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Padé approximant for refractive index and nonlocal envelope equations

Sh. Amiranashvili*, U. Bandelow, A. Mielke

Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, D10117 Berlin, Germany

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ABSTRACT

Padé approximation is superior to Taylor expansion when functions contain poles. This is especially important for response functions in complex frequency domain, where singularities are present and intimately related to resonances and absorption. Therefore, we introduce a rational Padé approximant for the complex medium refractive index $n(\omega)$. The approximant is calculated using only local information of medium dispersion properties close to a carrier frequency ω_0 . In return it typically offers an accurate global representation of medium dispersion and absorption. Moreover, the fulfillment of the causality principle and the Kramers–Kronig relation can be established. In practice, our results are relevant if $n(\omega)$ is known only for $\omega \simeq \omega_0$ whereas optical field is spectrally broad such that (i) the resonance absorption becomes important and (ii) a traditional polynomial dispersion operator diverges and induces huge errors. As an exemplary application we use the approximant to derive a nonlocal envelope model for ultrashort pulses. The model provides a natural bridge between the commonly used local envelope equations and the most general non-envelope models operating directly with the electric field.

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1. Introduction

An average evolution of the optical field can be described in terms of an envelope [1]. This strategy is especially successful when the field spectrum is narrow, centered around a carrier frequency ω_0 . A slowly-varying envelope approximation (SVEA) leads then to an envelope equation, e.g., the nonlinear Schrödinger equation (NSE) which is extremely efficient in nonlinear optics [2]. On the other hand, a recent progress in generation of few-cycle and sub-cycle optical pulses for which the SVEA does not apply awakened interest in new models [3–5]. Even for a longer pulse the SVEA can be broken down by self-focusing [6,7], a steep pulse edge [8], and supercontinuum generation [9,10]. In all such situations, effective models for spectrally broad pulses are necessary. They are derived assuming an unidirectional character of the pulse propagation instead of the SVEA.

Much effort has been directed toward deriving the propagation equations that abandon the envelope concept and operate directly with the pulse field (see [11,12] for a recent review). Many such models ignore dissipation and use a simplified medium response function [13–15]. In return, the deduced equations allow for exact treatment [16–20]. These simple models are opposed by the most general ones operating in the spectral domain and allowing for an

arbitrary dispersion [21–23]. Here, a priori, knowledge of the medium dispersion is assumed for all frequencies possibly generated by the pulse, which is difficult in applications. Finally, several generalized envelope equations have been developed [24,25,8,26]. They yield description of the pulse propagation, optical shocks, and supercontinuum generation [27–30,9,31]. The medium dispersion is taken into account by a standard polynomial dispersion operator like in the higher-order NSE.

Contrary to popular belief, the traditional dispersion operator cannot completely quantify dispersion for ultrashort pulses. This happens when the pulse spectral width becomes comparable with the optical transparency window [32]. Here, by including more terms in the dispersion operator one induces huge errors far from the carrier frequency. In this paper we overcome the divergence by introducing a Padé approximant for the medium refractive index. A local differential dispersion operator in the envelope equation is replaced with a nonlocal pseudodifferential one. With only the refractive index around ω_0 , we recover the dispersion, account for causality principle and resonance absorption, and obtain an adequate description of spectrally broad pulses.

2. Dispersion operator

Medium dispersion, i.e., a relation between the wave vector k and the circular frequency ω of an optical wave propagating into

* Corresponding author. Tel.: +493020372513.

E-mail address: shalva@wias-berlin.de (Sh. Amiranashvili).

a uniform medium, is often characterized by specifying the dispersion parameters

$$\beta_m + i\alpha_m = \left. \frac{d^m k(\omega)}{d\omega^m} \right|_{\omega=\omega_0}, \quad m = 0, 1, 2, \dots \quad (1)$$

The latter are traditionally associated with the Taylor expansion

$$k(\omega) = \sum_{m=0}^{\infty} \frac{\beta_m + i\alpha_m}{m!} (\omega - \omega_0)^m, \quad (2)$$

and with the differential operator

$$\mathfrak{D}_k = \sum_{m=0}^{\infty} \frac{\beta_m + i\alpha_m}{m!} (i\partial_t)^m, \quad (3)$$

acting on a complex wave amplitude in the envelope equation for pulse propagation. In practical applications \mathfrak{D}_k is truncated and contains only a finite number of terms.

As an example, we consider a linearly polarized electromagnetic pulse with the reference frequency ω_0 . In the case of a purely one-dimensional propagation along the z-axis the pulse electric field can be presented as [1]

$$E(z, t) = \frac{1}{2} \Psi(z, t) e^{-i\omega_0 t} + c.c., \quad (4)$$

where $\Psi(z, t)$ is the complex amplitude (or complex envelope) with respect to temporal oscillations in a given space point. In what follows we will also use the spectral representation of the complex amplitude

$$\Psi(z, t) = \sum_{\Omega} \Psi_{\Omega}(z) e^{-i\Omega t}, \quad (5)$$

and the corresponding representation of the electric field

$$E(z, t) = \frac{1}{2} \sum_{\omega} E_{\omega}(z) e^{-i\omega t} + c.c., \quad (6)$$

in which Eq. (4) yields that $\Psi_{\Omega} = E_{\omega_0+\Omega}$. In the spectral domain the operator \mathfrak{D}_k is defined by

$$(\mathfrak{D}_k \Psi)_{\Omega} = \left(\sum_{m=0}^{\infty} \frac{\beta_m + i\alpha_m}{m!} \Omega^m \right) \Psi_{\Omega}.$$

In a linear dispersive medium the complex amplitude $\Psi(z, t)$ is governed by a formally exact equation

$$i\partial_z \Psi + \mathfrak{D}_k \Psi = 0, \quad (7)$$

that immediately follows from the dispersion relation (2). Nonlinear effects are further included in Eq. (7) under assumption that only nonlinear terms oscillating with the carrier frequency ω_0 are of interest. For instance, in cubic media with the instantaneous nonlinear polarization

$$P^{\text{nonl}} = \chi^{(3)} E^3, \quad (8)$$

the corresponding generalization of Eq. (7) reads

$$i\partial_z \Psi + \mathfrak{D}_k \Psi + \frac{3\pi\chi^{(3)}\omega_0^2}{2c^2\beta_0} |\Psi|^2 \Psi = 0. \quad (9)$$

With respect to the expansion (3) the first two terms in \mathfrak{D}_k are related to the reference wave vector β_0 and the reference group velocity $1/\beta_1$. A standard redefinition

$$\Psi(z, t) = \psi(z, \tau) \exp(i\beta_0 z), \quad \tau = t - \beta_1 z \quad (10)$$

simplifies Eq. (9) to the form [2]

$$i\partial_z \psi + \sum_{m=0}^{\infty} \frac{i^m}{m!} \gamma_m \frac{\partial^m \psi}{\partial \tau^m} + \frac{3\pi\chi^{(3)}\omega_0^2}{2c^2\beta_0} |\psi|^2 \psi = 0, \quad (11)$$

where

$$\gamma_0 = i\alpha_0, \quad \gamma_1 = i\alpha_1, \quad \gamma_{m \geq 2} = \beta_m + i\alpha_m.$$

The sum in Eq. (11) is referred to as the dispersion operator. The SVEA presupposes that $\omega_0^{-1} \partial_t \ll 1$ and that one may break off the infinite sum. A suitable truncation of the dispersion operator transforms Eq. (11) into a generalized higher-order NSE.

3. Padé approximant

It is important to realize that both the linear Eq. (7) and the deduced NSE (11) may become incorrect for ultrashort optical pulses even if one keeps all dispersion parameters in the dispersion operator [32]. This happens because of the priory resonance nature of the medium response function $\epsilon(\omega)$, where $\omega^2 \epsilon(\omega) = k^2(\omega) c^2$. For instance, let us consider the Lorentz model for the complex response function

$$\epsilon(\omega) = 1 - \sum_s \frac{\bar{\omega}_s^2}{\omega^2 - \omega_s^2 + 2i\delta_s \omega}, \quad (12)$$

where s enumerates resonances, ω_s is the undamped resonance frequency, $\bar{\omega}_s$ is the plasma frequency, and δ_s is the phenomenological damping constant. We see that both $\epsilon(\omega)$ and the deduced $k(\omega)$ have singularity points in the complex frequency plane. Therefore the convergence radius of the Taylor expansion (2) is finite and determined by the singularity nearest to the carrier frequency ω_0 . If the pulse spectral width exceeds the convergence radius, Eq. (7) cannot be applied. In practice, this happens if the pulse spectral width is comparable with the optical transparency window [32]. Here, including more terms in Eq. (2) we improve accuracy for $\omega \approx \omega_0$ but cause huge errors for $\omega \simeq 2\omega_0$!

A basic inadequacy of the dispersion operator for spectrally broad pulses is ignored by all model equations in which the operator (3) is used [24,25,22,8,26]. Thereby the expansion (2) is presupposed to converge for all frequencies of interest thus implying a restriction on the spectral width and the pulse duration. This restriction is removed in this paper. For this purpose we first change from $\epsilon(\omega)$ and $k(\omega)$ to the complex refractive index

$$n(\omega) = \sqrt{\epsilon(\omega)} = \frac{k(\omega)c}{\omega}.$$

As we will see, the contribution of the dispersion effect to the final equation for pulse propagation is determined exclusively by the refractive index. Second, we replace the polynomial Taylor series with the rational Padé approximant for the medium refractive index

$$n(\omega) = \frac{p_0 + p_1(\omega - \omega_0) + p_2(\omega - \omega_0)^2 + \dots}{1 + q_1(\omega - \omega_0) + q_2(\omega - \omega_0)^2 + \dots}, \quad (13)$$

in which p_m and q_m are free parameters constricted by Eq. (1). They are uniquely determined from the set of known dispersion parameters $\beta_m + i\alpha_m$ after the degrees of the numerator and the denominator in Eq. (13) are specified (see, e.g., [33]). Representation (13) is locally identical to the standard Taylor expansion and therefore derivatives of $n(\omega)$ and $k(\omega)$ are approximated at least to the same accuracy. However, the available information can now be differently distributed between the numerator and the denominator making the Padé approximant (13) more flexible than the Taylor expansion.

A principle advantage of Eq. (13) over Eq. (2) is that the former accounts for complex singularities and with respect to real ω domain approximates $n(\omega)$ in a considerably larger frequency domain. Moreover, truncating the numerator and denominator of Eq. (13) to polynomials of the same power M (i.e., using the diagonal $[M/M]$ Padé approximant) we additionally ensure that $n(\omega)$ remains bounded for $\omega \rightarrow \infty$, as opposed by unbounded poly-

mial approximations. Among other things, this leads to a considerable reduction of the numerical stiffness when computing solutions of the envelope equations.

Fig. 1 shows an exemplary application of the [5/5] Padé approximant for the refractive index of fluoride glass. A double resonance Lorentz model (12) for $\epsilon(\omega)$ is used. Note, that $n(\omega) = \sqrt{\epsilon(\omega)}$ is not a rational function, nevertheless it is excellently reproduced by the rational approximant. Moreover, to a good accuracy the approximant yields $n(\omega) = 1 + O(\omega^{-2})$ for $\omega \rightarrow \infty$ in accord with the physical intuition. Using a suitable approximation order M one can even ensure validity of the causality principle and the Kramers–Kronig relation. To this end all singularities (and roots) of $n(\omega)$ must belong to the lower half-plane of the complex ω plane [34]. This general property is tested by examining the roots of both the numerator and the denominator in Eq. (13). Note, that $2M$ points should be checked for the $[M/M]$ approximant. These points represent a set of singularities and roots of $n(\omega)$ in a very complicated manner, no simple general statement concerning behavior of these points is available [33]. Therefore a numerical test was performed. If necessary, the approximation order was increased, resulting in the improving of the approximation. For the dispersion profile and the carrier frequency shown in Fig. 1 the diagonal $[M/M]$ Padé approximant supports the causality property for $M = 3, 5, 6, 8, 9, 10$ and does not support it for $M = 1, 2, 4, 7$.

In what follows we demonstrate that the differential dispersion operator in the higher-order NSE is naturally replaced by the pseudodifferential one, \mathfrak{D}_n . The latter is associated with Eq. (13) and defined in a spectral domain

$$(\mathfrak{D}_n \Psi)_\Omega = n(\omega_0 + \Omega) \Psi_\Omega. \quad (14)$$

Among other things, it is profitable to change from \mathfrak{D}_k to \mathfrak{D}_n because the refractive index $n(\omega) \neq 0$ for real frequencies and therefore \mathfrak{D}_n can be inverted as opposed by \mathfrak{D}_k . This is an important issue because some of the previously reported envelope equations for short pulses contain $(\beta_0 + i\beta_1 \partial_t)^{-1}$, a singular term resulting from the formal inversion of the truncated $\mathfrak{D}_k \approx \beta_0 + i\beta_1 \partial_t$. These questions are addressed in the next section in which an envelope equation is derived.

4. Applications

Rational approximations and in particular the Padé approximant give a new possibility to use the local information inherent

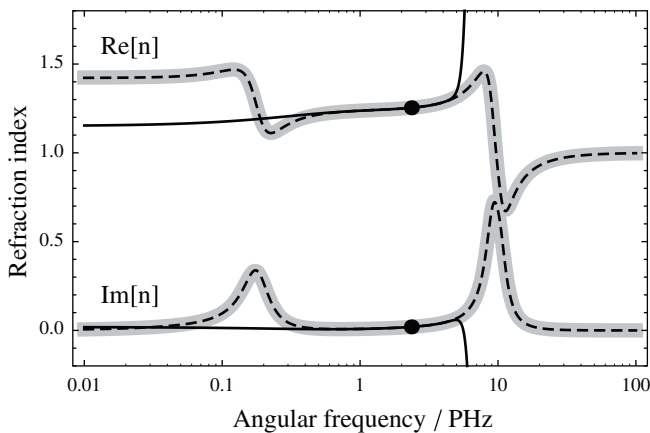


Fig. 1. Real and imaginary parts of the refractive index for fluoride glass (light thick lines, Ref. [32]) and [5/5] Padé approximant (dashed lines). Taylor expansion of the 11th order is also shown (solid lines). Both expansions are calculated at $\omega_0 = 2.355$ PHz (800 nm, thick points) and are locally identical. The Padé approximant offers better global properties, e.g., relative errors in the transparency window are $<1.5\%$.

in the dispersion parameters such that the global dispersion properties are respected. These approximations are of interest for mathematical modeling of spectrally broad pulses for which the resonances become important. As an example, we will consider applications to the commonly used envelope equations. The goal is to introduce a nonlocal envelope model that provides a natural bridge between the higher-order NSE (11) and the unidirectional reductions of the Maxwell equations [35,21–23].

The nonlocal envelope equation is further derived for a linearly polarized electromagnetic pulse propagating along the z -axis in a bulk cubic medium. We start with the model equation [1] for the pulse electric field $E(z, t)$

$$c^2 \partial_z^2 E - \partial_t^2 (E + 4\pi P) = 0, \quad (15)$$

in which the induced polarization $P(z, t)$ is decoupled into linear and nonlinear parts $P = P^{\text{lin}} + P^{\text{nonl}}$. The linear polarization contributes to the linear part of the electric displacement $D^{\text{lin}} = E + 4\pi P^{\text{lin}}$. The latter is related to the electric field via the medium response function $\epsilon(\omega)$, in the spectral representation $D_\omega^{\text{lin}} = \epsilon(\omega) E_\omega$. In what follows, the instantaneous cubic nonlinear polarization (8) is assumed for simplicity, such that Eq. (15) takes the form

$$c^2 \partial_z^2 E - \partial_t^2 D^{\text{lin}} = 4\pi \chi^{(3)} \partial_t^2 (E^3). \quad (16)$$

Eq. (16) is further considered in a weakly nonlinear limit, i.e., the nonlinear polarization on the right-hand-side is a small perturbation to the linear terms. An arbitrary linear dispersion is a major focus of interest for this work, that is why the diffraction effects are neglected and the simplest cubic nonlinearity is assumed. However, we note that an account of a more sophisticated nonlinear medium response will be straightforward. In what follows, we will use the complex amplitude representation (4). Having in mind an arbitrary pulse duration, however, we abandon the SVEA and only assume an unidirectional character of the pulse propagation (see below). The goal is to derive a closed equation for the complex amplitude $\Psi(z, t)$ compatible with Eq. (13).

The complex amplitude representation for the electric displacement reads (see Appendix)

$$D^{\text{lin}} = \frac{1}{2} (\mathfrak{D}_n^2 \Psi) e^{-i\omega_0 t} + \text{c.c.} \quad (17)$$

Therefore

$$\partial_t^2 D^{\text{lin}} = -\frac{1}{2} [(\omega_0 + i\partial_t)^2 \mathfrak{D}_n^2 \Psi] e^{-i\omega_0 t} + \text{c.c.},$$

where the last expression and Eq. (4) are further inserted into Eq. (16). For simplicity the third harmonic in the nonlinear term is neglected, though one may keep it in the spirit of [31]. We write the nonlinear term as

$$\partial_t^2 (E^3) \approx -\frac{3}{8} [(\omega_0 + i\partial_t)^2 |\Psi|^2 \Psi] e^{-i\omega_0 t} + \text{c.c.},$$

and transform Eq. (16) to the form

$$(ic\partial_z)^2 \Psi = \mathfrak{T}^2 (\mathfrak{D}_n^2 \Psi + 3\pi\chi^{(3)} |\Psi|^2 \Psi), \quad (18)$$

where $\mathfrak{T} = \omega_0 + i\partial_t$ is introduced for brevity sake.

Ignoring first the nonlinear term in Eq. (18) we obtain $(ic\partial_z \pm \mathfrak{T}\mathfrak{D}_n)\Psi = 0$ for the two counter-propagating waves. We take the first one as the main wave and account for the small contribution of the nonlinearity by putting $ic\partial_z \Psi \approx -\mathfrak{T}\mathfrak{D}_n \Psi$ and

$$[(ic\partial_z)^2 - \mathfrak{T}^2 \mathfrak{D}_n^2] \Psi \approx (ic\partial_z + \mathfrak{T}\mathfrak{D}_n)(-2\mathfrak{T}\mathfrak{D}_n \Psi). \quad (19)$$

Eq. (18) is then factorized and simplified to the form

$$ic\partial_z \Psi + (\omega_0 + i\partial_t) \left[\mathfrak{D}_n \Psi + \frac{3\pi\chi^{(3)}}{2} \mathfrak{D}_{1/n} |\Psi|^2 \Psi \right] = 0, \quad (20)$$

which is the generalized envelope equation we are interested in. The operator \mathfrak{D}_n was introduced in Eq. (14). It is bounded and its inverse $\mathfrak{D}_n^{-1} = \mathfrak{D}_{1/n}$ can be defined because the complex refractive index $n(\omega) \neq 0$ for real frequencies and $n(\omega) \rightarrow 1$ for $\omega \rightarrow \infty$. Note, that the definition of \mathfrak{D}_n^{-1} with only local dispersion parameters becomes possible due to the accurate choice of the representation (13).

Eq. (20) can be applied as long as the approximation (19) holds, the latter condition replaces SVEA. A similar condition was introduced in [24] and was referred to as the slowly-evolving wave approximation. It also ensures an unidirectional character of pulse propagation. In the rest of this section we demonstrate that Eq. (20) provides a natural bridge between the previously reported envelope models and the non-envelope unidirectional reductions of the Maxwell equations.

The identity $k(\omega)c = \omega n(\omega)$ yields for $\omega = \omega_0 + \Omega$ that

$$\mathfrak{D}_k = \frac{1}{c} (\omega_0 + i\partial_t) \mathfrak{D}_n.$$

We use this fact to return to \mathfrak{D}_k in Eq. (20) and formally allow for both \mathfrak{D}_k^{-1} and representation (3). Eq. (20) then takes the form

$$i\partial_z \Psi + \mathfrak{D}_k \Psi + \frac{3\pi\chi^{(3)}}{2c^2} (\omega_0 + i\partial_t)^2 \mathfrak{D}_k^{-1} |\Psi|^2 \Psi = 0, \quad (21)$$

which is a nonlinear counterpart of Eq. (7). Completely ignoring $i\partial_t$ in the nonlinear part of Eq. (21), approximating \mathfrak{D}_k^{-1} with $1/\beta_0$, and changing to $\psi(z, \tau)$ in accord with Eq. (10) we reduce Eq. (21) to the standard higher-order NSE (11). Generalizations of the higher-order NSE are obtained by a more accurate treatment of the nonlocal operator in the nonlinear part. For instance, assuming $i\partial_t \ll \omega_0$, using the approximations

$$(\omega_0 + i\partial_t)^2 \approx \omega_0^2 + 2i\omega_0\partial_t, \quad \mathfrak{D}_k^{-1} \approx (\beta_0 + i\beta_1\partial_t)^{-1},$$

and changing to $\psi(z, \tau)$ in accord with Eq. (10) we simplify Eq. (21) to the nonlocal envelope equation

$$i\partial_z \psi + \sum_{m=0}^{\infty} \frac{i^m}{m!} \gamma_m \frac{\partial^m \psi}{\partial \tau^m} + \frac{3\pi\chi^{(3)}}{2c^2} \frac{\omega_0^2 + 2i\omega_0\partial_t}{\beta_0 + i\beta_1\partial_t} |\psi|^2 \psi = 0,$$

proposed in [8,26]. One can further expand the fraction in the nonlinear term with respect to $i\partial_t$ and obtain a generalized NSE with the “optical shock” term (see, e.g., [9]).

Alternatively, taking the point of view of [24] one can explore the fact that the phase and the group velocities of the carrier wave are usually close to each other and put $\omega_0 \approx \beta_0/\beta_1$. Now truncating

$$(\omega_0 + i\partial_t)^2 \mathfrak{D}_k^{-1} \approx \frac{(\omega_0 + i\partial_t)^2}{\beta_0 + i\beta_1\partial_t} \approx \frac{1}{\beta_1} (\omega_0 + i\partial_t), \quad (22)$$

we simplify Eq. (21) to the local form

$$i\partial_z \Psi + \mathfrak{D}_k \Psi + \frac{3\pi\chi^{(3)}}{2c^2\beta_1} (\omega_0 + i\partial_t) |\Psi|^2 \Psi = 0.$$

Changing to $\psi(z, \tau)$ in accord with Eq. (10) we obtain another envelope equation with the optical shock term

$$i\partial_z \psi + \sum_{m=0}^{\infty} \frac{i^m}{m!} \gamma_m \frac{\partial^m \psi}{\partial \tau^m} + \frac{3\pi\chi^{(3)}}{2c^2\beta_1} (\omega_0 + i\partial_t) |\psi|^2 \psi = 0,$$

where an expansion of the cubic term with respect to $i\partial_t$ was formally avoided [24]. However, the Taylor expansion was still used when truncating \mathfrak{D}_k in Eq. (22).

We conclude, that the basic Eq. (20) contains all these envelope models as special cases. On the other hand, Eq. (20) is naturally

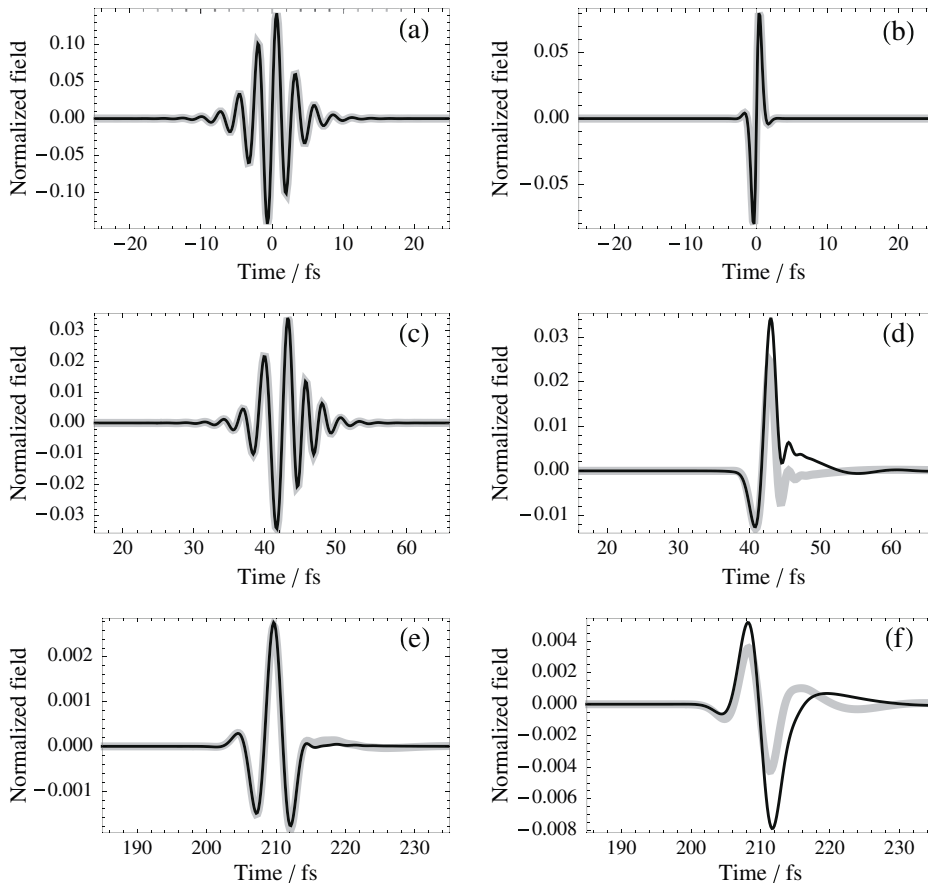


Fig. 2. Pulse electric field $E(z, t)$ normalized by $1/\sqrt{\chi^{(3)}}$ versus time for a few-cycle pulse (left) and a sub-cycle pulse (right) for a situation shown in Fig. 1. Solutions are evaluated from the Padé based propagation Eq. (20) (thin line) and the Taylor based Eq. (21) (thick line). (a and b) $z = 0$; (c and d) $z = 10 \mu\text{m}$; (e and f) $z = 50 \mu\text{m}$.

related to models derived for the spectral representation of fields [35,21–23]. Such models allow for arbitrary $k(\omega)$ without referring to the polynomial expansion (2). Using representation (5) we first rewrite Eq. (20) in the spectral form

$$i\partial_z \Psi_\Omega + k(\omega_0 + \Omega) \Psi_\Omega + \frac{3\pi\chi^{(3)}}{2c} \frac{\omega_0 + \Omega}{n(\omega_0 + \Omega)} (|\Psi|^2 \Psi)_\Omega = 0,$$

and then use Eqs. (4) and (6) to return to the electric field such that

$$i\partial_z E_\omega + k(\omega) E_\omega + \frac{2\pi\chi^{(3)}}{c} \frac{\omega}{n(\omega)} (E^3)_\omega = 0. \quad (23)$$

This model is essentially identical to the reduced equations for short pulses derived in [35,21] and is a special case of more general vectorial spectral models used in [22,23]. Also these models will benefit from the accurate choice of the rational representation (13) in cases where $k(\omega)$ is actually known only for $\omega \simeq \omega_0$.

Finally we note, that Eq. (20) can be solved numerically by a straightforward application of the split-step Fourier method [2]. Two exemplary solutions are shown in Fig. 2. The initial pulse is shaped as $1/\cosh$ and the reference frequency ω_0 corresponds to 800 nm. The diagonal Padé approximant depicted in Fig. 1 is used. The numerical solution is calculated using 2^{12} harmonics for $|t| < 256\pi/\omega_0$ and periodic boundary conditions. The propagation distance is 50 μm . For comparison, the solutions of the Taylor expansion based Eq. (21) in which an artificial spectral filter is implemented to avoid divergence are also shown. The initial pulse width at half maximum of the intensity envelope is 3.75 fs (Fig. 2a) and 0.75 fs (Fig. 2b). In the former case Eqs. (20) and (21) yield similar results. In the latter case the deviations caused by the broad pulse spectrum are clearly seen and Eq. (21) cannot be applied.

5. Conclusions

In conclusion let us summarize our findings. The approach that we wish to put forward is useful when the medium dispersion properties are known locally, i.e., only in an interval of frequencies. In such a situation, a nonlinear Schrödinger equation with a polynomial approximation of the dispersion operator is traditionally used. The operator may diverge both for ultrashort pulses and in the case of supercontinuum generation. A basic inadequacy of the polynomial dispersion operator is especially critical if the dispersion at resonances becomes important.

In this paper we show that full knowledge of the medium dispersion properties is not necessary to avoid divergence of the dispersion operator, one can still use a set of dispersion parameters for the pulse carrier frequency. However, one should construct a suitable rational approximation for the complex refractive index $n(\omega)$ and the corresponding nonlocal operator \mathfrak{D}_n . As opposed by the standard dispersion operator, \mathfrak{D}_n is bounded and invertible, has correct asymptotic behavior, and may even quantify dispersion beyond the original approximation interval. The approximation preserves both local information and global properties, e.g., one can guarantee the Kramers–Kronig relation and an accurate description of medium absorption. Our approach can be used both with the nonlocal envelope Eq. (20) and with the general non-envelope models.

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Appendix A. Group properties of \mathfrak{D}_n

In this Appendix we remind some useful facts concerning the linear medium response. A linear relation between the input signal $a(t)$ and the output $b(t)$ is given by the convolution

$$b(t) = a(t) + \int_0^\infty \tilde{f}(s) a(t-s) ds, \quad (24)$$

where the memory function $\tilde{f}(s)$ describes the induced medium response and the signal does not change for $f(s) = 0$. Eq. (24) implies a linear proportionality between the spectral components

$$b_\omega = f(\omega) a_\omega, \quad f(\omega) = 1 + \int_0^\infty \tilde{f}(s) e^{i\omega s} ds, \quad (25)$$

in which the spectral function $f(\omega)$ allows an analytic continuation into the complex ω plane. It is natural to assume that $f(\omega) \neq 0, \infty$ for $\text{Re}\omega \geq 0$ and that $f(\omega) = 1 + O(\omega^{-2})$ as $\omega \rightarrow \infty$ (see, e.g., [34]). In this respect $f(\omega)$ and $1/f(\omega)$ have the same analytic properties and a backward transformation from output to input is possible. Therefore, a set of the spectral functions is a commutative group with respect to multiplication.

If the input signal is centered around the carrier frequency ω_0 , we can introduce two complex amplitudes

$$a(t) = \frac{1}{2} A(t) e^{-i\omega_0 t} + \text{c.c.}, \quad b(t) = \frac{1}{2} B(t) e^{-i\omega_0 t} + \text{c.c.},$$

and rewrite Eq. (24) as

$$B(t) = A(t) + \int_0^\infty \tilde{f}(s) A(t-s) e^{i\omega_0 s} ds. \quad (26)$$

In a full analogy with Eq. (14) one can introduce an operator \mathfrak{D}_f such that

$$B = \mathfrak{D}_f A, \quad (\mathfrak{D}_f A)_\Omega = f(\omega_0 + \Omega) A_\Omega. \quad (27)$$

For a short-range memory function $\tilde{f}(s)$ one can further expand $A(t-s)$ into the Taylor series, insert the expansion into Eq. (26), and obtain a differential representation of \mathfrak{D}_f in analogy with Eq. (3)

$$\mathfrak{D}_f = \sum_{m=0}^\infty \frac{f^{(m)}(\omega_0)}{m!} (i\partial_t)^m, \quad (28)$$

where the derivatives $f^{(m)}(\omega_0)$ are calculated from the integral representation (25).

The mapping $f \rightarrow \mathfrak{D}_f$ provides a representation of the spectral functions group. Indeed, it is easy to check that if $f(\omega) = g(\omega)h(\omega)$ then

$$\mathfrak{D}_{gh} A = \mathfrak{D}_g (\mathfrak{D}_h A) = \mathfrak{D}_h (\mathfrak{D}_g A), \quad (29)$$

and one can invert \mathfrak{D}_f by defining $\mathfrak{D}_f^{-1} = \mathfrak{D}_{1/f}$.

For an ultrashort pulse the amplitude $A(t-s)$ changes rapidly and the Taylor expansion in Eq. (26) is not suitable. Eq. (28) should therefore be abandoned in favor of the spectral definition (27). Following these lines we use \mathfrak{D}_n defined by Eq. (14) with $n(\omega_0 + \Omega)$ given by Eq. (13). In particular, the group property (29) yields

$$\mathfrak{D}_\epsilon = \mathfrak{D}_n^2 \quad \text{and} \quad \mathfrak{D}_n^{-1} = \mathfrak{D}_{1/n}.$$

The latter equations are used when deriving Eqs. (17) and (20), respectively.

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