mathcal{E}(t, q_k)) \\ & \leq \limsup_{k \rightarrow \infty} \mathcal{I}(t, \hat{q}_k) - \liminf_{k \rightarrow \infty} \mathcal{I}(t, q_k) + \limsup_{k \rightarrow \infty} \mathcal{D}(z_k, \hat{z}_k) + \limsup_{k \rightarrow \infty} (\mathcal{C}(\hat{z}_k) - \mathcal{C}(z_k)) - \langle l(t), \hat{u} - u \rangle \end{aligned} \tag{3.24}$$

and estimate these limits in separate steps.

For a shorter notation in the subsequent steps, we now introduce the abbreviation  $[f < g] := \{x \in \Omega \mid f(x) < g(x)\}$  with an analogous meaning for  $\leq, >$  and  $\geq$ .

**Step 1:** We prove that  $\hat{z}_k \rightharpoonup \hat{z}$  in  $W^{1,r}(\Omega)$  as  $k \rightarrow \infty$ .

By construction the sequence  $(\hat{z}_k)_{k \in \mathbb{N}}$  is uniformly bounded in  $W^{1,r}(\Omega)$ . Thus, there is a weakly convergent subsequence  $\hat{z}_{k_l} \rightharpoonup \tilde{z} \in W^{1,r}(\Omega)$ . Due to the compact embedding this subsequence converges strongly in  $L^r(\Omega)$  and by Riesz' convergence theorem it has a further subsequence converging pointwise a.e. in  $\Omega$ . This last subsequence has to converge  $\hat{z}_{k_{l_m}} \rightarrow \hat{z}$  a.e. in  $\Omega$  by definition of  $\hat{z}_k$ . Hence, we obtain  $\tilde{z} = \hat{z}$  and therefore  $\hat{z}_k \rightharpoonup \hat{z}$  in  $L^r(\Omega)$ . Since  $(\hat{z}_k)_{k \in \mathbb{N}}$  is bounded in  $W^{1,r}(\Omega)$ , the same arguments also yield  $\hat{z}_k \rightharpoonup \hat{z}$  in  $W^{1,r}(\Omega)$ .

**Step 2:** We show that  $\limsup_{k \rightarrow \infty} (\mathcal{C}(\hat{z}_k) - \mathcal{C}(z_k)) \leq \mathcal{C}(\hat{z}) - \mathcal{C}(z)$ :

For the calculation of the limit, the domain  $\Omega$  is decomposed as follows:

$$\Omega = A_k \cup B_k \text{ with } B_k = [(\hat{z} - \delta_k)^+ > z_k] \text{ and } A_k = \Omega \setminus B_k.$$

Thereby it holds  $B_k = [(\hat{z} - \delta_k)^+ > z_k] \subset [(z - \delta_k)^+ > z_k] \subset [|z - z_k| \geq \delta_k]$ . By application of Markov's inequality in estimate (M) we can now determine  $(\delta_k)_{k \in \mathbb{N}}$  in such a way that  $\mathcal{L}^d([(\hat{z} - \delta_k)^+ > z_k]) \rightarrow 0$  as  $k \rightarrow \infty$ :

$$\mathcal{L}^d([(\hat{z} - \delta_k)^+ > z_k]) \leq \mathcal{L}^d([|z - z_k| \geq \delta_k]) \stackrel{(M)}{\leq} \frac{1}{\delta_k^r} \int_{\Omega} |z - z_k|^r dx \rightarrow 0,$$

if, for instance,  $\delta_k := \|z_k - z\|_{L^r(\Omega)}^{\frac{1}{r}}$ . Note that Markov's inequality is only applicable if  $\delta_k > 0$ . But  $\|z_k - z\|_{L^r(\Omega)} = 0$  implies  $\mathcal{L}^d([|z_k - z| > 0]) = 0$  and hence  $\mathcal{L}^d(B_k) \rightarrow 0$  as  $k \rightarrow \infty$  is guaranteed. For  $A_k = \Omega \setminus B_k$  we have  $\mathcal{L}^d(A_k) \rightarrow \mathcal{L}^d(\Omega)$  as  $k \rightarrow \infty$ . Using the characteristic functions of these sets

$$I_{A_k}(x) := \begin{cases} 1 & \text{if } x \in A_k \\ 0 & \text{if } x \in B_k \end{cases}$$

and Lemma 3.15 from below we find  $I_{A_k} \nabla z_k \rightharpoonup \nabla z$  in  $L^r(\Omega, \mathbb{R}^d)$ . By weak sequential lower semicontinuity we conclude

$$\begin{aligned} \limsup_{k \rightarrow \infty} (\mathcal{C}(\hat{z}_k) - \mathcal{C}(z_k)) &= \limsup_{k \rightarrow \infty} \int_{A_k} \frac{\kappa}{r} (|\nabla(\hat{z} - \delta_k)^+|^r - |\nabla z_k|^r) dx \\ &\leq \int_{\Omega} \frac{\kappa}{r} |\nabla \hat{z}|^r dx - \liminf_{k \rightarrow \infty} \int_{\Omega} \frac{\kappa}{r} |I_{A_k} \nabla z_k|^r dx \leq \mathcal{C}(\hat{z}) - \mathcal{C}(z). \end{aligned}$$

**Step 3:** Estimation of the remaining terms in line (3.24):

To calculate  $\limsup_{k \rightarrow \infty} \mathcal{I}(t, \hat{u}, \hat{z}_k)$  we choose a subsequence  $(\hat{z}_{k_l})_{l \in \mathbb{N}} \subset (\hat{z}_k)_{k \in \mathbb{N}}$  such that  $\hat{z}_{k_l} \rightharpoonup \hat{z}$   $\mathcal{L}^d$ -a.e.. Since  $W(x, e, \cdot) \in C^0([0, 1])$  cf. (H1) we have that  $W(\cdot, e(\hat{u}) + e_D(t), \hat{z}_{k_l}) \rightarrow W(\cdot, e(\hat{u}) + e_D(t), \hat{z})$   $\mathcal{L}^d$ -a.e.. Furthermore, by (H5) we infer that  $W(x, e(\hat{u}) + e_D(t), \hat{z}_{k_l}) \leq K(W(x, e(\hat{u}) + e_D(t), \hat{z}) + \tilde{K}) \in L^1(\Omega)$ . Then, the dominated convergence theorem gives  $\mathcal{I}(t, \hat{u}, \hat{z}_{k_l}) \rightarrow \mathcal{I}(t, \hat{u}, \hat{z})$ .

The estimate  $-\liminf_{k \rightarrow \infty} \mathcal{I}(t, q_k) \leq -\mathcal{I}(t, q)$  is obvious by the weak sequential lower semicontinuity of  $\mathcal{I}(t, \cdot)$ .

In view of the definition of the joint recovery sequence it holds  $\widehat{z}_k \leq z_k$  for every  $k \in \mathbb{N}$  and therefore

$$\lim_{k \rightarrow \infty} \mathcal{D}(z_k, \widehat{z}_k) = \lim_{k \rightarrow \infty} \int_{\Omega} R(x, \widehat{z}_k - z_k) \, dx = \int_{\Omega} R(x, \widehat{z} - z) \, dx = \mathcal{D}(z, \widehat{z}),$$

by continuity of  $R$ , since both  $z_k \rightarrow z$  and  $\widehat{z}_k \rightarrow \widehat{z}$  in  $L^1(\Omega)$  as  $k \rightarrow \infty$ .

Hence inequality (JRC\*) is proven. □

It remains to show the lemma applied in step 2 of the above proof.

**Lemma 3.15.** *If  $A_k \subset \Omega$ ,  $\mathcal{L}^d(A_k) \rightarrow \mathcal{L}^d(\Omega)$ , and  $f_k \rightarrow f$  in  $L^r(\Omega)$  as  $k \rightarrow \infty$ , then  $I_{A_k} f_k \rightarrow f$  in  $L^r(\Omega)$  as  $k \rightarrow \infty$ .*

*Proof.* Let  $\varphi \in L^{r'}(\Omega)$  and  $\varphi_k := I_{A_k} \varphi$ . Since  $\mathcal{L}^d(\Omega \setminus A_k) \rightarrow 0$  it holds  $\|\varphi_k - \varphi\|_{L^{r'}(\Omega)}^{r'} = \int_{\Omega \setminus A_k} |\varphi|^{r'} \, dx \rightarrow 0$ . Together with  $f_k \rightarrow f$  in  $L^r(\Omega)$  this implies  $\int_{\Omega} I_{A_k} f_k \varphi \, dx = \int_{\Omega} f_k \varphi_k \, dx \rightarrow \int_{\Omega} f \varphi \, dx$  for any  $\varphi \in L^{r'}(\Omega)$ . □

Next, we give an example on a weakly converging sequence, where the method (3.22) generates a weakly converging recovery sequence, that does not converge strongly.

**Example 3.16.** *Consider  $\Omega = \{x \in \mathbb{R}^2 \mid |x| < 1\}$  and  $z_k(x) = \min\{k|x|, 1/2\}$ ,  $k \in \mathbb{N}$ . Then  $z_k \rightarrow z = \frac{1}{2}$  in  $H^1(\Omega)$ . For  $\widehat{z} := \frac{1}{4}$  and  $(\widehat{z}_k)_{k \in \mathbb{N}}$  as in (3.22) it holds  $\widehat{z}_k \rightarrow \widehat{z}$  in  $H^1(\Omega)$ , but  $\|\widehat{z}_k - \widehat{z}\|_{H^1(\Omega)}^2 \rightarrow \frac{\pi}{16}$ .*

*However,  $(z_k)_{k \in \mathbb{N}}$  may not be a stable sequence. In [18, Sect. 6] it is conjectured that weakly convergent, stable sequences converge also strongly. Then construction (3.22) implies  $\widehat{z}_k \rightarrow \widehat{z}$  as well, and (JRC\*) would follow more easily.*

### 4 On the temporal regularity of energetic solutions

By the properties (S) & (E) and the uniform boundedness of the energy  $\mathcal{E}(t, q(t)) \leq E_*$  for all  $t \in [0, T]$  a general, but very weak result on the temporal regularity of an energetic solution  $q : [0, T] \rightarrow \mathcal{Q}$  is provided:

$$z \in \text{BV}([0, T], L^1(\Omega)) \cap L^\infty([0, T], W^{1,r}(\Omega)) \quad \text{and} \quad u \in L^\infty([0, T], W^{1,p}(\Omega, \mathbb{R}^d)).$$

The BV estimate is due to  $\text{Var}_{L^1(\Omega)}(z, [r, s]) \leq \frac{1}{\varrho_0} \text{Diss}_{\mathcal{D}}(z, [r, s]) < \infty$ , which is a consequence of the energy balance. In fact, the monotonicity  $z(t_1, x) \geq z(t_2, x)$  for  $t_1 < t_2$  implies  $\text{Var}_{L^1(\Omega)}(z, [r, s]) = \int_{\Omega} z(r, x) - z(s, x) \, dx \leq \mathcal{L}^d(\Omega)$ . The  $L^\infty$  bound for  $q = (u, z)$  in  $W^{1,p}(\Omega, \mathbb{R}^d) \times W^{1,r}(\Omega)$  is a consequence of the energy bound  $\mathcal{E}(t, q(t)) \leq E_*$ .

It was first obtained in [17] that the temporal regularity of the energetic solution can be improved, if  $\mathcal{E}$  has additional convexity properties. If for all  $t \in [0, T]$  the energy  $\mathcal{E}(t, \cdot)$  is strictly convex, one obtains that all energetic solutions are continuous in time. Furthermore, it is proven in [17] that even Lipschitz-continuity can be achieved for energies that are uniformly convex of the form

$$\mathcal{E}(t, \theta q_1 + (1-\theta)q_2) \leq \theta \mathcal{E}(t, q_1) + (1-\theta)\mathcal{E}(t, q_2) - c\theta(1-\theta)\|q_1 - q_2\|_{\mathcal{Q}}^\alpha \quad \text{for all } \theta \in [0, 1], q_1, q_2 \in \mathcal{Q}, \quad (4.1)$$

with some constant  $c > 0$  and  $\alpha = 2$ . In Sect. 4.2 we will see that (4.1) depends on the choice of  $\|\cdot\|_{\mathcal{Q}}$  and that uniform convexity is not restricted to the exponent  $\alpha = 2$ . In a lemma we provide properties of stored-energy densities  $W$  that lead to uniform convexity on sublevels with an exponent  $\alpha \geq 2$ . In such a situation we prove Hölder-continuity in time.

Before we go into the analysis we provide an example of an energy density  $W$  that satisfies all the assumptions from above and additionally the uniform convexity conditions that will be used later. The fact that joint convexity is compatible with damage models was first exploited in [27].

**Example 4.1.** *The simplest example for a suitable  $W$  generating a uniformly convex energy functional is given by*

$$W(x, e, z) = \frac{1}{2(1+\eta(1-z))^\gamma} e:\mathbb{B}:e + \frac{a}{2} z^2,$$

where  $\eta, a > 0$ ,  $\gamma \in (0, 1)$ , and  $\mathbb{B}$  is a symmetric and positive definite linear operator on  $\mathbb{R}_{\text{sym}}^{d \times d}$ . Such densities are discussed in detail in Sect. 5.1.

#### 4.1 Temporal continuity

The first result provides continuity in time, which means that energetic solutions cannot have jumps. The idea is to use that under the assumption of strict convexity energetic solutions  $q : [0, T] \rightarrow \mathcal{Q}$  have weak left and right limits  $q_+(t)$  and  $q_-(t)$  for all  $t$ . Moreover, it can be shown that  $q_-(t)$ ,  $q(t)$  and  $q_+(t)$  must be minimizers of the functional  $q \mapsto$

$\mathcal{E}(t, q) + \mathcal{D}(q_-(t), q)$ . By strict convexity one then concludes that all three values must be the same and weak continuity follows. Strong continuity is concluded by an argument of Visintin (cf. [29]), which allows us to convert weak convergence and energy convergence into strong convergence by exploiting the strict convexity once again.

We now develop the details. We first provide a result that does not explicitly use the strict convexity of  $\mathcal{E}(t, \cdot)$ ; for stable states  $q = (u, z) \in \mathcal{S}(t)$  it only requires the uniqueness of the minimizer of  $\mathcal{E}(t, \cdot, z)$ , which then is  $u$ .

**Lemma 4.2** (Jump relations). *Assume that  $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$  satisfies (E1)–(C2). Moreover,*

$$\forall t \in [0, T] \forall q = (u, z) \in \mathcal{S}(t) : \quad \{u\} = \underset{\tilde{u} \in \mathcal{U}}{\text{Argmin}} \mathcal{E}(t, \tilde{u}, z). \tag{4.2}$$

Then, for all  $t \in [0, T]$  the weak limits  $q_-(t) = \text{w-lim}_{\tau \rightarrow t^-} q(\tau)$  and  $q_+(t) = \text{w-lim}_{\tau \rightarrow t^+} q(\tau)$  (where  $q_-(0) := q(0)$  and  $q_+(T) = q(T)$ ) exist and satisfy

$$\mathcal{E}(t, q_-(t)) = \mathcal{E}(t, q(t)) + \mathcal{D}(q_-(t), q(t)), \quad \mathcal{E}(t, q(t)) = \mathcal{E}(t, q_+(t)) + \mathcal{D}(q(t), q_+(t)), \tag{4.3}$$

$$\text{and } \mathcal{D}(q_-(t), q_+(t)) = \mathcal{D}(q_-(t), q(t)) + \mathcal{D}(q(t), q_+(t)).$$

**Proof.** From  $\text{Diss}_{\mathcal{D}}(z, [0, T]) < \infty$  we conclude that the limits  $z_-(t) = \text{w-lim}_{\tau \rightarrow t^-} z(\tau)$  and  $z_+(t) = \text{w-lim}_{\tau \rightarrow t^+} z(\tau)$  exist, cf. [13]. Now, fix  $t$ , choose  $v_{\pm} \in \mathcal{U}$  and subsequences  $(t_k^{\pm})_{k \in \mathbb{N}}$  such that  $u(t_k^{\pm}) \rightharpoonup v_{\pm}$ , where  $t_k^{\pm} \rightarrow t$  with  $\pm(t_k^{\pm} - t) > 0$ . Then, (C2) guarantees  $(v_{\pm}, z_{\pm}(t)) \in \mathcal{S}(t)$ . Exploiting the assumption (4.2) we find that  $v_{\pm}$  are uniquely determined and cannot depend on the subsequence. Hence, the function  $u : [0, T] \rightarrow \mathcal{U}$  has the desired left-hand and right-hand limits  $u_{\pm}(t)$  in the weak sense.

To obtain the desired energy identities (4.3) we exploit the energy balance

$$\mathcal{E}(s, q(s)) + \text{Diss}_{\mathcal{D}}(z, [r, s]) = \mathcal{E}(r, q(r)) + \int_r^s \partial_{\tau} \mathcal{E}(\tau, q(\tau)) \, d\tau, \quad 0 \leq r < s \leq T.$$

For the first identity in (4.3) we let  $s = t$  and consider  $r \rightarrow t^-$ . Using the obvious relation  $\text{Diss}_{\mathcal{D}}(z, [r, t]) \rightarrow \mathcal{D}(z_-(t), z(t))$  we find

$$\mathcal{E}(t, q(t)) + \mathcal{D}(z_-(t), z(t)) \leq \limsup_{r \rightarrow t^-} \mathcal{E}(r, q(r)) \leq \mathcal{E}(t, q_-(t)) \leq \mathcal{E}(t, q(t)) + \mathcal{D}(z_-(t), z(t)),$$

where the second estimate follows from the stability  $\mathcal{E}(r, q(r)) \leq \mathcal{E}(r, q_-(t)) + \mathcal{D}(z(r), z_-(t))$  by taking the limit  $r \rightarrow t^-$ , while the third estimate is just the stability of  $q_-(t)$ . This establishes the first relation in (4.3).

The second relation in (4.3) follows by setting  $r = t$  and taking the limit  $s \rightarrow t^+$ :

$$\begin{aligned} \mathcal{E}(t, q_+(t)) + \mathcal{D}(z(t), z_+(t)) &\leq \liminf_{s \rightarrow t^+} \mathcal{E}(s, q(s)) + \mathcal{D}(z(t), z(s)) \\ &= \mathcal{E}(t, q(t)) + 0 \leq \mathcal{E}(t, q_+(t)) + \mathcal{D}(z(t), z_+(t)), \end{aligned}$$

where we first used lower semicontinuity (E1), then the energy balance, and finally the stability of  $q(t)$ . Thus, the second identity in (4.3) holds.

The third relation in (4.3) follows from (D1) and the first two identities:

$$\mathcal{D}(z_-(t), z_+(t)) \leq \mathcal{D}(z_-(t), z(t)) + \mathcal{D}(z(t), z_+(t)) = \mathcal{E}(t, q_-(t)) - \mathcal{E}(s, q_+(s)) \leq \mathcal{D}(z_-(t), z_+(t)),$$

where the last estimate uses the stability of  $q_-(t)$ . □

The next result provides the continuity of the energetic solutions under the assumption that the functional  $\mathcal{E}(t, \cdot) : \mathcal{Q} \rightarrow \mathbb{R}_{\infty}$  is strictly convex. In fact, the proof only uses the weaker property that for stable states  $q = (u, z) \in \mathcal{S}(t)$  the functional  $\tilde{q} \mapsto \mathcal{E}(t, \tilde{q}) + \mathcal{D}(z, \tilde{z})$  has a unique minimizer, see [22].

**Theorem 4.3** (Continuity by strict convexity). *Let the assumptions of the existence theorem 3.1 hold. Moreover, assume that  $W(x, \cdot, \cdot) : \mathbb{R}_{\text{sym}}^{d \times d} \times [0, 1] \rightarrow \mathbb{R}$  is strictly convex for a.a.  $x \in \Omega$ . Then, any energetic solution  $q : [0, T] \rightarrow \mathcal{Q}$  is (norm-) continuous with respect to time, i.e.  $q \in C^0([0, T], \mathcal{Q})$ .*

**Proof.** We first observe that for each  $t \in [0, T]$  the functional  $\mathcal{E}(t, \cdot)$  is strictly convex, since it is obtained by integration over the strictly convex density  $(e, z, A) \mapsto W(x, e+e_D(t, x), z) + \frac{k}{r}|A|^r$  and the linear term  $l(t)$  with arguments  $(e, z, A) = (e(u), z, \nabla z)$  depending linearly on  $(u, z) \in \mathcal{Q}$ . Moreover, for each  $z \in \mathcal{Z}$  the mapping  $\tilde{z} \mapsto \mathcal{D}(z, \tilde{z})$  is convex. Thus, for each  $t \in [0, T]$  the functional

$$\mathcal{Q} \ni \tilde{q} = (\tilde{u}, \tilde{z}) \mapsto \mathcal{E}(t, \tilde{q}) + \mathcal{D}(z_-(t), \tilde{z})$$

has a unique minimizer.

Exploiting the jump relations (4.3) we easily find that  $q_-(t)$ ,  $q(t)$ , and  $q_+(t)$  all provide the same value  $\mathcal{E}(t, q_-(t))$ , which must be the global minimum by the stability of  $q_-(t)$ . Hence, the three values must coincide, and Lemma 4.2 allows us to conclude weak continuity of  $q : [0, T] \rightarrow \mathcal{Q}$ , namely  $q(\tau) \rightharpoonup q(t)$  for  $\tau \rightarrow t$ .

Applying the jump relations (4.3) once again we also have  $\mathcal{E}(\tau, q(\tau)) \rightarrow \mathcal{E}(t, q(t))$  for  $\tau \rightarrow t$ . Fixing  $t$  and employing (2.2) we also obtain  $\mathcal{E}(t, q(\tau)) \rightarrow \mathcal{E}(t, q(t))$ . Thus, we are able to apply Proposition 4.4 below to the family  $V(\tau) = (e(u(\tau)) + e_D(t), z(\tau), A(\tau))$ , which provides the following strong convergence in  $L^p(\Omega, \mathbb{R}_{\text{sym}}^{d \times d}) \times L^r(\Omega) \times L^r(\Omega, \mathbb{R}^d)$ :

$$(e(u(\tau)) + e_D(t), z(\tau), \nabla z(\tau)) \rightarrow (e(u(\tau)) + e_D(t), z(\tau), \nabla z(t)).$$

Using Korn's inequality (3.7) the desired strong convergence  $q(\tau) \rightarrow q(t)$  in  $\mathcal{Q}$  follows. □

The following result was used in the proof above. Since it is only a slight variant of [29, §2 & Th. 8], we leave the details to the reader.

**Proposition 4.4.** *Let  $\Omega$  satisfy (A1) and  $\mathbf{C}$  be a nonempty, closed, convex subset of  $\mathcal{V} := L^p(\Omega, \mathbb{R}^N)$ ,  $1 \leq p < \infty$ ,  $N \geq 1$ . Let  $\phi : \Omega \times \mathbb{R}^N \rightarrow [0, \infty]$  be a Carathéodory function such that  $\phi(x, \cdot)$  is strictly convex on  $\mathbb{R}^N$  a.e. on  $\Omega$ . For  $V \in \mathbf{C}$  set  $\Phi(V) := \int_{\Omega} \phi(x, V(x)) \, dx$ . Then, the following holds:*

$$\left. \begin{array}{l} V_k \rightharpoonup V \text{ in } \mathcal{V}, \\ \Phi(V_k) \rightarrow \Phi(V), \end{array} \right\} \implies \left\{ \begin{array}{l} V_k \rightarrow V \text{ in } \mathcal{V}, \\ \phi(\cdot, V_k(\cdot)) \rightarrow \phi(\cdot, V(\cdot)) \text{ in } L^1(\Omega). \end{array} \right.$$

### 4.2 Temporal Hölder- and Lipschitz-continuity

In this section we generalize the ideas developed in [17, 21], where Lipschitz-continuity with respect to time was derived. Our generalization has two aspects. First we emphasize that the convexity properties can be formulated with respect to a norm  $\|\cdot\|_{\mathcal{V}}$  that may differ significantly from that in the underlying space  $\mathcal{X} \supset \mathcal{Q}$ . In particular, if  $\mathcal{X}$  is chosen as small as possible under preservation of the coercivity of  $\mathcal{E}$  (see (E1)), it may be advantageous to investigate the temporal regularity of energetic solutions with respect to the norm of a bigger Banach space  $\mathcal{V} \supset \mathcal{X}$ , since temporal regularity may improve. Second we generalize the notion of uniform convexity by allowing for a weaker lower bound in (4.4). Previous work asked  $\alpha = 2$  and  $\beta = 1$  and enforced the condition on all of  $\mathcal{Q}$ , while we only pose it on sublevels.

After we have established the main abstract result in Theorem 4.5, we will show how the main assumptions can be satisfied for integral functionals in Lemma 4.6. The effective use of the spaces  $\mathcal{V}$  and  $\mathcal{X}$  and applications to damage will be demonstrated in the Examples 5.5–5.7 in Sect. 5.

**Theorem 4.5** (Temporal Hölder-continuity). *Let  $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$  be a rate-independent system, where  $\mathcal{Q}$  is a closed convex subset of a Banach space  $\mathcal{X}$ . Let  $L_E(t) = \{q \in \mathcal{Q} \mid \mathcal{E}(t, q) \leq E\}$ . Assume that there is a Banach space  $\mathcal{V}$  and that there are constants  $\alpha \geq 2$ ,  $\beta \leq 1$  such that for all  $E_*$  there exist constants  $C_*$ ,  $c_* > 0$  so that for all  $t \in [0, T]$ ,  $q_0, q_1 \in L_{E_*}(t)$  and all  $\theta \in [0, 1]$  the following holds:*

$$\mathcal{E}(t, q_\theta) + \mathcal{D}(z_0, z_\theta) + c_*\theta(1-\theta)\|q_1 - q_0\|_{\mathcal{V}}^\alpha \leq (1-\theta)(\mathcal{E}(t, q_0) + \mathcal{D}(z_0, z_0)) + \theta(\mathcal{E}(t, q_1) + \mathcal{D}(z_0, z_1)) \quad (4.4a)$$

$$|\partial_t \mathcal{E}(t, q_1) - \partial_t \mathcal{E}(t, q_0)| \leq C_* \|q_1 - q_0\|_{\mathcal{V}}^\beta, \quad (4.4b)$$

where  $(u_\theta, z_\theta) = q_\theta = (1-\theta)q_0 + \theta q_1$ .

Then, any energetic solution  $q : [0, T] \rightarrow \mathcal{Q}$  of  $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$  is Hölder-continuous from  $[0, T]$  to  $\mathcal{V}$  with the exponent  $1/(\alpha-\beta)$ , i.e. there is a constant  $C_H > 0$  such that

$$\|q(s) - q(t)\|_{\mathcal{V}} \leq C_H |t-s|^{1/(\alpha-\beta)} \quad \text{for all } s, t \in [0, T]. \quad (4.5)$$

**Proof.** We proceed in three steps. First we derive an improved stability condition (S), where an additional term of the form  $c_*\theta(1-\theta)\|q_1 - q_0\|_{\mathcal{V}}^\alpha$  appears on the left-hand side. Second, following [17, 21], we derive an estimate for  $\|q(s) - q(t)\|_{\mathcal{V}}$  and finally we use a differential inequality to obtain (4.5).

*Step 1. Improved stability estimate:*

Choose  $E_*$  such that  $\mathcal{E}(t, q(t)) \leq E_*$  for all  $t$ . For fixed  $s, t \in [0, T]$  we apply (4.4a) with  $q_0 = q(t)$  and  $q_1 = q(s)$ . By the stability of  $q(t)$  we find

$$\mathcal{E}(t, q_0) \leq \mathcal{E}(t, q_\theta) + \mathcal{D}(z, z_\theta) \leq (1-\theta)\mathcal{E}(t, q_0) + \theta(\mathcal{E}(t, q_1) + \mathcal{D}(z_0, z_1)) - c_*\theta(1-\theta)\|q_1 - q_0\|_{\mathcal{V}}^\alpha.$$

After subtracting  $\mathcal{E}(t, q_0)$  from both sides we may divide by  $\theta$  and pass to the limit  $\theta \rightarrow 0^+$ . Recalling  $q_0 = q(t)$  and  $q_1 = q(s)$  this leads to

$$\mathcal{E}(t, q(t)) + c_*\|q(t) - q(s)\|_{\mathcal{V}}^\alpha \leq \mathcal{E}(t, q(s)) + \mathcal{D}(z(t), z(s)), \tag{4.6}$$

which is the desired improved stability estimate. (In fact, in place of  $q(s)$  we could have taken any  $\tilde{q}$  with  $\mathcal{E}(t, \tilde{q}) \leq E_*$ ; or vice versa, we could have weakened condition (4.4) by assuming it only for stable states.)

*Step 2. Estimate for  $\|q(t) - q(s)\|_{\mathcal{V}}$ :*

Now we assume  $0 \leq s \leq t \leq T$  and interchange the role of  $s$  and  $t$  in (4.6). Employing  $\mathcal{D}(z(s), z(t)) \leq \text{Diss}_{\mathcal{D}}(z; [s, t])$  and the energy balance we find

$$\begin{aligned} c_*\|q(t) - q(s)\|_{\mathcal{V}}^\alpha &\leq \mathcal{E}(s, q(t)) + \mathcal{D}(z(s), z(t)) - \mathcal{E}(s, q(s)) \\ &\leq \mathcal{E}(s, q(t)) - \mathcal{E}(t, q(t)) + \mathcal{E}(t, q(t)) + \text{Diss}_{\mathcal{D}}(z; [s, t]) - \mathcal{E}(s, q(s)) \\ &= \int_s^t \partial_\xi \mathcal{E}(\xi, q(t)) - \partial_\xi \mathcal{E}(\xi, q(\xi)) \, d\xi \leq \int_s^t C_* \|q(t) - q(\xi)\|_{\mathcal{V}}^\beta \, d\xi, \end{aligned}$$

where we used (4.4b) in the last estimate.

*Step 3. Hölder estimate:*

Putting  $h(\tau) := \int_{t-\tau}^t \|q(\xi) - q(t)\|_{\mathcal{V}}^\beta \, d\xi$  for  $\tau \in [0, t - s]$  yields  $h'(\tau) \leq (\frac{C_*}{c_*} h(\tau))^{\beta/\alpha}$ . Using  $h(0) = 0$  leads to  $h(\tau) \leq C_1 \tau^{\alpha/(\alpha-\beta)}$  with a constant  $C_1$  depending only on  $C_*, c_*, \alpha$ , and  $\beta$ . Hence we conclude

$$\|q(s) - q(t)\|_{\mathcal{V}} = h'(t-s)^{1/\beta} \leq \left(\frac{C_*}{c_*} h(t-s)\right)^{1/\alpha} \leq \left(\frac{C_* C_1}{c_*} h(t-s)\right)^{1/\alpha} (t-s)^{1/(\alpha-\beta)},$$

which is the desired result. □

We now discuss a few results which are useful to establish the assumptions in (4.4) for integral functionals.

**Lemma 4.6** (On the convexity assumptions).

(A) Assume that  $\mathcal{D}(z_0, \cdot) : \mathcal{Z} \rightarrow [0, \infty]$  and  $\mathcal{C} : \mathcal{Q} \rightarrow \mathbb{R}_\infty$  are convex and that  $\mathcal{W} : \mathcal{Q} \rightarrow \mathbb{R}_\infty$  satisfies the following:

$$\begin{aligned} \forall E_* \exists C_W, c_w > 0 \forall q_0, q_1 \text{ with } \mathcal{W}(q_0), \mathcal{W}(q_1) \leq w_* \forall \theta \in [0, 1] : \\ \mathcal{W}((1-\theta)q_0 + \theta q_1) + c_w \theta(1-\theta)\|q_1 - q_0\|_{\mathcal{V}}^\alpha \leq (1-\theta)\mathcal{W}(q_0) + \theta\mathcal{W}(q_1). \end{aligned} \tag{4.7}$$

Then, with  $\mathcal{E}(t, \cdot) = \mathcal{W} + \mathcal{C}$  condition (4.4a) holds.

(B) For  $j \in \{1, \dots, m\}$  let  $V_j \in \{\mathbb{R}, \mathbb{R}^d, \mathbb{R}_{\text{sym}}^{d \times d}, \mathbb{R}^{d \times d}\}$  and let  $\mathbb{V} := \times_{j=1}^m V_j$ . Assume that  $\mathbb{W} : \Omega \times \mathbb{V} \rightarrow [0, \infty]$  is a Carathéodory function and that there exist  $k \in \{0, 1, \dots, m\}$ ,  $C_1, c_1, c_0 > 0$  and  $p_j > 1$  with  $p_j \geq 2$  for  $j \leq k$  and  $p_j < 2$  for  $j > k$  such that for a.a.  $x \in \Omega$  and all  $b, b^0, b^1 \in \mathbb{V}$  the following estimates hold:

$$\mathbb{W}(x, b) \geq c_0 \sum_{j=1}^m |b_j|^{p_j} - C_1, \tag{4.8a}$$

$$\begin{aligned} \mathbb{W}(x, (1-\theta)b^0 + \theta b^1) + c_1 \theta(1-\theta) \left( \sum_{j=1}^k |b_j^1 - b_j^0|^{p_j} + \sum_{j=k+1}^m \frac{|b_j^1 - b_j^0|^2}{(1 + \mathbb{W}(x, b^0) + \mathbb{W}(x, b^1))^{\gamma_j}} \right) \\ \leq (1-\theta)\mathbb{W}(x, b^0) + \theta\mathbb{W}(x, b^1), \end{aligned} \tag{4.8b}$$

where  $\gamma_j = (2-p_j)/p_j \in (0, 1)$ . Then, with  $\mathcal{V} = \times_{j=1}^m L^{p_j}(\Omega)$  and  $\mathcal{W}(v) = \int_\Omega \mathbb{W}(x, v(x)) \, dx$  the condition (4.7) holds with  $\alpha = \max\{p_1, \dots, p_k, 2\}$ .

(C) Assume that for a.a.  $x \in \Omega$  we have  $\mathbb{W}(x, \cdot) \in C^1(\mathbb{V})$  and that there is a constant  $c_* > 0$  such that the following holds for all  $b^0, b^1 \in \mathbb{V}$  :

$$\begin{aligned} & \mathbb{W}(x, b^1) - \mathbb{W}(x, b^0) - \partial_b \mathbb{W}(b^0) \cdot (b^1 - b^0) \\ & \geq c_* \sum_{j=1}^k |b_j^1 - b_j^0|^{p_j} + c_* \sum_{j=k+1}^m \frac{|b_j^1 - b_j^0|^2}{(1 + \mathbb{W}(x, b^0) + \mathbb{W}(x, b^1))^{\gamma_j}} \end{aligned} \tag{4.9}$$

for  $p_j, \gamma_j$  as in part (B). Then  $\mathbb{W}$  satisfies (4.8b).

(D) Let  $\mathbb{P} : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}$  be a Carathéodory function satisfying

$$|\mathbb{P}(x, b)| \leq C_2 \mathbb{W}(x, b) + C_3, \tag{4.10a}$$

$$|\mathbb{P}(b^1) - \mathbb{P}(b^0)| \leq C_4 \sum_{j=1}^m (1 + \mathbb{W}(x, b^0) + \mathbb{W}(x, b^1))^{\delta_j} |b_j^1 - b_j^0|, \tag{4.10b}$$

where  $\delta_j = (p_j - 1)/p_j \in (0, 1)$  and  $\mathbb{W}$  fulfills (4.8). For  $\mathcal{W}(v) < \infty$  define  $\mathcal{P}(v) = \int_{\Omega} \mathbb{P}(x, v(x)) \, dx$ . Then, for each  $E_*$  there exists  $C_*^{\mathbb{P}}$  such that for all  $v_0, v_1 \in \mathcal{V}$  with  $\mathcal{W}(v_0), \mathcal{W}(v_1) \leq E_*$  we have  $|\mathcal{P}(v_1) - \mathcal{P}(v_0)| \leq C_*^{\mathbb{P}} \|v_1 - v_0\|_{\mathcal{V}}$ .

Proof. Part (A) follows simply by using the convexity of  $\mathcal{D}(z, \cdot)$  and  $\mathcal{C}$  and adding it to the estimate provided by (4.7).

For Part (B) be first note that  $\mathcal{W}(v^0), \mathcal{W}(v^1) \leq E_*$  together with (4.8a) implies that there is a constant  $\Lambda_*$  such that

$$\|v_j^n\|_{L^{p_j}(\Omega)} \leq \Lambda_* \quad \text{for } n \in \{0, 1\} \text{ and } j = 1, \dots, m.$$

Setting  $b^n = v^n(x)$  and integrating both sides of (4.8b) over the domain  $\Omega$  it remains to estimate the left-hand side from below. For  $j > k$  we derive a so-called reverse Hölder's inequality for the the quotient  $u^2/N^{-\gamma}$  via

$$\int_{\Omega} u^{2/(1+\gamma)} \, dx \leq \left( \int_{\Omega} u^2/N^{\gamma} \, dx \right)^{1/(1+\gamma)} \left( \int_{\Omega} N \, dx \right)^{\gamma/(1+\gamma)}$$

where  $u = |v_j^1(x) - v_j^0|$  and  $N = 1 + \mathbb{W}(v^0) + \mathbb{W}(v^1)$ . This provides the lower bound

$$\begin{aligned} & (1-\theta)\mathcal{W}(v^0) + \theta\mathcal{W}(v^1) - \mathcal{W}((1-\theta)v^0 + \theta v^1) \\ & \geq c_1 \theta (1-\theta) \left( \sum_{j=1}^k \|v_j^1 - v_j^0\|_{L^{p_j}}^{p_j} + \sum_{j=k+1}^m \frac{\|v_j^1 - v_j^0\|_{L^{p_j}}^2}{(C^d(\Omega) + 2E_*)^{\gamma_j}} \right). \end{aligned}$$

Since  $\alpha = \max\{p_1, \dots, p_k, 2\}$  the desired condition (4.7) follows from  $\|v_j^1 - v_j^0\|_{L^{p_j}}^{\rho} \geq \|v_j^1 - v_j^0\|_{L^{p_j}}^{\alpha} / (2\Lambda_*)^{\alpha - \rho}$  and from  $\rho \leq \alpha$ .

To establish Part (C) we let  $b^{\theta} = (1-\theta)b^0 + \theta b^1$  and apply (4.9) with  $b^0$  replaced by  $b^{\theta}$ . Dropping  $x$  for notational simplicity and using  $b^1 - b^{\theta} = (1-\theta)(b^1 - b^0)$  we find

$$\begin{aligned} & \mathbb{W}(b^1) - \mathbb{W}(b^{\theta}) - (1-\theta)\partial_b \mathbb{W}(b^{\theta}) \cdot (b^1 - b^0) \\ & \geq c_* \sum_{j=1}^k (1-\theta)^{p_j} |b_j^1 - b_j^0|^{p_j} + c_* (1-\theta)^2 \sum_{j=k+1}^m \frac{|b_j^1 - b_j^0|^2}{(1 + \mathbb{W}(b^1) + \mathbb{W}(b^{\theta}))^{\gamma_j}}. \end{aligned} \tag{4.11}$$

Similarly, we may replace  $b^1$  by  $b^0$  in (4.9) and  $b^0$  by  $b^{\theta}$ , respectively, and find, using  $b^0 - b^{\theta} = -\theta(b^1 - b^0)$ ,

$$\begin{aligned} & \mathbb{W}(b^0) - \mathbb{W}(b^{\theta}) + \theta\partial_b \mathbb{W}(b^{\theta}) \cdot (b^1 - b^0) \\ & \geq c_* \sum_{j=1}^k \theta^{p_j} |b_j^1 - b_j^0|^{p_j} + c_* \theta^2 \sum_{j=k+1}^m \frac{|b_j^1 - b_j^0|^2}{(1 + \mathbb{W}(b^0) + \mathbb{W}(b^{\theta}))^{\gamma_j}}. \end{aligned} \tag{4.12}$$

Multiplying (4.11) by  $\theta$  and (4.12) by  $1-\theta$  and adding the results, the term with the partial derivative cancels and we obtain

$$(1-\theta)\mathbb{W}(b^0) + \theta\mathbb{W}(b^1) - \mathbb{W}(b^\theta) \geq c_* \sum_{j=1}^k (\theta(1-\theta)^{p_j} + (1-\theta)\theta^{p_j}) |b_j^1 - b_j^0|^{p_j} + c_* \theta(1-\theta) \sum_{j=k+1}^m A_j(\theta, b^1, b^0) |b_j^1 - b_j^0|^2$$

where  $A_j(\theta, b^1, b^0) = \frac{1-\theta}{(1 + \mathbb{W}(b^1) + \mathbb{W}(b^\theta))^{\gamma_j}} + \frac{\theta}{(1 + \mathbb{W}(b^0) + \mathbb{W}(b^\theta))^{\gamma_j}}$ .

Since  $\theta(1-\theta)^{p_j} + (1-\theta)\theta^{p_j} \geq \theta(1-\theta)/2^{p_j}$  it suffices to estimate the terms  $A_j$  from below. Letting  $w_n = \mathbb{W}(b^n)$  convexity gives  $\mathbb{W}(b^\theta) \leq (1-\theta)w_0 + \theta w_1$ . Using  $\theta \in [0, 1]$  we find

$$A_j(\theta, b^1, b^0) \geq \frac{1-\theta}{(1 + (1+\theta)w_1 + (1-\theta)w_0)^{\gamma_j}} + \frac{\theta}{(1 + (2-\theta)w_1 + \theta w_0)^{\gamma_j}} \geq \left( \frac{1-\theta}{(1+\theta)^{\gamma_j}} + \frac{\theta}{(2-\theta)^{\gamma_j}} \right) \frac{1}{(1 + w_1 + w_0)^{\gamma_j}} \geq \frac{(2/3)^{\gamma_j}}{(1 + w_1 + w_0)^{\gamma_j}}$$

Thus, (4.8) is established and Part (C) is proved.

Part (D) follows by a direct application of Hölder’s inequality providing

$$|\mathcal{P}(v^1) - \mathcal{P}(v^0)| \leq C_4 \sum_{j=1}^m (\mathcal{L}^d(\Omega) + 2E_*)^{\delta_j} \|v_j^1 - v_j^0\|_{L^{p_j}} \leq C_*^{\mathcal{P}} \|v^1 - v^0\|_{\mathcal{V}}$$

with  $C_*^{\mathcal{P}} = \max\{(\mathcal{L}^d(\Omega) + 2E_*)^{\delta_j} \mid j = 1, \dots, m\}$ . □

Part (D) will be applied to  $\mathcal{P}(q) = \partial_t \mathcal{E}(t, q)$  which is given in (3.12). Clearly, the linear term involving  $\dot{l}(t)$  can be estimated directly. Thus, for fixed  $t \in [0, T]$  the density  $\mathbb{P}$  has the form  $\mathbb{P}(x, e, z) = \partial_e W(x, e + e_D(t, x)) : \dot{e}_D(t, x)$ , where  $e_D$  is given in (A2) of (3.2), see before Corollary 5.4 for more details.

## 5 Examples

In this section we give examples on stored elastic energy densities that are well known from engineering literature and that satisfy the hypotheses stated in Sect. 3.1. Moreover, we provide examples fitting to the setup of Lemma 4.6. To simplify notations we drop the explicit dependence on the material coordinates  $x \in \Omega$ . Of course, the results can be generalized to heterogeneous materials, if all the estimates are uniform as assumed in the previous sections.

### 5.1 Elastic energy densities with additional convexity properties

Sects. 5.1.1 and 5.1.2 deal with examples on the different types of convexity. They all use Part (C) of Lemma 4.6.

#### 5.1.1 Examples on joint convexity, strict convexity and uniform convexity

In the modeling of damage the inner variable often influences the stored elastic energy density in form of a product. The function  $W$  analyzed in the following was first introduced in [27]. There, it was shown that such product can be jointly convex in the two variables  $e$  and  $z$ . With regard to Lemma 4.6 we summarize several properties of  $W$  in the next lemma.

**Lemma 5.1.** For  $g \in C^2([0, 1], (0, 1])$ ,  $a \geq 0$  and  $\mathbb{B} \in \mathbb{R}^{(d \times d) \times (d \times d)}$  symmetric and positive definite let

$$W(e, z) := \frac{1}{2g(z)} e : \mathbb{B} : e + \frac{a}{2} z^2,$$

where we further assume  $1 = g(0) > g(1) > 0$ ,  $g'(z) \leq 0$  and  $g''(z) \leq -\gamma \leq 0$  for  $z \in [0, 1]$ . Then,  $W : \mathbb{R}_{\text{sym}}^{d \times d} \times [0, 1] \rightarrow \mathbb{R}$  is convex and there exists a constant  $C > 0$  such that for all  $e, \hat{e}, z$ , and  $\hat{z}$  we have

$$|\partial_e W(e, z)| \leq C (W(e, z) + 1), \tag{5.1}$$

$$|\partial_e W(e, z) - \partial_e W(\hat{e}, \hat{z})| \leq C |e - \hat{e}| + C (1 + W(e, z) + W(\hat{e}, \hat{z}))^{1/2} |z - \hat{z}|. \tag{5.2}$$

If additionally  $a > 0$  and  $\gamma > 0$ , then there exists  $c_* > 0$  such that

$$W(\hat{e}, \hat{z}) - W(e, z) - \partial_e W(e, z) : (\hat{e} - e) - \partial_z W(e, z) (\hat{z} - z) \geq \frac{c_*}{2} (|\hat{e} - e|^2 + |\hat{z} - z|^2). \tag{5.3}$$

**Proof.** The estimates (5.1) and (5.2) follow easily from the linear structure  $\partial_e W(e, z) = \frac{1}{g(z)} \mathbb{B} : e$  and the positive definiteness of  $\mathbb{B}$ , namely  $W(e, z) \geq c_1 |e|^2$  for all  $e$  and  $z$ .

To establish the convexity properties we calculate the Hessian  $D^2W$  explicitly. Omitting the argument  $z$  in  $g$  and its derivatives we obtain

$$D^2W(e, z) \left[ \begin{pmatrix} E \\ Z \end{pmatrix}, \begin{pmatrix} E \\ Z \end{pmatrix} \right] = \frac{1}{g^3} (gE - g'Ze) : \mathbb{B} : (gE - g'Ze) + \frac{-g''}{2g^2} e : \mathbb{B} : e Z^2 + aZ^2, \tag{5.4}$$

which provides convexity since all terms on the right-hand side are nonnegative.

To derive strict convexity we let  $\delta(z) = g'(z)/g(z) \in [-\delta_0, \delta_0]$  and use  $g''(z) \leq -\gamma < 0$  to find  $c_2, c_3 > 0$  such that

$$D^2W(e, z) \left[ \begin{pmatrix} E \\ Z \end{pmatrix}, \begin{pmatrix} E \\ Z \end{pmatrix} \right] \geq c_2 |E - \delta Z e|^2 + c_3 |e|^2 Z^2 + aZ^2 \geq \frac{c_2 \varepsilon}{1 + \varepsilon} |E|^2 + (c_3 - \varepsilon \delta_0^2 c_2) |e|^2 Z^2 + aZ^2.$$

Choosing  $\varepsilon = c_3 / (\delta_0^2 c_2)$  we obtain (5.3) with  $c_* = \min\{a, c_2 c_3 / (c_3 + \delta_0^2)\}$  by classical convexity arguments. □

The above lemma states that the stored energy density  $W(e, z) = \frac{1}{\eta - z} e : \mathbb{B} : e + az^2/2$  with  $\eta > 1, a \geq 0$ , and  $\mathbb{B}$  symmetric and positive definite is convex. For  $a = 0$ , it is not strictly convex, since  $W(0, z) = 0$  for  $z \in [0, 1]$ . For  $a > 0$  we gain strict convexity but still do not have uniform convexity for  $W$  on  $\mathbb{R}_{\text{sym}}^{d \times d} \times [0, 1]$ , since  $g'' \equiv 0$ , i.e.,  $\gamma = 0$ . For  $C^2$  functions uniform convexity is equivalent to  $D^2W(e, z) \left[ \begin{pmatrix} E \\ Z \end{pmatrix}, \begin{pmatrix} E \\ Z \end{pmatrix} \right] \geq c_* (|E|^2 + Z^2)$  for some fixed  $c_* > 0$ . However, inserting  $(E, Z) = (\delta e, 1)$  into the formula (5.4) gives  $D^2W(e, z) \left[ \begin{pmatrix} \delta e \\ 1 \end{pmatrix}, \begin{pmatrix} \delta e \\ 1 \end{pmatrix} \right] = a$ , while  $|\delta e|^2 + 1$  may be arbitrarily big, since  $\delta(z) = g'(z)/g(z) = -1/(\eta - z) < 0$ .

### 5.1.2 More examples on uniform convexity

Here we construct an example for uniform convex stored elastic energy densities that have variables being parts of the strain tensor, like its deviator or its trace. The following lemma provides a rich set of uniformly convex functions.

**Lemma 5.2.** *Let  $V \in \{\mathbb{R}, \mathbb{R}^d, \mathbb{R}_{\text{sym}}^{d \times d}, \mathbb{R}^{d \times d}\}$  have the scalar product  $A_1 \cdot A_2 \in \mathbb{R}$  for all  $A_1, A_2 \in V$ . For  $\kappa, \varepsilon > 0$ , and  $p \in (1, \infty)$  let  $Z_{p\kappa\varepsilon}(A) := \frac{\kappa}{p} (\varepsilon + |A|^2)^{\frac{p}{2}}$  for  $A \in V$ . Then there exist constants  $c_{p\kappa\varepsilon}, C_p, k_{p\kappa\varepsilon} > 0$  and  $\lambda_p \in \{0, \varepsilon\}$  such that for all  $A_1, A_2, A \in V$  we have*

$$Z_{p\kappa\varepsilon}(A_1) - Z_{p\kappa\varepsilon}(A_2) \geq \partial_A Z_{p\kappa\varepsilon}(A_2) \cdot (A_1 - A_2) + c_{p\kappa\varepsilon} (\lambda_p + |A_1| + |A_2|)^{p-2} |A_1 - A_2|^2, \tag{5.5}$$

$$|\partial_A Z_{p\kappa\varepsilon}(A)| \leq C_p (Z_{p\kappa\varepsilon}(A) + 1), \tag{5.6}$$

$$|\partial_A Z_{p\kappa\varepsilon}(A_1) - \partial_A Z_{p\kappa\varepsilon}(A_2)| \leq \begin{cases} k_{p\kappa\varepsilon} |A_1 - A_2| & \text{if } 1 < p < 2 \\ k_{p\kappa\varepsilon} (\sqrt{\varepsilon} + |A_1| + |A_2|)^{p-2} |A_1 - A_2| & \text{if } p \geq 2 \end{cases} \tag{5.7}$$

**Proof.** In the proof we omit the subscripts  $p, \kappa$ , and  $\varepsilon$ . Direct computations give

$$\partial_A Z(A_2) \cdot A_1 = \kappa (\varepsilon + |A_2|^2)^{\frac{p-2}{2}} A_2 \cdot A_1,$$

$$\partial_A^2 Z(A_2)[A_1, A_3] = (p-2) \kappa (\varepsilon + |A_2|^2)^{\frac{p-4}{2}} (A_2 \cdot A_1)(A_2 \cdot A_3) + \kappa (\varepsilon + |A_2|^2)^{\frac{p-2}{2}} A_1 \cdot A_3.$$

Estimate (5.5) can be verified by a Taylor expansion of  $\xi \mapsto Z(A_2 + \xi(A_1 - A_2))$  in the point  $\xi = 0$  with a remainder term of order 2 using the ideas of [10]. Estimate (5.6) is obtained, with  $C_p = p^{(p-1)/p}$ , via

$$|\partial_A Z(A)| \leq \kappa (\varepsilon + |A|^2)^{\frac{p-2}{2}} (\varepsilon + |A|^2)^{\frac{1}{2}} = (pZ(A))^{(p-1)/p} \leq C_p (Z(A) + 1).$$

In the following we carry out the proof of estimate (5.7) using a Taylor expansion of  $f(\xi) := \partial_A Z(A_2 + \xi(A_1 - A_2))$  in the point  $\xi = 0$  with a remainder term of order 1:

$$|\partial_A Z(A_1) - \partial_A Z(A_2)| = |f(1) - f(0)| \leq \int_0^1 \left| \frac{df(\xi)}{d\xi} \right| d\xi.$$

We let  $A^\xi := A_2 + \xi(A_1 - A_2)$ . For  $1 < p \leq 2$  we have

$$\begin{aligned} \left| \frac{df(\xi)}{d\xi} \right| &= \left| \partial_A^2 Z(A^\xi)[A_1 - A_2, \cdot] \right| \leq \left( (2-p)\kappa(\varepsilon + |A^\xi|^2)^{\frac{p-4}{2}} |A^\xi|^2 + \kappa(\varepsilon + |A^\xi|^2)^{\frac{p-2}{2}} \right) |A_1 - A_2| \\ &\leq (3-p)\kappa(\varepsilon + |A^\xi|^2)^{\frac{p-2}{2}} |A_1 - A_2| \leq (3-p)\kappa\varepsilon^{\frac{p-2}{2}} |A_1 - A_2|. \end{aligned}$$

This provides the upper estimate in (5.7). Similarly, for  $p \geq 2$  we have

$$\begin{aligned} \left| \frac{df(\xi)}{d\xi} \right| &= \left| \partial_A^2 Z_{p\kappa\varepsilon}(A^\xi)[A_1 - A_2, \cdot] \right| \\ &\leq (p-1)\kappa(\varepsilon + |A^\xi|^2)^{\frac{p-2}{2}} |A_1 - A_2| \leq (p-1)\kappa(\sqrt{\varepsilon} + |A_1| + |A_2|)^{p-2} |A_1 - A_2|, \end{aligned}$$

which is the lower estimate in (5.7). □

We introduce linear operators  $g_i, g_j, g_{kl} : \mathbb{R}_{\text{sym}}^{d \times d} \rightarrow \mathbb{R}_{\text{sym}}^{d \times d}$  in the form

deviator:  $g_i(e) = e^D := e - \frac{\text{tr } e}{d} \text{Id}$  (5.8a)

volumetric strain:  $g_j(e) := \frac{\text{tr } e}{d} \text{Id}$  (5.8b)

$kl$ -th component of  $e$ :  $g_{kl}(e) := e_{kl} M_{kl}$  for  $k, l \in 1, \dots, d$ , where  $M_{kl}$  (5.8c)

has the entry 1 at positions  $kl$  and  $lk$  and 0 else.

These operators are orthogonal projections with respect to the scalar product  $e : \hat{e} = \sum_{k,l} e_{kl} \hat{e}_{kl}$  and form building blocks for our examples. The following lemma provides convexity properties by combining these  $g_i$  with Lemma 5.2. Here, the index  $i$  indicates that the operator is defined by one of the expressions in (5.8).

**Lemma 5.3.** For  $1 < q, p_i, r, \tilde{r} < \infty, \varepsilon_q, \varepsilon_i, \kappa_q, \kappa_i, \kappa, \tilde{\kappa} > 0, \tilde{\varepsilon} \geq 0$ , and  $N \in \mathbb{N}_0$  let

$$\begin{aligned} \overline{W}(t, x, e, z, A) &:= \widehat{W}(e + e_D(t, x), z) + Z_{q\kappa_q\varepsilon_q}(e + e_D(t, x)) \\ &\quad + \sum_{i=1}^N Z_{p_i\kappa_i\varepsilon_i}(g_i(e + e_D(t, x))) + Z_{\tilde{r}\tilde{\kappa}\tilde{\varepsilon}}(z) + Z_{r\kappa_0}(A), \end{aligned} \tag{5.9}$$

where  $\widehat{W}$  is as in Lemma 5.1 with  $\gamma, a > 0$  and the linear operators  $g_i : \mathbb{R}_{\text{sym}}^{d \times d} \rightarrow \mathbb{R}_{\text{sym}}^{d \times d}$  are as in (5.8). Then,  $\overline{W}$  and  $\partial_t \overline{W}$  satisfy the counterparts to (4.8) and (4.10), respectively, namely there exist constants  $c_*, C_* > 0$  such that

$$\begin{aligned} &c_1\theta(1-\theta) \left( T_2(E, w^{1,0}) + T_2(Z, w^{1,0}) + T_q(E, w^{1,0}) + \sum_{i=1}^N T_{p_i}(G_i, w^{1,0}) + T_{\tilde{r}}(Z, w^{1,0}) + T_r(A, w^{1,0}) \right) \\ &\leq \theta \overline{W}(t, x, e^1, z^1, A^1) + (1-\theta) \overline{W}(t, x, e^0, z^0, A^0) - \overline{W}(t, x, \theta(e^1, z^1, A^1) + (1-\theta)(e^0, z^0, A^0)), \\ &|\partial_e \overline{W}(t, x, e^1, z^1, A^1) - \partial_e \overline{W}(t, x, e^0, z^0, A^0)| \\ &\leq C_* \left( S_2(E, w^{1,0}) + S_2(Z, w^{1,0}) + S_q(E, w^{1,0}) + \sum_{i=1}^N S_{p_i}(G_i, w^{1,0}) + S_{\tilde{r}}(Z, w^{1,0}) + S_r(A, w^{1,0}) \right), \end{aligned}$$

where  $E = |e^1 - e^0|, Z = |z^1 - z^0|, G_i = |g_i(e^1 - e^0)|, A = |A^1 - A^0|, w^{1,0} = \overline{W}(t, x, e^1, z^1, A^1) + \overline{W}(t, x, e^0, z^0, A^0)$  and where  $T_p, S_p$  are defined via

$$T_p(\xi, w) = \begin{cases} |\xi|^p & \text{for } p \geq 2, \\ |\xi|^2 / (1+w)^{(2-p)/p} & \text{for } p \in [1, 2], \end{cases} \quad S_p(\xi, w) = \begin{cases} (1+w)^{(p-1)/p} |\xi| & \text{for } p \geq 2, \\ |\xi| & \text{for } p \in [1, 2]. \end{cases}$$

**Proof.** We let  $\mathbb{W}(e, \tilde{e}, g_1, \dots, g_N, z, A) := \widehat{W}(e, z) + Z_{q\kappa_q\varepsilon_q}(\tilde{e}) + \sum_{i=1}^N Z_{p_i\kappa_i\varepsilon_i}(g_i) + Z_{\tilde{r}\tilde{\kappa}\tilde{\varepsilon}}(z) + Z_{r\kappa_0}(A)$  and  $\hat{b}(e, z, A) := (e, e, g_1(e), \dots, g_N(e), z, A)$ . Then, we have the relation  $\overline{W}(t, x, e, z, A) = \mathbb{W}(\hat{b}(e + e_D(t, x), z, A))$  and, by the chain

rule,  $\partial_t \overline{W}(t, x, e, z, A) = \mathbb{P}(\widehat{b}(e+e_D(t, x)), z, A; \dot{e}_D(t, x))$  with  $\mathbb{P}(b; \dot{e}) = \partial_e \mathbb{W}(b) : \dot{e} + \partial_{\varepsilon} \mathbb{W}(b) : \dot{\varepsilon} + \sum_1^N \partial_{g_j} \mathbb{W}(b) : g_j(\dot{e})$ . We also used the fact that each  $g_i$  is linear, self-adjoint and idempotent.

As a consequence it suffices to show the desired estimates for each term  $Z_{p_j \kappa_j \varepsilon_j}$  separately. For  $i = 1, 2$  let  $A_i$  be the corresponding  $j$ th component of the vectors  $b^i$ . For simplicity, we suppress the subscript  $j$  in the sequel. Inequality (4.8a) is obvious, so that we only prove (4.8b) in detail by showing (4.9). From (5.5) in Lemma 5.2 we derive for  $p \geq 2$  that  $Z_{p\kappa\varepsilon}(A_1) - Z_{p\kappa\varepsilon}(A_2) \geq \partial_A Z_{p\kappa\varepsilon}(A_2) \cdot (A_1 - A_2) + c_{p\kappa\varepsilon} |A_1 - A_2|^p$  and for  $1 < p < 2$  :

$$\begin{aligned} Z_{p\kappa\varepsilon}(A_1) - Z_{p\kappa\varepsilon}(A_2) &\geq \partial_A Z_{p\kappa\varepsilon}(A_2) \cdot (A_1 - A_2) + c_{p\kappa\varepsilon} (\lambda_p + |A_1| + |A_2|)^{p-2} |A_1 - A_2|^2 \\ &\geq \partial_A Z_{p\kappa\varepsilon}(A_2) \cdot (A_1 - A_2) + (\kappa/p)^{\frac{2-p}{p}} (\kappa/p)^{\frac{p-2}{p}} 3^{\frac{(p-1)(p-2)}{p}} (\lambda_p^p + (\varepsilon + |A_1|^2)^{\frac{p}{2}} + (\varepsilon + |A_2|^2)^{\frac{p}{2}})^{\frac{p-2}{p}} |A_1 - A_2|^2 \\ &\geq \partial_A Z_{p\kappa\varepsilon}(A_2) \cdot (A_1 - A_2) + (\kappa/p)^\gamma 3^{-\gamma(p-1)} \max\{1, \lambda_p^p\}^{-\gamma} (1 + \mathbb{W}(b^1) + \mathbb{W}(b^2))^{-\gamma} |A_1 - A_2|^2 \end{aligned}$$

with  $\gamma = (2-p)/p$ . In view of (5.3) this proves (4.8b).

Estimate (4.10a) holds, since  $|\partial_t \widehat{W}(e+e_D, z)| = |\partial_t \widehat{W}(e+e_D, z) : \dot{e}_D| \leq c_D \tilde{c}(\overline{W}(t, e, z, A) + 1)$  by (5.1) and since

$$|\partial_A Z_{p\kappa\varepsilon}(A)| = |\kappa(\varepsilon + |A|^2)^{\frac{p-2}{2}} A| \leq C_{p\kappa\varepsilon} (1 + Z_{p\kappa\varepsilon}(A)).$$

Inequality (4.10b) follows from (5.2) together with (5.7), since in the case  $p \geq 2$  we have

$$\begin{aligned} |\partial_A Z_{p\kappa\varepsilon}(A_1) - \partial_A Z_{p\kappa\varepsilon}(A_2)| &\leq k_{p\kappa\varepsilon} (\varepsilon^{\frac{p}{2p}} + (\varepsilon + |A_1|^2)^{\frac{p}{2p}} + (\varepsilon + |A_2|^2)^{\frac{p}{2p}})^{p-2} |A_1 - A_2| \\ &\leq k_{p\kappa\varepsilon} 3^{\frac{(p-1)(p-2)}{p}} (\varepsilon^{\frac{p}{2}} + (\varepsilon + |A_1|^2)^{\frac{p}{2}} + (\varepsilon + |A_2|^2)^{\frac{p}{2}})^{\frac{p-2}{p}} |A_1 - A_2| \\ &\leq k_{p\kappa\varepsilon} 3^{\frac{(p-1)(p-2)}{p}} \max\{1, \varepsilon^{\frac{p-2}{2}}\} (1 + \mathbb{W}(b^1) + \mathbb{W}(b^2))^{\frac{p-1}{p}} |A_1 - A_2|. \end{aligned}$$

□

As a first simple consequence we obtain Lipschitz-continuity of energetic solutions for a reasonably wide class of stored-energy densities  $\widehat{W}$ .

**Corollary 5.4.** *If the assumptions of Theorem 3.1 hold with  $1 < r \leq 2$  and if  $W$  is given as in Lemma 5.1, then all energetic solutions  $q : [0, T] \rightarrow \mathcal{Q}$  satisfy  $q \in C^{Lip}([0, T], \mathcal{V})$  with  $\mathcal{V} = H^1(\Omega, \mathbb{R}^d) \times W^{1,r}(\Omega)$ .*

Using the results of Lemmata 5.1 and 5.2 we now discuss the effective use of the spaces  $\mathcal{X}$  and  $\mathcal{V}$  for particular examples of the type introduced in Lemma 5.3. We emphasize that different choices for  $\mathcal{V}$  lead to different results that cannot be obtained by just using one space  $\mathcal{V}$ . We start with the simpler case of time-independent Dirichlet-conditions.

**Example 5.5** (Time-independent Dirichlet-conditions). Assume that

$$W(x, e, z, A) := \frac{|e + e_D(x)|^2}{2(1 + 2(1 - z))^{\frac{1}{2}}} + |\text{tr}(e + e_D(x))|^p + |A|^3 + |z|^2 \quad \text{with } p > 2. \tag{5.10}$$

The process is driven by time-dependent volume forces  $l \in C^1([0, T], H^{-1}(\Omega, \mathbb{R}^d))$  so that the free energy is given by  $\mathcal{E}(t, u, z) := \int_{\Omega} W(x, e(u), z, \nabla z) dx - \langle l(t), u + u_D \rangle$ . The existence of an energetic solution to  $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$  is proved using  $\mathcal{Q} := \{(u, z) \in \mathcal{X} \mid 0 \leq z \leq 1 \text{ a.e. in } \Omega\}$  with  $\mathcal{X} := \{u \in H^1(\Omega, \mathbb{R}^d) \mid \text{tr } e(u) \in L^p(\Omega), u = 0 \text{ on } \Gamma_D\} \times W^{1,3}(\Omega)$ .

Let  $q = (u, z)$ ,  $\widehat{q} = (\widehat{u}, \widehat{z}) \in L_{E_*}(t)$  with  $e = e(u)$  and  $\widehat{e} = e(\widehat{u})$ . Due to the time-independent Dirichlet-conditions we have  $\partial_t \mathcal{E}(t, u, z) = \langle \dot{l}(t), u + u_D \rangle$  and (4.4b) takes the form

$$|\partial_t \mathcal{E}(t, q) - \partial_t \mathcal{E}(t, \widehat{q})| \leq c_l \|u - \widehat{u}\|_{H^1} \leq c_l \|q - \widehat{q}\|_{\mathcal{V}} \tag{5.11}$$

for any norm  $\|\cdot\|_{\mathcal{V}}$  satisfying  $\|u\|_{H^1} \leq \|(u, z)\|_{\mathcal{V}}$ . Using the uniform convexity inequalities (5.3), (5.5) and Lemma 4.6 Parts (B), (C) we find for  $\theta \in (0, 1)$  and  $(u_\theta, z_\theta) = (\theta \widehat{u} + (1-\theta)u, \theta \widehat{z} + (1-\theta)z)$

$$\begin{aligned} \theta \mathcal{E}(t, \widehat{u}, \widehat{z}) + (1-\theta) \mathcal{E}(t, u, z) - \mathcal{E}(t, u_\theta, z_\theta) \\ \geq \frac{\theta(1-\theta)c_*}{1+c_K^2} (\|e - \widehat{e}\|_{L^2}^2 + \|\text{tr}(e - \widehat{e})\|_{L^p}^p + \|\nabla z - \nabla \widehat{z}\|_{L^3}^3 + \|z - \widehat{z}\|_{L^2}^2). \end{aligned} \tag{5.12}$$

We introduce four different norms, namely for  $j, k \in \{0, 1\}$  we let

$$\|(u, z)\|_{\mathcal{V}_{jk}} := \|e(u)\|_{L^2} + \|\text{tr } e(u)\|_{L^j} + \|z\|_{L^k},$$

where  $\|\tau\|_{\mathcal{U}_0} = 0$ ,  $\|\tau\|_{\mathcal{U}_1} = \|\tau\|_{L^p}$ ,  $\|z\|_{\mathcal{Z}_0} = \|z\|_{L^2}$ ,  $\|\tau\|_{\mathcal{Z}_1} = \|z\|_{W^{1,3}}$ .

We may define the associated Banach space  $\mathcal{V}_{jk}$  and find the embeddings  $\mathcal{V}_{11} \subset \mathcal{V}_\alpha \subset \mathcal{V}_{00}$  for  $\alpha = '01'$  or  $'10'$ .

In light of (5.11) we are free to drop the second or third term in the right-hand side of (5.12), which leads to the lower estimate

$$\theta \mathcal{E}(t, \hat{q}) + (1-\theta) \mathcal{E}(t, q) - \mathcal{E}(t, q_\theta) \geq \theta(1-\theta) c_{jk} \|\hat{q} - q\|_{\mathcal{V}_{jk}}^{\alpha_{jk}} \tag{5.13}$$

with  $\alpha_{00} = 2$ ,  $\alpha_{10} = p$ ,  $\alpha_{01} = 3$ , and  $\alpha_{11} = \max\{3, p\}$ . Applying Theorem 4.5 gives the following temporal regularities:

$$q \in C^{\text{Lip}}([0, T], \mathcal{V}_{00}) \cap C^{1/2}([0, T], \mathcal{V}_{01}) \cap C^{1/(p-1)}([0, T], \mathcal{V}_{10}) \cap C^{1/(\max\{p,3\}-1)}([0, T], \mathcal{V}_{11}).$$

Next we show the importance of the effective use of the space  $\mathcal{V}$  in the case of time-dependent Dirichlet-conditions.

**Example 5.6** (Time-dependent Dirichlet-conditions). We consider  $\overline{W}(t, x, e, z, A) := W(x, e + e_D(t, x), z, A)$  with  $W$  from the previous example. Now,  $\mathcal{E} : [0, T] \times \mathcal{Q} \rightarrow \mathbb{R}$  reads  $\mathcal{E}(t, u, z) := \int_\Omega \overline{W}(t, x, e(u), z, \nabla z) dx$  with  $\mathcal{Q}$  as above. For  $\mathcal{V}$  we again use the spaces  $\mathcal{V}_{jk}$  which provide the same convexity powers  $\alpha_{jk}$  in (4.4a) as in Example 5.5.

It remains to study the powers  $\beta_{jk}$  for the power  $\partial_t \mathcal{E}(t, q)$  in (4.4b). Thereto we consider states  $q = (u, z)$ ,  $\hat{q} = (\hat{u}, \hat{z}) \in L_{E_*}(t)$  with  $e = e(u)$ ,  $\hat{e} = e(\hat{u})$ . Lemma 5.1 applies to  $\widehat{W}(t, x, e, z) = \frac{|e+e_D(t,x)|^2}{2(1+2(1-z))^{1/2}} + |z|^2$ . Hence estimate (5.2) and Hölder's inequality with  $\tilde{p} = 2$  yield

$$\begin{aligned} \left| \partial_t \int_\Omega \widehat{W}(t, x, e, z) - \widehat{W}(t, x, \hat{e}, \hat{z}) dx \right| &\leq C c_D \mathcal{L}^d(\Omega)^{\frac{1}{2}} \|e - \hat{e}\|_{L^2} + C (\mathcal{L}^d(\Omega) + \mathcal{E}(t, q) + \mathcal{E}(t, \hat{q}))^{\frac{1}{2}} \|z - \hat{z}\|_{L^2} \\ &\leq C(1 + c_D) (\mathcal{L}^d(\Omega) + 2E_*)^{\frac{1}{2}} (\|e - \hat{e}\|_{L^2} + \|z - \hat{z}\|_{L^2}), \end{aligned}$$

where  $\nabla z$  and  $\nabla \hat{z}$  disappear because of  $\partial_t$ . Hölder's inequality for  $\tilde{p} = p$  and (4.10b) lead to

$$\left| \partial_t \int_\Omega |\text{tr}(e + e_D(t, x))|^p - |\text{tr}(\hat{e} + e_D(t, x))|^p dx \right| \leq C_4 c_D (\mathcal{L}^d(\Omega) + 2E_*)^{\frac{p-1}{p}} \|\text{tr}(e - \hat{e})\|_{L^p}. \tag{5.14}$$

Adding these two estimates and putting  $\tilde{C} := (C + C_4)(1 + c_D)(\mathcal{L}^d(\Omega) + 2E_*)^{\frac{p-1}{p}}$  results in

$$|\partial_t \mathcal{E}(t, q) - \partial_t \mathcal{E}(t, \hat{q})| \leq C (\|e - \hat{e}\|_{L^2} + \|\text{tr}(e - \hat{e})\|_{L^p} + \|z - \hat{z}\|_{L^2}) \leq \tilde{C} \|q - \hat{q}\|_{\mathcal{V}_{1k}} \text{ for } k = 0 \text{ or } 1. \tag{5.15}$$

Using the uniform convexity (5.13) (for  $j = 1$ ) and applying Theorem 4.5 gives the following temporal regularities:

$$q \in C^{1/(p-1)}([0, T], \mathcal{V}_{10}) \cap C^{1/(\max\{p,3\}-1)}([0, T], \mathcal{V}_{11})$$

for all energetic solutions of  $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ . Thus, the largest Hölder exponent we can reach is  $1/2$ .

The result can still be improved by strengthening (5.14). For example, for  $2 \leq p \leq 4$  we have

$$\begin{aligned} \left| \partial_t \int_\Omega |\text{tr}(e + e_D)|^p - |\text{tr}(\hat{e} + e_D)|^p dx \right| &\leq \int_\Omega C(1 + |\text{tr}(e + e_D)| + |\text{tr}(\hat{e} + e_D)|)^{p-2} |\text{tr}(e - \hat{e})| dx \\ &\leq C \|1 + |\text{tr}(e + e_D)| + |\text{tr}(\hat{e} + e_D)|\|_{L^2}^{p-2} \|e - \hat{e}\|_{L^2} \leq C_4 c_D (\mathcal{L}^d(\Omega) + 2E_*)^{\frac{p-1}{p}} \|q - \hat{q}\|_{\mathcal{V}_{00}}, \end{aligned}$$

where we first used (5.7), then the Cauchy-Schwarz inequality, and then  $\| |\text{tr}(\hat{e} + e_D)|^{p-2} \|_{L^2} \leq C \|\text{tr}(\hat{e} + e_D)\|_{L^p}$ , which holds for  $2 < p \leq 4$ . Thus, for  $p \in [2, 4]$  we obtain Lipschitz continuity of  $q = (u, z) : [0, T] \rightarrow \mathcal{V}_{00} = H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ .

Note that in the above examples it is not possible to reduce the powers 3 or  $p$  by estimating  $\|\nabla z - \nabla \hat{z}\|_{L^3(\Omega)}^3$  and  $\|\text{tr}(e - \hat{e})\|_{L^p}^p$  by a lower norm, since the application of Hölder's inequality only changes the Lebesgue norm, but not its power:  $\|Z\|_{L^3}^3 \geq \mathcal{L}^d(\Omega)^{-1/6} \|Z\|_{L^2}^3$ . Moreover, an interpolation  $\|Z\|_{L^q} \leq \|Z\|_{L^r}^\theta \|Z\|_{L^3}^{1-\theta}$  with  $1 < r < q < 3$  and  $\frac{1}{q} = \frac{\theta}{p} + \frac{(1-\theta)}{3}$  even leads to larger powers. Furthermore, for  $p > 4$  the term  $\|\text{tr}(e - \hat{e})\|_{L^p}^p$  in estimate (5.12) cannot be dropped, as we did it with  $\|\nabla z - \nabla \hat{z}\|_{L^3(\Omega)}^3$ , since it appears in the estimates (5.14), (5.15) because of the time-dependent Dirichlet conditions.

In the following example we demonstrate how additional regularity results can be obtained by using two different spaces for  $\mathcal{V}$  and by subsequent interpolation.

**Example 5.7 (Interpolation).** We consider

$$W(x, e, z, A) := \frac{|e + e_D(x)|^2}{2(1+2(1-z))^{1/2}} + |e + e_D(x)|^p + |e + e_D(x)|^4 + |A|^3 + |z|^2 \quad \text{with } 2 < p < 4$$

and  $\mathcal{E}(t, u, z) := \int_{\Omega} W(t, x, e(u), z, \nabla z) dx - \langle l(t), u + u_D \rangle$  with  $l \in C^1([0, T], H^{-1}(\Omega))$ . Existence of solutions follows in  $\mathcal{Q} := \{(u, z) \in \mathcal{X} \mid 0 \leq z \leq 1 \text{ in } \Omega\}$  with  $\mathcal{X} := \{u \in H^1(\Omega, \mathbb{R}^d) \mid e(u) \in L^4(\Omega, \mathbb{R}^{d \times d}), u = 0 \text{ on } \Gamma_D\} \times W^{1,3}(\Omega)$ .

The strategy from the examples above shows that energetic solutions  $q = (u, z) : [0, T] \rightarrow \mathcal{Q}$  have the temporal regularity

$$\begin{aligned} u &\in C^{\text{Lip}}([0, T], H^1(\Omega, \mathbb{R}^d)) \cap C^{1/(p-1)}([0, T], W^{1,p}(\Omega, \mathbb{R}^d)) \cap C^{1/3}([0, T], W^{1,4}(\Omega, \mathbb{R}^d)), \\ z &\in C^{\text{Lip}}([0, T], L^2(\Omega)) \cap C^{1/2}([0, T], W^{1,3}(\Omega)). \end{aligned}$$

Thereby it holds  $u(t) \in H^1(\Omega, \mathbb{R}^d) \cap W^{1,4}(\Omega, \mathbb{R}^d)$  and hence  $u \in W^{1,q}(\Omega, \mathbb{R}^d)$  for all  $2 < q < 4$ . By the interpolation theorem we obtain  $\|u(t)\|_{W^{1,q}(\Omega, \mathbb{R}^d)} \leq \|u(t)\|_{H^1(\Omega, \mathbb{R}^d)}^{\theta} \|u(t)\|_{W^{1,4}(\Omega, \mathbb{R}^d)}^{(1-\theta)}$  with  $\frac{1}{q} = \frac{\theta}{2} + \frac{(1-\theta)}{4}$  and hence

$$\|u(s) - u(t)\|_{W^{1,q}(\Omega, \mathbb{R}^d)} \leq \|u(s) - u(t)\|_{H^1(\Omega, \mathbb{R}^d)}^{\theta} \|u(s) - u(t)\|_{W^{1,4}(\Omega, \mathbb{R}^d)}^{(1-\theta)} \leq C|s - t|^{\theta + \frac{1-\theta}{3}}.$$

In particular, if  $q = p$  we have  $\theta = \frac{4-p}{p}$  and therewith the new Hölder-exponent  $h = \theta + \frac{1-\theta}{3} = \frac{8-p}{3p}$ , which satisfies  $h > \frac{1}{p-1}$  for all  $p \in (2, 4)$ . Thus we have obtained by interpolation that  $u \in C^{(8-p)/(3p)}([0, T], W^{1,p}(\Omega, \mathbb{R}^d))$ .

## 5.2 Damage of concrete

In the style of [6, p. 319], where a model describing the damage of concrete is introduced, we consider here a stored elastic energy density of the form

$$W(e, z) := \mu|e|^2 + \varphi_-(\text{tr}(-e)^+) + z\varphi_+(\text{tr}(e)^+), \quad (5.16)$$

where  $\mu > 0$  is the shear modulus. The functions  $\varphi_{\pm} : [0, \infty) \rightarrow [0, \infty)$  only see the volume changes. They are convex and continuously differentiable with  $\varphi_{\pm}(0) = 0$  and  $|\varphi'_{\pm}(x)| \leq c(\varphi_{\pm}(x) + \hat{c})$  for constants  $c, \hat{c} > 0$ . Since damage mostly occurs under extension and compression corresponds to  $\text{tr}(e) < 0$ , the function  $\varphi_-$  is not coupled to damage. However,  $\varphi_+$  is premultiplied by  $z$ , since tension forces in concrete easily produce damage.

It is obvious that  $W : \mathbb{R}_{\text{sym}}^{d \times d} \times [0, 1] \rightarrow \mathbb{R}$  satisfies (H1), (H3) and (H5). Convexity condition (H2) holds, since  $\text{tr}(\cdot)$  is linear,  $\varphi_{\pm}$  are monotone and convex and  $(\pm \cdot)^+$  are convex as well. To demonstrate (H4) we use  $\partial_e(\pm \text{tr}(e)^+) : \tilde{e} = \text{sgn}(\pm \text{tr}(e)^+) \text{Id} : \tilde{e}$ . Applying the chain rule on  $\varphi_{\pm}(\text{tr}(\pm e(u))^+)$  we conclude that

$$\begin{aligned} |\partial_e W(e, z)| &= |2\mu e + \varphi'_-(\text{tr}(-e)^+) \text{sgn}(-\text{tr}(e)^+) \text{Id} + z\varphi'_+(\text{tr}(e)^+) \text{sgn}(\text{tr}(e)^+) \text{Id}| \\ &\leq \mu(|e|^2 + 1) + dc(\varphi_-(\text{tr}(-e)^+) + \hat{c}) + zdc_+(\varphi_+(\text{tr}(e)^+) + \hat{c}) \leq \max\{1, dc\}(W(e, z) + \max\{1, \hat{c}\}). \end{aligned}$$

## 5.3 Ramberg-Osgood materials

This section deals with Ramberg-Osgood materials, which are defined by energy densities composed similarly to (5.9), but formulated in terms of the complementary energy density depending on the stresses instead of the strains. Anyhow, in the following it is explained that the corresponding stored elastic energy density of Ramberg-Osgood materials cannot be controlled by (H3) together with (H4\*) but does satisfy (H3) together with estimate (H4). As introduced in [25], Ramberg-Osgood materials can be described by a constitutive relation of a power-law type formulated in terms of the complementary energy density

$$W_{\text{cp}} : \mathbb{R}^{d \times d} \rightarrow \mathbb{R} : \sigma \mapsto \frac{1}{2} \sigma : \mathbb{A} : \sigma + \frac{a}{p'} |\sigma^D|^{p'}, \quad (5.17)$$

which depends on the linearized 2nd Piola-Kirchhoff stress tensor  $\sigma$  and its deviatoric part  $\sigma^D := \sigma - \frac{1}{d} \text{tr} \sigma \text{Id}$ . Thereby  $a \in \mathbb{R}^+$ ,  $2 < p' < \infty$ , and  $\mathbb{A} \in \mathbb{R}^{(d \times d) \times (d \times d)}$  is symmetric, positive definite with constants  $0 < c_1^{\mathbb{A}} < c_2^{\mathbb{A}}$  such that  $c_1^{\mathbb{A}} |e|^2 \leq e : \mathbb{A} : e \leq c_2^{\mathbb{A}} |e|^2$ . The complementary energy and the stored elastic energy, which depends on the strain tensor  $e \in \mathbb{R}_{\text{sym}}^{d \times d}$ , are linked by a Legendre transform, i.e.:

$$W(e) = \sup_{\sigma \in \mathbb{R}_{\text{sym}}^{d \times d}} \{\sigma : e - W_{\text{cp}}(\sigma)\} \quad \text{so that} \quad \partial_e W(e) = \sigma \quad \text{and} \quad \partial_{\sigma} W_{\text{cp}}(\sigma) = e. \quad (5.18)$$

See [30] Chap. 51 and [3] Prop. IX 2.1. for more details. This relation together with (5.17) yields  $e = \partial_\sigma W_{cp}(\sigma) = \mathbb{A}(x) : \sigma + a|\sigma^D|^{p'-2}\sigma^D$ , which is used to check the hypotheses (H2)-(H4). In view of the first relation in (5.18), convexity is easily obtained for  $W(\cdot)$ . Furthermore, we derive the coercivity inequality:

$$\begin{aligned} W(e) &\geq \sup_{\sigma \in \mathbb{R}_{sym}^{d \times d}} \left\{ \sigma : e - \frac{c_2^{\mathbb{A}}}{2} |\sigma|^2 - \frac{a}{p'} |\sigma^D|^{p'} \right\} \\ &= \sup_{\sigma \in \mathbb{R}_{sym}^{d \times d}} \left\{ \sigma^D : e^D - \frac{c_2^{\mathbb{A}}}{2} |\sigma^D|^2 - \frac{a}{p'} |\sigma^D|^{p'} + \frac{1}{d^2} \text{tr } \sigma \text{tr } e - \frac{c_2^{\mathbb{A}}}{2} (\text{tr } \sigma)^2 \right\} \\ &= \sup_{t \in \mathbb{R}} \left\{ \frac{t}{d^2} \text{tr } e - \frac{c_2^{\mathbb{A}}}{2} t^2 \right\} + \sup_{\tau \in \mathbb{R}_{dev}^{d \times d}} \left\{ \tau : e^D - \frac{c_2^{\mathbb{A}}}{2} |\tau|^2 - \frac{a}{p'} |\tau|^{p'} \right\} \\ &= \frac{1}{2d^4 c_2^{\mathbb{A}}} (\text{tr } e)^2 + \sup_{t \geq 0} \left\{ t |e^D| - \frac{c_2^{\mathbb{A}}}{2} t^2 - \frac{a}{p'} t^{p'} \right\} \geq \frac{1}{2d^4 c_2^{\mathbb{A}}} (\text{tr } e)^2 + \sup_{t \geq 0} \left\{ t |e^D| - t^{p'} \left( \frac{2a}{p'} \right) + C_1 \right\} \\ &= \frac{1}{2d^4 c_2^{\mathbb{A}}} (\text{tr } e)^2 + \frac{|e^D|^p}{p(2a)^{p-1}} - C_1 \geq \min \left\{ \frac{1}{2d^4 c_2^{\mathbb{A}}}, \frac{1}{p(c_2^{\mathbb{A}} + a)^{p-1}} \right\} |e|^p - C_2, \end{aligned} \tag{5.19}$$

where Young's inequality  $t^2 \leq bt^{p'} + C_b$  has been used for the second estimate. The last inequality results from  $1 < p \leq 2$ . Hence, (H3) holds for the exponent  $p = \frac{p'}{p'-1}$ . On the other hand we obtain with the same technique

$$\begin{aligned} W(e) &\leq \sup_{\sigma \in \mathbb{R}_{sym}^{d \times d}} \left\{ \sigma : e - \frac{c_1^{\mathbb{A}}}{2} |\sigma|^2 - \frac{a}{p'} |\sigma^D|^{p'} \right\} \leq \frac{(\text{tr } e)^2}{2c_1^{\mathbb{A}} d^4} + \frac{|e^D|^2}{8c_1^{\mathbb{A}}} + \frac{(p' - 1)|e^D|^p}{2p'(2a)^{p-1}} \\ &\leq 3 \max \left\{ \frac{1}{2c_1^{\mathbb{A}} d^4}, \frac{1}{8c_1^{\mathbb{A}}}, \frac{(p' - 1)}{2p'(2a)^{p-1}} \right\} (|e|^2 + 2), \end{aligned}$$

which yields  $|\partial_e W(e)| \leq c(|e| + \tilde{c})$  due to convexity. Thus, (H3) and (H4\*) are not satisfied for the same exponent. But (H3) in combination with (H4) holds, since (5.19) gives

$$|\partial_e W(e)| \leq c(|e| + \tilde{c}) \leq c \left( \left( \frac{1}{c_0} (W(e) + C_2) \right)^{1/p} + \tilde{c} \right) \leq c_1 (W(e) + \tilde{c}_1).$$

**Acknowledgements** The authors are grateful for helpful discussions with Dorothee Knees and Tomáš Roubíček. The research was supported by DFG within the Research Training Group 1128 *Analysis, Numerics and Optimization of Multiphase Problems*.

### References

- [1] G. Bouchitté, A. Mielke, and T. Roubíček, A complete-damage problem at small strains, *Z. Angew. Math. Phys. (ZAMP)* **60**(2), 205–236 (2009).
- [2] B. Dacorogna, *Direct Methods in the Calculus of Variations* (Springer-Verlag, Berlin, 1989).
- [3] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems* (North Holland, Amsterdam, 1976).
- [4] G. Francfort and A. Garroni, A variational view of partial brittle damage evolution, *Arch. Ration. Mech. Anal.* **182**, 125–152 (2006).
- [5] G. Francfort and A. Mielke, Existence results for a class of rate-independent material models with nonconvex elastic energies, *J. Reine Angew. Math.* **595**, 55–91 (2006).
- [6] M. Frémond, *Non-Smooth Thermomechanics* (Springer-Verlag, Berlin, 2002).
- [7] M. Frémond and B. Nedjar, Damage, gradient of damage and principle of virtual power, *Int. J. Solids Struct.* **33**, 1083–1103 (1996).
- [8] K. Hackl and H. Stumpf, Micromechanical concept for the analysis of damage evolution in thermo-viscoelastic and quasi-static brittle fracture, *Int. J. Solids Struct.* **30**, 1567–1584 (2003).
- [9] L. M. Kachanov, *Introduction to Continuum Damage Mechanics*, second edition, *Mechanics of Elastic Stability*, Vol. 10 (Kluwer Academic Publishers, Dordrecht, 1990).
- [10] D. Knees, On the regularity of weak solutions of quasi-linear elliptic transmission problems on polyhedral domains, *Z. Anal. Anwendungen* **23**, 509–546 (2004).
- [11] V. Kondrat'ev and O. Oleinik, Boundary-value problems for the system of elasticity theory in unbounded domains. Korn's inequalities, *Russian Math. Surveys* **43**(5), 65–119 (1988).
- [12] E. Lorentz and A. Benallal, Gradient constitutive relations: numerical aspects and applications to gradient damage, *Comput. Methods Appl. Mech. Eng.* **194**, 5191–5220 (2005).

- [13] A. Mainik and A. Mielke, Existence results for energetic models for rate-independent systems, *Calc. Var. Partial Differ. Equ.* **22**, 73–99 (2005).
- [14] A. Mainik and A. Mielke, Global existence for rate-independent gradient plasticity at finite strain, *J. Nonlinear Sci.* **19**(3), 221–248 (2009).
- [15] M. Marcus and V. J. Mizel, Absolute continuity on tracks and mappings of Sobolev spaces, *Arch. Ration. Mech. Anal.* **45**, 294–320 (1972).
- [16] A. Mielke and T. Roubíček, Rate-independent damage processes in nonlinear elasticity, *Math. Models Methods Appl. Sci.* **16**, 177–209 (2006).
- [17] A. Mielke and F. Theil, On rate-independent hysteresis models, *Nonlinear Differ. Equ. Appl. (NoDEA)* **11**, 151–189 (2004).
- [18] A. Mielke, Complete-damage evolution based on energies and stresses, *Discr. Cont. Dynam. Systems Ser. S* (2009), To appear. WIAS preprint 1414.
- [19] A. Mielke, L. Paoli, and A. Petrov, On the existence and approximation for a 3D model of thermally induced phase transformations in shape-memory alloys, *SIAM J. Math. Anal.* **41**(4), 1388–1414 (2009).
- [20] A. Mielke and A. Petrov, Thermally driven phase transformation in shape-memory alloys, *Adv. Math. Sci. Appl. (Gakkotosho Tokyo)* **17**, 667–685 (2007).
- [21] A. Mielke and R. Rossi, Existence and uniqueness results for a class of rate-independent hysteresis problems, *Math. Models Methods Appl. Sci.* **17**, 81–123 (2007).
- [22] A. Mielke and T. Roubíček, Rate-Independent Systems: Theory and Application (In preparation, 2009).
- [23] A. Mielke, T. Roubíček, and U. Stefanelli,  $\Gamma$ -limits and relaxations for rate-independent evolutionary problems, *Calc. Var. Partial Differ. Equ.* **31**, 387–416 (2008).
- [24] A. Mielke, T. Roubíček, and J. Zeman, Complete damage in elastic and viscoelastic media and its energetics, *Comput. Methods Appl. Mech. Eng.* (2009), Published online. WIAS preprint 1285.
- [25] W. R. Osgood and W. Ramberg, Description of stress-strain curves by three parameters, Tech. Rep. 902, National Bureau of Standards, Washington, 1943, NACA Technical Note.
- [26] W. Pompe, Korn's first inequality with variable coefficients and its generalization, *Comment. Math. Univ. Carolinae* **44**(1), 57–70 (2003).
- [27] T. Roubíček, Rate independent processes in viscous solids at small strains, *Math. Methods Appl. Sci.* **32**(7), 825–862 (2009); Erratum *Math. Methods Appl. Sci.* **32**(16), 2176 (2009).
- [28] G. Seregin, On the regularity of the minimizers of some variational problems of plasticity theory, *St. Petersburg Math. J.* **4**(5), 989–1020 (1993).
- [29] A. Visintin, Strong convergence results related to strict convexity, *Commun. Partial Differ. Equ.* **9**(5), 439–466 (1984).
- [30] E. Zeidler, *Nonlinear Functional Analysis and its Applications. III Variational Methods and Optimization* (Springer-Verlag, New York, 1985),