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Global Existence for Rate-Independent Gradient Plasticity at Finite Strain

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Abstract We provide a global existence result for the time-continuous elastoplasticity problem using the energetic formulation. For this, we show that the geometric nonlinearities arising from the multiplicative decomposition of the strain can be controlled via polyconvexity and a priori stress bounds in terms of the energy density. While temporal oscillations are controlled via energy dissipation, the spatial compactness is obtained via regularizing terms involving gradients of the internal variables.

Keywords Energetic rate-independent systems · Energetic solution · Finite-strain elastoplasticity · Multiplicative decomposition of the strain · Lie group of plastic strain · Dissipation distance · Nonlocal theory via gradient terms

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Dedicated to Sir John Ball on the occasion of his 60th birthday.

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1 Introduction

The theory of elastostatics at finite strains has seen a rapid development within the last decades. The fundamental work on polyconvex materials developed in Ball (1976) provided a basis for a general theory that allows us to treat the geometric nonlinearities arising in physically correct models. In particular, a stored-energy density $W : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty \overset{\text{def}}{=} \mathbb{R} \cup \{\infty\}$ has to satisfy (i) objectivity and (ii) local non-self-interpenetration:

(i) $W(QF) = W(F)$ for $Q \in \mathrm{SO}(d)$, $F \in \mathbb{R}^{d\times d}$,
(ii) $W(F) = \infty$ for $\det F \le 0$, and $W(F) \to \infty$ for $\det F \to 0^+$.

Hence, the proper domain of W is the general linear group $\mathrm{GL}^+(d) \overset{\text{def}}{=} \{F \in \mathbb{R}^{d\times d} \mid \det F > 0\}$, which already highlights an underlying Lie group structure.

Approximately at the same time, the theory of elastoplasticity obtained a sound mathematical basis starting from Moreau (1974), see also Temam (1985), Alber (1998), Han and Reddy (1999) for surveys on further developments. However, this theory is restricted to the case of small strains and the so-called additive split $\mathrm{e}(u) = \frac{1}{2}(\nabla u + (\nabla u)^{\mathsf{T}}) = \mathrm{e}_{\mathrm{el}} + \mathrm{e}_{\mathrm{pl}}$, as it fundamentally depends on the methods of convex analysis in Hilbert spaces.

Elastoplasticity at finite strain is usually based on the multiplicative decomposition $\nabla\varphi = F = F_{\mathrm{el}} F_{\mathrm{pl}}$, introduced in Lee (1969). This decomposition reflects the Lie group structure of $\mathrm{GL}^+(d)$, where the elastic part F_{el} will contribute to the energy storage whereas the plastic tensor $P = F_{\mathrm{pl}}$ evolves according to a plastic flow rule. The plastic tensor maps the material frame (crystallographic lattice) onto itself and is usually assumed to lie in the special linear group $\mathrm{SL}(d) \overset{\text{def}}{=} \{P \in \mathbb{R}^{d\times d} \mid \det P = 1\}$.

Meanwhile, these models are heavily used in engineering and are quite successful in predicting macroscopic deformation processes like deep drawing and other forming processes, see, e.g. Simo and Ortiz (1985), Miehe and Stein (1992), Neff and Wieners (2003). A major advance in the mathematical approach to this field was the observation in Ortiz and Stainier (1999) that the time-incremental problems in rate-independent and in the viscoplastic case can be written as a minimization problem for the sum of the increments in the stored energy and in the dissipated energy. This idea opened up the rich toolbox of the direct methods in the calculus of variations and lead, in particular, to the analysis of static microstructures in elastoplasticity, see Mühlhaus and Aifantis (1991), Ortiz and Repetto (1999), Ortiz et al. (2000), Carstensen et al. (2002), Miehe (2003), Conti and Theil (2005), Gürses et al. (2006).

In this paper, we follow a similar spirit but our aim is to develop a theory for the time-continuous problem, which in the classical setting consists of the elastic equilibrium equation and the plastic flow law. The difficulty is to find a formulation that allows us to use functional analytical tools that are compatible with the strong nonlinearities generated by the Lie group structures resulting from $\mathrm{GL}^+(d)$ and $\mathrm{SL}(d)$. We use here the theory of energetic solutions for rate independent systems as developed in Mielke et al. (2002), Mielke (2005). The recently developed geometric formulation on abstract topological spaces (cf. Francfort and Mielke 2006; Mielke et al. 2008; Mielke 2008) was strongly motivated by the present application, and thus provides the first mathematical foundation to treat the existence theory for time-dependent finite-strain elastoplasticity. Having at hand the quite technical existence result developed there, this paper focuses more on the modeling aspects arising from the Lie group structures and their interaction with the functional analysis needed to establish the assumptions of the abstract existence result.

To be more specific, we introduce some notations. Let $\varphi : \Omega \to \mathbb{R}^d$ denote the deformation, $P : \Omega \to \mathrm{SL}(d)$ the plastic tensor, and $p : \Omega \to \mathbb{R}^m$ some hardening variables. Then we assume that the stored-energy functional takes the form

$$\widetilde{\mathcal{E}}(t,\varphi,P,p) = \int_\Omega W\big(x, \nabla\varphi P^{-1}, p, \nabla P, \nabla p\big)\,\mathrm{d}x - \big\langle \ell(t), \varphi \big\rangle.$$

Here, the gradients $(\nabla P, \nabla p)$ will be essential to provide compactness and prevent formation of microstructures. In our quasistatic setting, we will assume that

$$\varphi(t) \text{ minimizes the energy } \widetilde{\mathcal{E}}\big(t,\cdot,P(t),p(t)\big). \tag{1.1}$$

The evolution of the plastic variables P and p is governed by the plastic flow rule which will be assumed to be formulated by a dissipation potential $R(x,P,p,\dot P,\dot p)$ such that

$$0 \in \partial^{\mathrm{sub}}_{(\dot P,\dot p)} R(x,P,p,\dot P,\dot p) + \begin{pmatrix} \partial_P W(\cdots) - \mathrm{div}(\partial_{\nabla P} W(\cdots)) \\ \partial_p W(\cdots) - \mathrm{div}(\partial_{\nabla p} W(\cdots)) \end{pmatrix}. \tag{1.2}$$

Here, $R(x,P,p,\cdot,\cdot)$ is convex on the tangent space and $\partial^{\mathrm{sub}}_{(\dot P,\dot p)} R$ denotes the corresponding subdifferential. This flow rule is rate independent if $R(x,P,p,\cdot,\cdot)$ is positively homogeneous of degree 1, i.e., $R(x,P,p,\lambda(\dot P,\dot p)) = \lambda R(x,P,p,\dot P,\dot p)$. As a consequence, the subdifferential $\partial^{\mathrm{sub}}_{(\dot P,\dot p)} R(x,P,p,\cdot,\cdot)$ is positively homogeneous of degree 0, i.e., it depends on the direction of the rate $(\dot P,\dot p)$ but not on its norm. Thus, if (φ,P,p) solves (1.1) and (1.2) for the loading ℓ, then for all $\lambda > 0$ the process $t \mapsto (\varphi(\lambda t), P(\lambda t), p(\lambda t))$ solves (1.1) and (1.2) for the loading $t \mapsto \ell(\lambda t)$.

As will be explained in Sect. 2.2, solutions of rate-independent systems are usually not continuous, so we need a weak form of the flow rule (1.2). For this purpose, the (infinitesimal) dissipation metric R is replaced by the associated dissipation distance D; see (2.4) for the definition. The dissipation functional

$$\mathcal{D}(P_0,p_0,P_1,p_1) = \int_\Omega D\big(x, P_0(x), p_0(x), P_1(x), p_1(x)\big)\,\mathrm{d}x$$

measures the minimal amount of energy dissipated when going from the state (P_0,p_0) to (P_1,p_1). An important fact is that $\mathcal{D}$ satisfies the (unsymmetric) triangle inequality. The need of an abstract formulation avoiding Banach spaces is already seen here: the axiom of plastic indifference (see (Sy2), Sect. 2.3) implies that the function $P \mapsto D(x,\mathbf{1},p_0,P,p_1)$ has only logarithmic growth (cf. also Šilhavý 2004). Thus, $\mathcal{D}$ cannot be coercive on any L^p space.

The energetic formulation of rate-independent systems provides a weak form of the system (1.1) and (1.2). For this, we choose a state space $\mathcal{Q}_0$ for $\boldsymbol{q} = (\varphi,P,p)$ by identifying suitable weakly closed subsets of Sobolev spaces over Ω. For given

Dirichlet boundary conditions,

$$\varphi(t,x) = g_{\mathrm{Dir}}(t,x) \quad \text{for } (t,x) \in [0,T] \times \Gamma_{\mathrm{Dir}} \tag{1.3}$$

a mapping $\boldsymbol{q} = (\varphi, P, p) : [0,T] \to \mathcal{Q}_0$ is called *energetic solution* if for all $t \in [0,T]$ the *stability condition* (S) and the *energy balance* (E) hold:

$$\begin{aligned} &\text{(S)} \quad \widetilde{\mathcal{E}}\big(t, \boldsymbol{q}(t)\big) \le \widetilde{\mathcal{E}}(t, \widehat{\boldsymbol{q}}) + \mathcal{D}\big(\boldsymbol{q}(t), \widehat{\boldsymbol{q}}\big) \quad \text{for all } \widehat{\boldsymbol{q}} \in \mathcal{Q}_0 \text{ satisfying (1.3)}, \\ &\text{(E)} \quad \widetilde{\mathcal{E}}\big(t, \boldsymbol{q}(t)\big) + \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{q}; [0,t]) = \widetilde{\mathcal{E}}\big(0, \boldsymbol{q}(0)\big) + \int_0^t \pi(s)\,\mathrm{d}s, \end{aligned} \tag{1.4}$$

where $\mathrm{Diss}_{\mathcal{D}}(\boldsymbol{q}; [r,s]) = \sup\{\sum_1^N \mathcal{D}(P(\tau_{j-1}), p(\tau_{j-1}), P(\tau_j), p(\tau_j)) \mid$ all partitions of $[r,s]\}$ and $\pi : [0,T] \to \mathbb{R}$ is the power of the external loadings:

$$\pi(t) = -\big\langle \dot{\ell}(t), \varphi(t) \big\rangle - \textstyle\int_{\Gamma_{\mathrm{Dir}}} \tau(t,x) \cdot \dot{g}_{\mathrm{Dir}}(t,x)\,\mathrm{d}x, \tag{1.5}$$

with τ being the normal stress on the boundary.

In Sect. 2, we follow Mielke (2003) for discussing the mechanical modeling of elastoplasticity and for explaining why the concept of energetic solutions can be seen as a weak version of the classical plasticity formulation. The major advantage of (S) and (E) is that it avoids derivatives and is based solely on the functionals $\widetilde{\mathcal{E}}$ and $\mathcal{D}$, which need not be smooth or even continuous. In Sect. 3, we formulate precise assumptions on W, D, and g_{Dir} that allow us to construct solutions in suitable Sobolev spaces. The main result is Theorem 3.1 which states the global existence of energetic solutions for a large class of models of elastoplasticity. In particular, we show that the classical cases of kinematical hardening and isotropic hardening are included; see Examples 3.2 and 3.3.

The existence theory is based on an abstract result for *rate-independent energetic systems* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$, where $\mathcal{Q}$ are topological spaces (here: weakly closed subsets of a reflexive Banach space $\boldsymbol{Q}$ equipped with the weak topology). This theory was developed in Mielke and Theil (1999), Mielke et al. (2002), Mainik and Mielke (2005), Francfort and Mielke (2006) and successively generalized to fit the application in finite-strain plasticity. In Sect. 4, we provide the recent version of Mielke et al. (2008), Mielke (2008) with all the abstract assumptions. This abstract approach relies on incremental minimization. For a finite partition $0 = t_0 < t_1 < \cdots < t_K = T$ and an initial state, we solve iteratively

$$\boldsymbol{q}_k \text{ minimizes } \boldsymbol{q} \mapsto \widetilde{\mathcal{E}}(t_k, \boldsymbol{q}) + \mathcal{D}(\boldsymbol{q}_{k-1}, \boldsymbol{q}). \tag{1.6}$$

These minimization problems are close to the ones used in the engineering papers mentioned above, the difference being that we use the dissipation distance $\mathcal{D}$ whereas most other works approximate this using R and some explicit predictors. Here, it is crucial that $\mathcal{D}$ is an extended quasi-distance on $\mathcal{Z}$; see (4.D1). In particular, the triangle inequality is essential to derive a priori bounds and to employ a generalized version of Helly's selection principle, cf. Mainik and Mielke (2005), Francfort and Mielke (2006).

In Mielke (2004), it is shown that (1.6) can be solved for arbitrarily large K even without the regularizing terms $(\nabla P, \nabla p)$ but under severe restrictions on $\widetilde{W}$ and D.

In Mielke and Müller (2006), where the term $(\operatorname{curl} P)P^{\mathsf{T}}$ is used for regularization, again the solvability of (1.6) is established for more general $\widetilde{W}$ and D. Here, we use full regularization via $(\nabla P, \nabla p)$ which is also common in engineering models, cf. Fleck and Hutchinson (1997), Gurtin (2000, 2002).

In Sect. 5, we then show that all assumptions are satisfied in our elastoplastic setting that includes kinematic hardening (cf. Example 3.2) as well as isotropic hardening (cf. Example 3.3). Our theory relies on sufficiently strong hardening to obtain coercivity, and thus failure effects like localization or fracture are prevented. Similarly, the regularization via $(\nabla P, \nabla p)$ prevents the formation of microstructure, cf. Carstensen et al. (2002), Bartels et al. (2004).

Some aspects of the geometric nonlinearities arising through the Lie group structures of finite strains and the multiplicative decomposition were already treated in Francfort and Mielke (2006) and Mielke and Müller (2006), respectively. Here, we combine and generalize these results in a unified setting. The first nontrivial ingredient is the treatment of time-dependent boundary conditions g_{Dir}. For this, we seek for $\varphi(t,\cdot)$ in the form

$$\varphi(t,x) = g_{\mathrm{Dir}}\big(t, y(x)\big) \quad \text{with } y \in \mathcal{Y} \overset{\text{def}}{=} \big\{ y \in \mathrm{W}^{1,q_Y}\big(\Omega;\mathbb{R}^d\big) \;\big|\; y|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id} \big\}$$

and set $\mathcal{E}(t,y,P,p) = \widetilde{\mathcal{E}}(t, g_{\mathrm{Dir}}(t,\cdot)\circ y, P, p)$. Hence, the integrand $W(x, F_{\mathrm{el}}, P, p, \nabla P, \nabla p)$ of $\mathcal{E}$ depends on the product

$$F_{\mathrm{el}} = \nabla g_{\mathrm{Dir}}\big(t, y(x)\big)\nabla y(x) P(x)^{-1}. \tag{1.7}$$

To allow for finite-strain elasticity, we assume that W is polyconvex in F_{el}. For the lower semicontinuity, we need to show that the minors $\mathbb{M}_s$ of order $s \in \{1,\dots,d\}$ of the term in (1.7) are weakly continuous. For this, we use the Cauchy–Binet relations

$$\mathbb{M}_s\big(GFP^{-1}\big) = \mathbb{M}_s(G)\,\mathbb{M}_s(F)\,\mathbb{M}_s\big(P^{-1}\big) = \frac{1}{\det P}\,\mathbb{M}_s(G)\,\mathbb{M}_s(F)\,\mathbb{K}_{d-s}(P)^{\mathsf{T}}$$

(see (5.4) for the definition of the generalized cofactor matrices $\mathbb{K}_s$), the celebrated weak continuity of $\mathbb{M}_s(\nabla y)$ (cf. Reshetnyak 1967; Ball 1976), and strong convergence of P, which is obtained from coercivity of W in ∇P.

The second nontrivial ingredient that is induced by the multiplicative structure of $\mathrm{GL}^+(d)$ is the control of the power of the external forces $\partial_t\mathcal{E}(t,\boldsymbol{q})$, which allows us to replace $\pi(t)$ in (1.5) by $\partial_t\mathcal{E}(t,\boldsymbol{q}(t))$. The crucial assumption is the multiplicative stress control $|\partial_F W(F)F^{\mathsf{T}}| \le c_1^W(W(F)+c_0^W)$, which was introduced in Bauman et al. (1991) and popularized in Ball (2002). The Lie group structure of $\mathrm{GL}^+(d)$ implies that the Kirchhoff stress $\partial_F W(F)F^{\mathsf{T}}$ lies in $\mathrm{gl}(d) = \mathrm{T}_{\mathbf{1}}\mathrm{GL}^+(d)$, and hence is more intrinsic than other stress measures. We obtain the formula

$$\partial_t\mathcal{E}(t,\boldsymbol{q}) = \int_\Omega \partial_F W(F_{\mathrm{el}}, P, p, \nabla P, \nabla p)F_{\mathrm{el}}^{\mathsf{T}} : V\,\mathrm{d}x$$

with $V(t,x) = \nabla g_{\mathrm{Dir}}(t,y(x))^{-1}\nabla\dot{g}_{\mathrm{Dir}}(t,y(x))$ and F_{el} from (1.7). Under suitable assumptions on g_{Dir}, this allows us to derive an estimate for $\partial_t\mathcal{E}(t,\boldsymbol{q})$ in terms of $\mathcal{E}(t,\boldsymbol{q})$ and to deduce further helpful continuity properties of $\partial_t\mathcal{E}$.

In the final Sect. 6, we discuss some aspects of the developed theory and give several possible generalizations. The plastic tensor P can be chosen from general subset of $\mathrm{GL}^+(d)$, which does not have a group structure or a manifold structure. Such situations occur in crystal plasticity with infinite latent hardening. Moreover, it is possible to include a condition that forbids (global) self-interpenetration. Thus, the present work develops a general existence theory for finite-strain elastoplasticity under the assumption that there is enough hardening and that there is a full gradient regularization. Using this basis, it should be possible to address the challenging questions how solutions behave if the hardening or the gradient regularization are very small and disappear in the limit. One expects localization effects like shearbands or fracture and formation of microstructure, respectively. Until now, these phenomena were investigated only for static deformation plasticity (e.g. Conti and Ortiz 2005) or for time-dependent systems under the assumption of small strains (e.g. Dal Maso et al. 2006) or rigid single-slip plasticity (cf. Conti and Theil 2005).

2 Modeling via Energetic Solutions

In this section, we discuss the modeling issues involved in our treatment of the problem. We start by introducing the general concept of generalized standard materials, which includes elastoplasticity as a special case. The next subsection discusses the mathematical issues of rate-independent systems and introduces the concept of energetic solutions as weak formulation. Finally, we show how elastoplasticity fits into this framework and how the symmetry assumptions lead to the associated Lie group structures.

2.1 Generalized Standard Materials

This mechanical theory of materials with internal variables was initiated in Halphen and Nguyen (1975) and found a very consistent formulation in Ziegler and Wehrli (1987). We just give the mathematical framework here and refer to Maugin (1992), Frémond (2002) for more details.

We consider an elastic body $\Omega \subset \mathbb{R}^d$ which is bounded and has a Lipschitz boundary $\partial\Omega$. A deformation is a mapping $\varphi : \Omega \to \mathbb{R}^d$ such that the deformation gradient $F(x) = \nabla\varphi(x)$ exists for a.e. $x \in \Omega$ and satisfies $F(x) \in \mathrm{GL}^+(d)$. Moreover, in this subsection, we simplify by assuming that φ satisfies the time-independent displacement boundary conditions $\varphi(t, x) = g_{\mathrm{Dir}}(x)$ on $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$.

The internal state at a material point $x \in \Omega$ is described by the internal variable $z(x) \in Z$, where Z is a smooth submanifold of some $\mathbb{R}^m$, e.g. $Z = \mathbb{S}^2 \subset \mathbb{R}^3$ if z denotes the magnetization. We consider quasistatic deformation processes, i.e., inertia terms are neglected, that are governed by two principles. First, we have energy storage which gives rise to the equilibrium equations, and second, we have dissipation due to changes in z which gives rise to a flow rule. Energy storage is described by the Gibbs energy

$$\mathcal{E}(t, \varphi, z) = \int_\Omega \widetilde{W}\big(x, \nabla\varphi(x), z(x), \nabla z(x)\big)\,\mathrm{d}x - \big\langle \ell(t), \varphi \big\rangle, \tag{2.1}$$

where $\langle \ell(t), \varphi\rangle = \int_\Omega f_{\mathrm{ext}}(t,x)\cdot\varphi(x)\,\mathrm{d}x + \int_{\Gamma_{\mathrm{Neu}}} h_{\mathrm{ext}}(t,x)\cdot\varphi(x)\,\mathrm{d}a(x)$ denotes the process-time dependent loading and $\Gamma_{\mathrm{Neu}} = \partial\Omega\backslash\Gamma_{\mathrm{Dir}}$. The elastic equilibrium is imposed in the form

$$\mathrm{D}_\varphi \mathcal{E}\big(t, \varphi(t), z(t)\big) = 0, \tag{2.2}$$

which may be rephrased as a static boundary-value problem for φ at each $t \in [0,T]$.

The dissipation is modeled by a dissipation potential $R : \Omega\times\mathrm{T}Z \to [0,\infty]$ such that $\partial_{\dot z} R(x,z,\dot z) \in \mathrm{T}_z^*Z$ describes the dissipational friction force due to changing z. Here, R will be nonnegative and convex in $\dot z \in \mathrm{T}_z Z$. Moreover, R is allowed to take the value ∞ to forbid motions in certain directions, which is useful to describe unidirectional processes like damage or isotropic hardening. Finally, $R(x,z,\cdot) : \mathrm{T}_z Z \to [0,\infty]$ may be nonsmooth, in particular at $\dot z = 0$, to allow for activated processes where a certain threshold (e.g., yield stress) has to be overcome to generate $\dot z \neq 0$. Thus, we will use the multi-valued subdifferential $\partial_{\dot z}^{\mathrm{sub}} R(x,z,\dot z)$ to indicate the set of all possible friction forces. From now on, we will mostly drop the explicit dependence of $\widetilde{W}$ and R on $x \in \Omega$ to simplify notation. Nevertheless, the theory applies to general heterogeneous material models.

This dissipation force will be equilibrated with the thermomechanical restoring force from the Gibbs energy, which gives the so-called Biot equation:

$$0 \in \partial_{\dot z}^{\mathrm{sub}} R\big(z(t,x), \dot z(t,x)\big) + \mathrm{D}_z\mathcal{E}\big(t,\varphi(t),z(t)\big)(x), \tag{2.3}$$

where $\mathrm{D}_z\mathcal{E}(t,\varphi,z) = \partial_z \widetilde{W}(\nabla\varphi, z, \nabla z) - \mathrm{div}(\partial_{\nabla z}\widetilde{W}(\nabla\varphi,z,\nabla z))$. If we define by $g(z,\cdot) : \mathrm{T}_z^*Z \to \mathrm{T}_z Z$ the possibly multi-valued inverse of the monotone operator $\partial_{\dot z}^{\mathrm{sub}} R(z,\cdot)$, this force balance can be transformed into the flow rule

$$\dot z(t,x) \in g\big(z(t,x), -\mathrm{D}_z\mathcal{E}\big(t,\varphi(t),z(t)\big)(x)\big).$$

Equations (2.2) and (2.3) form the system for the quasistatic evolution of generalized standard materials. In the case that $R(x,z,\dot z)$ is independent of z and has p-growth with $p > 1$, there is a fairly well-developed mathematical theory (cf., e.g. Alber 1998). Only few results are known for the case where R or g depend explicitly on z, cf., e.g. Mielke and Rossi (2007).

In this work, we are interested in the rate-independent case. This case is a degenerate limit that occurs naturally if we have an R that is nonsmooth at $\dot z = 0$, viz., $\partial_{\dot z}^{\mathrm{sub}} R(z,0)$ is a nontrivial convex set. In the limit of slow external loading $\ell(t) = \widetilde{\ell}(\delta t)$ with $0 < \delta \ll 1$ we may rescale time and (2.3) gives $0 \in \partial_{\dot z}^{\mathrm{sub}} R(z, \delta\dot z) + \mathrm{D}_z\mathcal{E}$. Thus, the formal limit $\delta \to 0$ leads to

$$0 \in \partial_{\dot z}^{\mathrm{sub}} R_1(z,\dot z) + \mathrm{D}_z\mathcal{E}\big(t,\varphi(t),z(t)\big), \quad \text{where } R_1(z,v) \overset{\mathrm{def}}{=} \lim_{\delta\to 0}\frac{1}{\delta} R(z,\delta v)$$

is the limiting dissipation potential, which is positively homogeneous of degree 1, namely $R_1(z,\lambda v) = \lambda R_1(z,v)$ for all $\lambda > 0$.

2.2 Energetic Formulation for Rate-Independent Systems

Assuming rate independency of R in (2.3), we have the positive homogeneity

$$\partial^{\mathrm{sub}}_{\dot z} R(z,v) = \partial^{\mathrm{sub}}_{\dot z} R(z,\lambda v) \qquad \text{for all } \lambda > 0.$$

The subdifferential $\partial^{\mathrm{sub}}_{\dot z} R(x,z,\cdot)$ is still a monotone operator, but it is discontinuous and it is not surjective. These two properties lead to several mathematical difficulties. The latter property implies that we cannot expect to have solutions for arbitrary initial conditions since $\mathrm{D}_z\mathcal{E}(0,\varphi(0),z(0))$ should lie in the range of the subdifferential. This will lead us to a stability condition that has to be satisfied for all $t \in [0,T]$. Moreover, by multiplying (2.2) with $\dot\varphi$ and (2.3) with $\dot z$, we obtain the natural energy balance

$$\mathcal{E}\big(t,\varphi(t),z(t)\big) + \int_0^t \int_\Omega R(z,\dot z)\,\mathrm{d}x\,\mathrm{d}t = \mathcal{E}\big(0,\varphi(0),z(0)\big) + \int_0^t \partial_s\mathcal{E}\big(s,\varphi(s),z(s)\big)\,\mathrm{d}s.$$

In the best case, $R(x,z,\cdot)$ is uniformly coercive and we obtain an a priori estimate for z of the type $\int_0^T \|\dot z(t)\|_{\mathrm{L}^1(\Omega)}\,\mathrm{d}t \le C$. However, for constructing solutions, one has to pass to the limit generating limit functions that have bounded variation with values in $\mathrm{L}^1(\Omega)$ but which are not always absolutely continuous. Thus, rate-independent systems need a solution concept that allows for jumps.

For this purpose, we note that $R : \mathrm{T}Z \to [0,\infty]$ plays the role of an (infinitesimal) Finsler-type metric on the manifold Z. We may associate to R the corresponding distance $D : Z \times Z \to [0,\infty]$, which we call dissipation distance in our context:

$$D(z_0,z_1) = \inf\left\{ \int_0^1 R\big(z(s),\dot z(s)\big)\,\mathrm{d}s \;\middle|\; z \in \mathrm{C}^1\big([0,1],Z\big),\, z(0)=z_0,\, z(1)=z_1 \right\}. \tag{2.4}$$

More precisely, D is an extended quasi-distance, namely the value ∞ and the unsymmetry $D(z_0,z_1) \neq D(z_1,z_0)$ are allowed. Most importantly, the (ordered) triangle inequality

$$D(z_1,z_3) \le D(z_1,z_2) + D(z_2,z_3)$$

holds as an immediate consequence from the definition. The triangle inequality is a seemingly simple but crucial tool to derive a priori estimates. The replacement of the (local) dissipation metric R by the (global) dissipation distance was the essential step to generalize the theory from linear spaces with translation-invariant dissipation potentials (i.e., R independent of z) to a truly geometric setting. This necessary generalization was done first in Mielke (2003) in the context of finite-strain plasticity.

The energetic formulation is based on a proper choice of the state space $\mathcal{Q} = \mathcal{Y} \times \mathcal{Z}$ for the state $\boldsymbol{q} = (\varphi, z)$. In addition to the energy functional $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty$, we define the dissipation distance

$$\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0,\infty]; \qquad \mathcal{D}(z_0,z_1) = \int_\Omega D\big(x, z_0(x), z_1(x)\big)\,\mathrm{d}x$$

and call a process $\boldsymbol{q} : [0,T] \to \mathcal{Q}$ an *energetic solution* for the rate-independent system $(\mathcal{Q},\mathcal{E},\mathcal{D})$ if for all $t \in [0,T]$ the global stability (S) and the energy balance (E) hold:

$$\text{(S)}\quad \mathcal{E}\big(t,\boldsymbol{q}(t)\big) \le \mathcal{E}(t,\widehat{\boldsymbol{q}}) + \mathcal{D}\big(\boldsymbol{q}(t),\widehat{\boldsymbol{q}}\big) \quad \text{for all } \widehat{\boldsymbol{q}} \in \mathcal{Q},$$

$$\text{(E)}\quad \mathcal{E}\big(t,\boldsymbol{q}(t)\big) + \mathrm{Diss}_{\mathcal{D}}\big(\boldsymbol{q};[0,t]\big) = \mathcal{E}\big(0,\boldsymbol{q}(0)\big) + \int_0^t \partial_s \mathcal{E}\big(s,\boldsymbol{q}(s)\big)\,\mathrm{d}s,$$

where $\mathrm{Diss}_{\mathcal{D}}(\boldsymbol{q};[r,s])$ is defined after (1.4).

The important connection between this energetic formulation and the differential system (2.2) and (2.3) is that every energetic solution that is sufficiently smooth, in particular differentiable in time, also solves the latter system. The opposite is only true under strong convexity assumptions, see Mielke and Theil (2004). To establish the former result, one first observes that inserting $\widehat{\boldsymbol{q}} = \boldsymbol{q}(t) + \varepsilon(\widehat{u},\widehat{z})$ into (S) gives the local stability

$$\big\langle \mathrm{D}_\varphi \mathcal{E}\big(t,\varphi(t),z(t)\big),\widehat{u}\big\rangle + \big\langle \mathrm{D}_\varphi \mathcal{E}\big(t,\varphi(t),z(t)\big),\widehat{z}\big\rangle + R(z,\widehat{z}) \ge 0$$

for all $(\widehat{u},\widehat{z})$. This implies (2.2) and $0 \in \partial^{\mathrm{sub}}_{\dot z} R(z,0) + \mathrm{D}_\varphi \mathcal{E}(t,\varphi(t),z(t))$. Taking the time derivative of (E) gives

$$\big\langle \mathrm{D}_\varphi \mathcal{E}\big(t,\varphi(t),z(t)\big),\dot z\big\rangle + \big\langle \mathrm{D}_\varphi \mathcal{E}\big(t,\varphi(t),z(t)\big),\dot z\big\rangle + R(z,\dot z) = 0.$$

Since $R(z,\cdot)$ is positively homogeneous of degree 1, this implies (2.3).

The advantage of the energetic formulation is that we can allow for solutions having jumps in time. Moreover, the functional $\mathcal{E}$ need not be Gâteaux differentiable.

2.3 Elastoplasticity at Finite Strain

We now specialize the generalized standard materials to finite-strain elastoplasticity by introducing the plastic tensor $P = F_{\mathrm{pl}} \in \mathrm{GL}^+(d)$, which maps the material space $\mathrm{T}_x\Omega$ (crystallographic lattice) onto itself. This leads to a multiplicative group structure and it appears natural to take P as an element of a Lie group $\mathfrak{P} \subset \mathrm{GL}^+(d)$ (however, see Sect. 6 for more general cases). Moreover, there may be additional plastic variables like hardening variables, slip strains, etc., which are combined into a vector $p \in \mathbb{R}^m$. Thus, in the general notations from above, we let $z = (P,p) \in Z = \mathfrak{P} \times \mathbb{R}^m$. We will use A as a place holder for $\nabla z = (\nabla P, \nabla p)$.

The stored energy density W is usually assumed to be the sum of an elastic part W_{el} and a part $W_{\mathrm{h,r}}$ including hardening and regularizing terms, namely

$$\widetilde{W}(x,F,P,p,A) = W_{\mathrm{el}}(x,F,P) + W_{\mathrm{h,r}}(x,P,p,A). \tag{2.5}$$

As above, the dissipation potential R will depend on $(z,\dot z) = (P,p,\dot P,\dot p)$.

Following Mielke (2003), the functions $\widetilde{W}$, W_{el}, and R have to satisfy the invariance principles (Sy1) and (Sy2), which have to hold for all $x \in \Omega$, $F \in \mathrm{GL}^+(d)$, $(P,p) \in \mathfrak{P} \times \mathbb{R}^m$, and A:

(Sy1) *Objectivity (frame indifference)*

$$\widetilde{W}(x,QF,P,p,A) = \widetilde{W}(x,F,P,p,A) \quad \text{for all } Q \in \mathrm{SO}(d).$$

(Sy2) *Plastic indifference*

$$\left.\begin{aligned} W_{\mathrm{el}}(x, F\widetilde{P}, P\widetilde{P}) &= W_{\mathrm{el}}(x, F, P) \\ R(x, P\widetilde{P}, p, \dot{P}\widetilde{P}, \dot{p}) &= R(x, P, p, \dot{P}, \dot{p}) \end{aligned}\right\} \quad \text{for all } \widetilde{P} \in \mathfrak{P}.$$

In (Sy1), each $Q \in \mathrm{SO}(d)$ is a rotation of the surrounding Euclidean space $\mathbb{R}^d$, and thus Q acts on $F = \nabla\varphi$ from the left, whereas P and p are not changed. In (Sy2), the invariance with respect to the "additional plastic deformation" $\widetilde{P}$ is only postulated for W_{el} and not for $W_{\mathrm{h,r}}$, since hardening is exactly the mechanism that destroys plastic indifference.

The consequence of (Sy2) is that W_{el} and R can be written in a reduced form via

$$W_{\mathrm{el}}(x, F, P) = \widehat{W}\big(x, FP^{-1}\big) \quad \text{and} \quad R(x, P, p, \dot{P}, \dot{p}) = \widehat{R}\big(x, p, \dot{P}P^{-1}, \dot{p}\big),$$

where $\dot{P}P^{-1}$ is the right translation of $\dot{P} \in \mathrm{T}_P\mathfrak{P}$ to the Lie algebra $\mathfrak{p} = \mathrm{T}_{\mathbf{1}}\mathfrak{P}$.

In fact, we will not use the additive decomposition (2.5), but merely assume that $\widetilde{W}$ is given in the form

$$\widetilde{W}(x, F, P, p, A) = W\big(x, FP^{-1}, P, p, A\big). \tag{2.6}$$

It seems that the form on the right-hand side is not more specific as the one on the left-hand side. However, in Sect. 3, we will make different assumptions on the dependence of W with respect to the variable $F_{\mathrm{el}} = FP^{-1}$ and P.

We now define the dissipation distance $D(x, \cdot, \cdot)$ on $Z \times Z$ as in (2.4). The plastic indifference (Sy2) implies that the dissipation distance D is right-invariant, namely

$$D(x, P_1, p_1, P_2, p_2) = D\big(x, \mathbf{1}, p_1, P_2P_1^{-1}, p_2\big) \quad \text{for all } x, P_1, P_2, p_1, p_2. \tag{2.7}$$

With these choices for W, R, and D the relevant equations (in differential form) for finite-strain elastoplasticity are exactly the elastic equilibrium (2.2) and the internal force balance (2.3) in Sect. 2.1. The remaining choice of the state space $\mathcal{Q}$ will be addressed in the next section. Then the formulation via (S) and (E), as given in Sect. 2.2, defines the energetic solutions for the elastoplastic problem.

3 Assumptions and Results

We formulate the precise assumption here. For notational simplicity, we omit volume and surface forces, i.e., we let $\ell \equiv 0$. Instead, the process will be driven by time-dependent Dirichlet data $g_{\mathrm{Dir}}(t, \cdot)$. See Remark 3.4 for the simple changes to be done, if forces have to be included. Moreover, we will assume $\mathfrak{P} = \mathrm{SL}(d)$ as this is the most important case and as it avoids complications involving additional terms "det P" appearing otherwise.

The domain $\Omega \subset \mathbb{R}^d$ is bounded and has a Lipschitz boundary. The Dirichlet part Γ_{Dir} of the boundary is assumed to have positive surface measure. The time-dependent Dirichlet data are imposed via a function $g_{\mathrm{Dir}} : [0, T] \times \Gamma_{\mathrm{Dir}} \to \mathbb{R}^d$, and we assume that it can be extended to all of $\mathbb{R}^d$ as follows:

$$\begin{aligned} &g_{\mathrm{Dir}} \in \mathrm{C}^1\big([0,T]\times\mathbb{R}^d;\mathbb{R}^d\big), \qquad \nabla g_{\mathrm{Dir}} \in \mathrm{BC}^1\big([0,T]\times\mathbb{R}^d, \mathrm{Lin}\big(\mathbb{R}^d;\mathbb{R}^d\big)\big)\\ &\text{and} \quad \big|\nabla g_{\mathrm{Dir}}(t,x)^{-1}\big| \le C \quad \text{for all } (t,x)\in[0,T]\times\mathbb{R}^d, \end{aligned} \tag{3.1}$$

where "BC^1" stands for bounded and once continuously differentiable. Thus, for each $t\in[0,T]$ the mapping $g_{\mathrm{Dir}}(t,\cdot):\mathbb{R}^d\to\mathbb{R}^d$ is a global diffeomorphism. The desired deformation $\varphi:[0,T]\times\Omega\to\mathbb{R}^d$ is searched in the form of a composition

$$\varphi(t,x) = g_{\mathrm{Dir}}\big(t, y(t,x)\big) \quad \text{with } y(t,\cdot)\in\mathcal{Y},$$

where the space of admissible deformations y is given by

$$\mathcal{Y} \overset{\text{def}}{=} \{y\in \boldsymbol{Y} \mid y|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id}\} \quad \text{with } \boldsymbol{Y} = \mathrm{W}^{1,q_{\boldsymbol{Y}}}\big(\Omega;\mathbb{R}^d\big)$$

and $q_{\boldsymbol{Y}} \in\,]d,\infty[$ to be specified later. The composition $\varphi = g_{\mathrm{Dir}}(t,\cdot)\circ y$ leads to a multiplicative split of the deformation gradient

$$\nabla\varphi(t,x) = \nabla g_{\mathrm{Dir}}\big(t,y(x)\big)\nabla y(x)$$

due to the classical chain rule. This multiplicative decomposition is perfectly compatible with the assumption of finite strains expressed through the Lie group structure of $\mathrm{GL}^+(d)$. In Habeck and Schuricht (2006), multiplicative variations $\varphi_\varepsilon = (\mathrm{id}+\varepsilon u)\circ\varphi_0$ were used to study the contact forces between elastic bodies under finite-strain deformations.

The internal variable will be $z=(P,p)\in\mathrm{SL}(d)\times\mathbb{R}^m$, and the space $\mathcal{Z}$ is chosen as

$$\begin{aligned} &\mathcal{Z} \overset{\text{def}}{=} \big\{(P,p)\in\boldsymbol{Z} \,\big|\, P(x)\in\mathrm{SL}(d) \text{ a.e. in } \Omega\big\}\\ &\quad\text{with } \boldsymbol{Z} = \big(\mathrm{L}^{q_{\boldsymbol{P}}}\big(\Omega;\mathbb{R}^{d\times d}\big)\cap\mathrm{W}^{1,r}\big(\Omega;\mathbb{R}^{d\times d}\big)\big)\times\big(\mathrm{L}^{q_{\boldsymbol{p}}}\big(\Omega;\mathbb{R}^m\big)\cap\mathrm{W}^{1,r}\big(\Omega;\mathbb{R}^m\big)\big), \end{aligned}$$

with $q_{\boldsymbol{P}}, q_{\boldsymbol{p}}, r\in\,]1,\infty[$ to be specified later (see Sect. 6 for the physically relevant case $r=1$). Clearly, $\boldsymbol{Y}$, $\boldsymbol{Z}$, and $\boldsymbol{Q}=\boldsymbol{Y}\times\boldsymbol{Z}$ are separable, reflexive Banach spaces, and $\mathcal{Y}$, $\mathcal{Z}$, and $\mathcal{Q}=\mathcal{Y}\times\mathcal{Z}$ are weakly closed subsets of the corresponding Banach spaces. The stored-energy functional $\mathcal{E}$ and the dissipation distance $\mathcal{D}$ take the forms

$$\begin{aligned} \mathcal{E}(t,y,z) &\overset{\text{def}}{=} \int_\Omega W\big(x, \nabla g_{\mathrm{Dir}}\big(t,y(x)\big)\nabla y(x)P(x)^{-1}, z(x), \nabla z(x)\big)\,\mathrm{d}x,\\ \mathcal{D}(z_0,z_1) &\overset{\text{def}}{=} \int_\Omega D\big(x, z_0(x), z_1(x)\big)\,\mathrm{d}x. \end{aligned}$$

To define the conditions on D and W, we use the notion of a *normal integrand*. If U is a topological space and $\mathcal{B}(U)$ its Borel σ-algebra, then a function $f:\Omega\times U\to\mathbb{R}_\infty$ is called a normal integrand, if

(NI1) f is $\mathcal{L}_\Omega\times\mathcal{B}(U)$ measurable,
(NI2) for a.a. $x\in\Omega$: $f(x,\cdot):U\to\mathbb{R}_\infty$ is lower semicontinuous,

where $\mathcal{L}_\Omega$ denotes the (Lebesgue) measurable subsets of Ω. Note that for each measurable mapping $u : \Omega \to U$ the composition $x \mapsto f(x, u(x))$ is measurable. We define the *domain* dom f via

$$\operatorname{dom} f \overset{\text{def}}{=} \big\{(x,u) \in \Omega \times U \;\big|\; f(x,u) < \infty\big\}.$$

For the extended quasi-distances $D(x,\cdot,\cdot)$, we impose the conditions

$$D : \Omega \times \big(\mathrm{SL}(d) \times \mathbb{R}^m\big)^2 \to [0,\infty] \quad \text{is a normal integrand;} \tag{3.2a}$$

$$\forall x \in \Omega,\ z_1, z_2 \in \mathrm{SL}(d) \times \mathbb{R}^m : \quad D(x, z_1, z_2) = 0 \iff z_1 = z_2; \tag{3.2b}$$

$$\forall x \in \Omega,\ z_1, z_2, z_3 \in \mathrm{SL}(d) \times \mathbb{R}^m : \quad D(x, z_1, z_3) \le D(x, z_1, z_2) + D(x, z_2, z_3). \tag{3.2c}$$

The conditions on W are much more involved. In particular, they include coercivity assumptions and convexity assumptions to obtain lower semicontinuity. To shorten notation, we let $L^{(d,m)} \overset{\text{def}}{=} \mathbb{R}^{d\times d\times d} \times \mathbb{R}^{m\times d}$ and use A as a placeholder for $\nabla z = (\nabla P, \nabla p) \in L^{(d,m)}$. The function $\mathbb{M} : \mathbb{R}^{d\times d} \to \mathbb{R}^{\mu_d}$ with $\mu_d = \sum_{s=1}^{d} \binom{d}{s}^2 = \binom{2d}{d} - 1$ maps a matrix to all its minors (subdeterminants).

$$\begin{aligned}
&\exists\, \mathbb{W} : \Omega \times \mathbb{R}^{\mu_d} \times \mathrm{SL}(d) \times \mathbb{R}^m \times L^{(d,m)} \to \mathbb{R}_\infty :\\
&\quad \text{(i)} \quad \mathbb{W} \text{ is a normal integrand,}\\
&\quad \text{(ii)} \quad \forall (x,F,z,A) : \ W\big(x,F,z,A\big) = \mathbb{W}(x, \mathbb{M}(F), z, A),\\
&\quad \text{(iii)} \quad \forall (x,z) : \mathbb{W}(x,\cdot,z,\cdot) : \mathbb{R}^{\mu_d} \times L^{(d,m)} \to \mathbb{R}_\infty \quad \text{is convex;}
\end{aligned} \tag{3.3a}$$

$$\begin{aligned}
&\exists\, c > 0,\ h \in \mathrm{L}^1(\Omega),\ q_{\boldsymbol{F}}, q_{\boldsymbol{P}}, q_{\boldsymbol{p}}, r > 1\ \forall (x,F,P,p,A) \in \operatorname{dom} W :\\
&\quad W(x,F,P,p,A) \ge h(x) + c\big(|F|^{q_F} + |P|^{q_P} + |p|^{q_p} + |A|^r\big).
\end{aligned} \tag{3.3b}$$

$$\begin{aligned}
&\exists\, c_0^W \in \mathbb{R},\ c_1^W > 0,\ \delta > 0,\ \text{modulus of continuity } \omega :]0,\delta[\to]0,\infty[\\
&\forall (x,F,z,A) \in \operatorname{dom} W\ \forall N \in \mathcal{N}_\delta \overset{\text{def}}{=} \big\{N \in \mathbb{R}^{d\times d} \mid |N - \mathbf{1}| < \delta\big\} :\\
&\quad W(x,\cdot,z,A) \text{ is differentiable on } \mathcal{N}_\delta F \text{ and}\\
&\quad \text{(i)} \quad \big|\partial_F W(x,F,z,A)F^{\mathsf{T}}\big| \le c_1^W\big(W(x,F,z,A) + c_0^W\big)\\
&\quad \text{(ii)} \quad \big|\partial_F W(x,F,z,A)F^{\mathsf{T}} - \partial_F W(x,NF,z,A)(NF)^{\mathsf{T}}\big|\\
&\qquad\qquad \le \omega\big(|N - \mathbf{1}|\big)\big(W(x,F,z,A) + c_0^W\big).
\end{aligned} \tag{3.3c}$$

Thus, (3.3a) implies that the mapping $F \mapsto W(x,F,z,A)$ is polyconvex, cf. Ball (1976). The notion G-quasiconvexity introduced in Buliga (2002) combines ideas of convexity and Lie groups, but so far it is not yet developed far enough to provide existence results.

In (3.3c), a modulus of continuity ω is a nondecreasing function with $\omega(\rho) \to 0$ for $\rho \to 0^+$, and $\mathcal{N}_\delta F$ means $\{NF \mid N \in \mathcal{N}_\delta\} \subset \mathbb{R}^{d\times d}$. The usefulness of constitutive assumption (3.3c)(i) is emphasized in Ball (2002). We call (i) a *multiplicative stress control*, since the Kirchhoff stress tensor $\partial_F W(x,F,z,A)F^{\mathsf{T}}$ is a "multiplicative stress" and it is estimated uniformly in terms of the energy W. The multiplicative nature is seen from the Lie group structure of $\mathrm{GL}^+(d)$ via

$$\partial_F W(x,F,z,A)F^\mathsf{T}{:}H = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\bigl(W\bigl(x,\mathrm{e}^{\varepsilon H}F,z,A\bigr) - W(x,F,z,A)\bigr).$$

Thus, $\partial_F W(x,F,z,A)F^\mathsf{T}$ lies in the Lie algebra $\mathrm{gl}(d)^* = \mathrm{T}^*_{\mathbf{1}}\mathrm{GL}^+(d)$, whereas the first Piola–Kirchhoff tensor $\partial_F W$ lies in $\mathrm{T}^*_F\mathrm{GL}^+(d)$.

Assumption (3.3c)(ii) states that we even have uniform continuity, if we use the energy as a weight. It generalizes the assumptions in Francfort and Mielke (2006), where twice differentiability was needed. The importance of these conditions is their full compatibility with polyconvexity and with the physically desirable conditions $W(x,F,z,A)=\infty$ for $\det F\le 0$ and $W(x,F,z,A)\to\infty$ for $\det F\to 0^+$, see our examples below.

We need one more condition that we give in two versions, one relating to kinematic hardening and the other to isotropic hardening. These alternative conditions should be seen as prototypical situations that have to be adjusted to the concrete plasticity model under investigation. The first condition is simple, but more restrictive concerning the applications in elastoplasticity:

$$D : \varOmega\times\bigl(\mathrm{SL}(d)\times\mathbb{R}^m\bigr)^2 \to [0,\infty[\quad \text{is a Carathéodory function,} \tag{3.4a}$$

$$\begin{aligned}&\exists\, h\in \mathrm{L}^1(\varOmega),\ C>0,\ q_1\in\bigl[1,q_{\boldsymbol{P}}\bigl[,\ q_2\in\bigl[1,q_{\boldsymbol{p}}\bigl[:\\ &\quad \bigl|D(x,P_0,p_0,P_1,p_1)\bigr| \le h(x) + C\bigl(|P_0|^{q_1}+|P_1|^{q_1}+|p_0|^{q_2}+|p_1|^{q_2}\bigr).\end{aligned} \tag{3.4b}$$

The second condition is more complicated, since it involves D and W. We set $\mathbb{D}_\rho(x) \overset{\text{def}}{=} \{(z_0,z_1) \mid D(x,z_1,z_2)<\infty,\ |z_0|,|z_1|\le\rho\}$ and $\mathbb{D}(x) = \operatorname{dom} D(x,\cdot) = \bigcup_{\rho>0}\mathbb{D}_\rho(x)$, and make the following assumptions:

$$r>d; \tag{3.5a}$$

$$D(x,\cdot,\cdot):\mathbb{D}(x)\to[0,\infty[\quad \text{is continuous;} \tag{3.5b}$$

$$\forall M>0\,\exists\rho>0\,\forall x\in\varOmega\,\forall z_0,z_1\in\mathbb{D}_\rho(x){:}\ D(x,z_0,z_1)\le M; \tag{3.5c}$$

$$\left.\begin{aligned}&\text{there exists a } v^*\in\mathbb{R}^m \text{ such that the following holds:}\\ &\text{(i) } \exists\, c_0^W,\ \text{modulus of continuity } \omega\ \ \forall\delta>0,\ (x,F,P,p,A)\in\operatorname{dom}W:\\ &\qquad \bigl|W(x,F,P,p+\delta v^*,A)-W(x,F,P,p,A)\bigr|\\ &\qquad\quad \le \omega(\delta)\bigl(W(x,F,P,p,A)+c_0^W\bigr),\\ &\text{(ii) } \forall\delta,\rho>0\,\exists\varepsilon>0\,\forall x\in\varOmega\ \forall z,z_0,z_1:\\ &\qquad |z-z_0|\le\varepsilon \text{ and } (z_0,z_1)\in\mathbb{D}_\rho(x) \quad\Longrightarrow\quad \bigl(z,z_1+(0,\delta v^*)\bigr)\in\mathbb{D}(x).\end{aligned}\right\} \tag{3.5d}$$

We now formulate our existence result, which will be proved in Sect. 5.

Theorem 3.1 *Let the spaces $\mathcal{Q}=\mathcal{Y}\times\mathcal{Z}\subset \boldsymbol{Y}\times\boldsymbol{Z}=\boldsymbol{Q}$ and the functionals $\mathcal{E}$ and $\mathcal{D}$ be defined as above. The integrability powers $q_{\boldsymbol{Y}}, q_{\boldsymbol{F}}, q_{\boldsymbol{P}}, q_{\boldsymbol{p}}$, and r satisfy*

$$\frac{1}{q_{\boldsymbol{F}}}+\frac{1}{q_{\boldsymbol{P}}}=\frac{1}{q_{\boldsymbol{Y}}}<\frac{1}{d},\quad q_{\boldsymbol{p}}>1,\quad \textit{and}\quad r>1. \tag{3.6}$$

Moreover, the conditions (3.1), (3.2), (3.3) *hold and, additionally, either* (3.4) *or* (3.5).

Let $\boldsymbol{q}_0 = (y_0, z_0) \in \mathcal{Q}$ *be a stable initial condition, i.e.*,

$$\mathcal{E}(0, \boldsymbol{q}_0) < \infty \quad \textit{and} \quad \forall \widehat{\boldsymbol{q}} \in \mathcal{Q}\colon \mathcal{E}(0, \boldsymbol{q}_0) \le \mathcal{E}(0, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}_0, \widehat{\boldsymbol{q}}).$$

Then there exists an energetic solution $\boldsymbol{q} : [0, T] \to \mathcal{Q}$ *for* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ *with* $\boldsymbol{q}(0) = \boldsymbol{q}_0$ *such that* $\boldsymbol{q} : [0, T] \to \boldsymbol{Q}$ *is measurable.*

The condition $r > d$ in (3.5a) is of technical nature. Using the ideas of Thomas (2008), which are developed for a damage model, it is expected that the result can be extended to all $r > 1$. The assumptions (3.4) and (3.5) should be seen as prototypes adapted to the two examples below. The only thing we need is that the generated functionals $\mathcal{E}$ and $\mathcal{D}$ satisfy the abstract property (4.4). The following two examples show that the theory is applicable to rate-independent elastoplasticity at finite strain. In particular, all the geometric nonlinearities arising from the multiplicative decomposition can be handled in this framework.

Example 3.2 (Kinematic hardening) We do not need the variable $p \in \mathbb{R}^m$ here, i.e., we set $m = 0$ in the above:

$$W(x, F, P, A) = W_{\text{el}}\big(F P^{-1}\big) + c_1 |P|^{q_P} + c_2 |\nabla P|^r,$$

where the elastic part is chosen to be polyconvex in the form

$$W_{\text{el}}(F) = \begin{cases} c_1 |F|^{q_F} + \frac{c_2}{(\det F)^\gamma} & \text{for } \det F_{\text{el}} > 0, \\ \infty & \text{otherwise,} \end{cases} \qquad \text{with } c_1, c_2, \gamma > 0.$$

It is easy to see that (3.3a) and (3.3b) are satisfied. The Kirchhoff tensor $K = \partial_F W F^\top$ in (3.3c) only depends on F and takes the simple form

$$K(F) = c_1 q_{\boldsymbol{F}} |F|^{q_F - 2} F F^\top - \frac{c_2 \gamma}{(\det F)^\gamma} \mathbf{1}.$$

Hence, (3.3c)(i) immediately holds with $c_0^W = 0$ and $c_1^W = \max\{q_{\boldsymbol{F}}, \gamma\sqrt{d}\}$. Moreover, condition (ii) also holds, since K can be differentiated once again giving $|\partial_F K(F)[HF]| \le C W(F) |H|$.

For the dissipation density D, we choose any left-invariant distance on the Lie group $\mathrm{SL}(d)$, viz.,

$$D(x, P_0, P_1) = d_{\mathrm{SL}}\big(P_1 P_0^{-1}\big) \quad \text{with } d_{\mathrm{SL}} : \mathrm{SL}(d) \to [0, \infty[\,,$$

where d_{SL} is generated by a norm R on the Lie algebra $\mathrm{sl}(d) \overset{\text{def}}{=} \mathrm{T}_{\mathbf{1}}\mathrm{SL}(d)$ via

$$d_{\mathrm{SL}}(P_1) \\ = \inf\left\{ \int_0^1 R\big(\dot{P}(s) P(s)^{-1}\big)\,\mathrm{d}s \,\middle|\, P \in \mathrm{C}^1\big([0,1], \mathrm{SL}(d)\big),\ P(0) = \mathbf{1},\ P(1) = P_1 \right\}.$$

Clearly this D satisfies the plastic indifference condition (2.7). According to Mielke (2002), the mapping d_{SL} is continuous, is strictly positive for $P \neq \mathbf{1}$, satisfies the triangle inequality $d_{\mathrm{SL}}(P_1 P_0) \leq d_{\mathrm{SL}}(P_0) + d_{\mathrm{SL}}(P_1)$, and allows for the bounds

$$\delta|\Sigma| \leq R(\Sigma) \leq d_{\mathrm{SL}}\big(Q\,\mathrm{e}^{\Sigma}\big) \leq C + R(\Sigma) \quad \text{for } \Sigma = \Sigma^{\mathsf{T}} \text{ and } Q \in \mathrm{SO}(d), \tag{3.7}$$

with $\delta, C > 0$; see Mielke (2002), Hackl et al. (2003). Thus, conditions (3.2) and (3.4) are fulfilled.

This shows that Theorem 3.1 is applicable for the case of kinematic hardening.

Example 3.3 (Isotropic hardening) We now use the scalar parameter $p \in \mathbb{R}$ to measure the amount of hardening, i.e., we have $m = 1$ in the abstract setting of Sect. 2.3. For the stored-energy density, we take the form

$$W(x, F, P, p, A) = W_{\mathrm{el}}\big(F P^{-1}\big) + c_1 \exp(c_2 p) + c_5|\nabla P|^r + c_6|\nabla p|^r + \chi_{\mathcal{P}}\big((P, p)\big),$$

where W_{el} is as in Example 3.2. Here, $\chi_{\mathcal{P}}$ is the characteristic function with $\chi_{\mathcal{P}}((P, p)) = 0$ for $(P, p) \in \mathcal{P}$ and ∞ otherwise. Before we specify the set $\mathcal{P}$, we define, using d_{SL} from above, the dissipation distance

$$D(x, P_0, p_1, P_1, p_1) = \begin{cases} d_{\mathrm{SL}}(P_1 P_0^{-1}) & \text{for } p_1 \geq p_0 + d_{\mathrm{SL}}(P_1 P_0^{-1}), \\ \infty & \text{otherwise,} \end{cases}$$

which again satisfies (2.7). We let $\mathcal{P} \overset{\text{def}}{=} \{(P, p) \mid D(\mathbf{1}, 0, P, p) < \infty\}$ and obtain $\mathcal{P} = \{(P, p) \mid p \geq d_{\mathrm{SL}}(P)\}$. Using (3.7) and $|Q\,\mathrm{e}^{\Sigma}| = |\mathrm{e}^{\Sigma}| \leq \mathrm{e}^{|\Sigma|}$, we find for $P = Q\,\mathrm{e}^{\Sigma}$, the coercivity estimate

$$c_1 \mathrm{e}^{c_2 p} \;\geq\; \frac{1}{2}\big(c_1 \mathrm{e}^{c_2 p} + c_1 \mathrm{e}^{c_2 \delta |\Sigma|}\big) \;\geq\; c_7 |p|^{q_p} + c_8 |P|^{q_{\boldsymbol{P}}}$$

with arbitrary $q_p > 1$ and $q_{\boldsymbol{P}} = c_2 \delta$. Thus, conditions (3.2) and (3.3) hold.

We now show condition (3.5). Part (a) can be achieved by taking $r > d$, and (3.5b) and (3.5c) hold automatically. For (3.5d), the vector $v^* = 1$ is the obvious choice. In fact, in (i) the estimate reduces to

$$\big|\, c_1 \exp\big(c_2(p + \delta)\big) - c_1 \exp(c_2 p) \,\big| = \omega(\delta) c_1 \exp(c_2 p) \quad \text{with } \omega(\delta) = \big|\mathrm{e}^{c_2 \delta} - 1\big|.$$

Condition (ii) is also valid, since d_{SL} is continuous.

Thus, we have shown that elastoplasticity with isotropic hardening and gradient regularization is covered by Theorem 3.1, and hence the existence of energetic solutions is guaranteed.

Remark 3.4 Time-dependent loading can also be added to $\mathcal{E}$ in the form indicated in (2.1). However, we again have to substitute $\varphi(x) = g_{\mathrm{Dir}}(t, y(x))$ and assume f_{ext} and h_{ext} such that $\ell \in \mathrm{C}^1([0, T], (\mathrm{W}^{1,q_Y}(\Omega; \mathbb{R}^d))^*)$ holds. Then the above theorem remains true without any change; see Francfort and Mielke (2006).

4 Abstract Existence Result

Our existence theory for elastoplasticity is based on the abstract theory of energetic solutions for rate-independent processes on topological spaces. This theory is developed in Mainik and Mielke (2005), Francfort and Mielke (2006); see also the survey Mielke (2005) and the further developments in Mielke et al. (2008). We use a slightly adapted version as we are in a concrete Banach space setting. Thus, we will use notions like weak and strong convergence, coercivity, and boundedness, which are not available in the general topological setting.

We consider two reflexive and separable Banach spaces $\boldsymbol{Y}$ and $\boldsymbol{Z}$ and weakly closed subsets $\mathcal{Y}$ and $\mathcal{Z}$, respectively. The state space for the full system is then given by $\mathcal{Q} = \mathcal{Y} \times \mathcal{Z} \subset \boldsymbol{Q} \overset{\text{def}}{=} \boldsymbol{Y} \times \boldsymbol{Z}$, and the states are denoted by $\boldsymbol{q} = (y, z)$. The evolution is described in terms of the stored-energy functional $\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}_\infty$ and the dissipation distance $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty]$. The triple $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ is called a *rate-independent energetic system.*

For the stored-energy functional $\mathcal{E}$ impose two general conditions:

Compactness of energy sublevels:

$$\forall t \in [0,T]\, \forall E \in \mathbb{R} : L_{t,E} := \big\{\boldsymbol{q} \in \mathcal{Q} \,\big|\, \mathcal{E}(t,\boldsymbol{q}) \le E\big\} \quad \text{is bounded and weakly closed in } \boldsymbol{Q}. \tag{4.E1}$$

Uniform control of the power $\partial_t \mathcal{E}$:

$$\begin{gathered}\exists\, c_0^E \in \mathbb{R}\, \exists\, c_1^E > 0\, \forall (t_*, \boldsymbol{q}) \in \operatorname{dom} \mathcal{E} : \\ \mathcal{E}(\cdot, \boldsymbol{q}) \in \mathrm{C}^1\big([0,T]\big) \quad \text{and} \quad \big|\partial_t \mathcal{E}(t,\boldsymbol{q})\big| \le c_1^E \big(c_0^E + \mathcal{E}(t,\boldsymbol{q})\big) \quad \text{for all } t.\end{gathered} \tag{4.E2}$$

Using a simple Gronwall argument, we see that (4.E2) implies the bound

$$(t,\boldsymbol{q}) \in \operatorname{dom}\mathcal{E} \quad \Longrightarrow \quad \forall t_1, t_2 \in [0,T] : \ \mathcal{E}(t_1,\boldsymbol{q}) + c_0^E \le \mathrm{e}^{c_1^E |t_2 - t_1|}\big(\mathcal{E}(t_2,\boldsymbol{q}) + c_0^E\big). \tag{4.1}$$

For the dissipation distance $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty]$, we impose two general conditions:

Extended quasi-distance:

$$\begin{aligned}&\text{(i) } \forall z_1, z_2 \in \mathcal{Z} : \ \mathcal{D}(z_1, z_2) = 0 \Longleftrightarrow z_1 = z_2, \\ &\text{(ii) } \forall z_1, z_2, z_3 \in \mathcal{Z} : \ \mathcal{D}(z_1, z_3) \le \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3).\end{aligned} \tag{4.D1}$$

Weak lower semi-continuity:

$$z_k \rightharpoonup z, \ \widehat{z}_k \rightharpoonup \widehat{z} \quad \Longrightarrow \quad \mathcal{D}(z, \widehat{z}) \le \liminf_{k\to\infty} \mathcal{D}(z_k, \widehat{z}_k). \tag{4.D2}$$

Note that (4.D1) and (4.D2) imply the following condition (4.2), which is used in Francfort and Mielke (2006).

Lemma 4.1 *If* (4.D1) *and* (4.D2) *hold, then we also have the following*:

$$\textit{if } (z_k)_{k\in\mathbb{N}} \textit{ is bounded and if } \min\big\{\mathcal{D}(z_k, z), \mathcal{D}(z, z_k)\big\} \to 0, \ \textit{then } z_k \rightharpoonup z. \tag{4.2}$$

Proof To prove condition (4.2), we take any bounded sequence $(z_k)_{k\in\mathbb{N}}$ and z in $\mathcal{Z}$. By choosing a subsequence, we find $\widehat{z}\in\mathcal{Z}$ with $z_{k_n}\rightharpoonup\widehat{z}$ and either (i) $\mathcal{D}(z_{k_n},z)\to 0$ or (ii) $\mathcal{D}(z,z_{k_n})\to 0$. Let us consider the case (i); the case (ii) is analogous. Using (4.D2), we find

$$0\le\mathcal{D}(\widehat{z},z)\le\liminf_{n\to\infty}\mathcal{D}(z_{k_n},z)=0.$$

Hence, we have $\mathcal{D}(\widehat{z},z)=0$, and the positivity (4.D1)(i) gives $z=\widehat{z}$ and $z_{k_n}\rightharpoonup z=\widehat{z}$. As the limit z is unique, we conclude that even the whole sequence converges (without taking a subsequence). □

To formulate the existence result, we need to impose additional conditions which provide a suitable compatibility between the two functionals $\mathcal{E}$ and $\mathcal{D}$. For this, we define the *set of stable states at time* t via

$$\begin{aligned}\mathcal{S}(t)&\overset{\text{def}}{=}\big\{\boldsymbol{q}\in\mathcal{Q}\,\big|\,\mathcal{E}(t,\boldsymbol{q})<\infty,\ \forall\widehat{\boldsymbol{q}}\in\mathcal{Q}\colon\mathcal{E}(t,\boldsymbol{q})\le\mathcal{E}(t,\widehat{\boldsymbol{q}})+\mathcal{D}(\boldsymbol{q},\widehat{\boldsymbol{q}})\big\},\\ \mathcal{S}_{[0,T]}&\overset{\text{def}}{=}\bigcup_{t\in[0,T]}\{t\}\times\mathcal{S}(t)\subset[0,T]\times\mathcal{Q}.\end{aligned}$$

Moreover, we define the notion of a *stable sequence* $(t_k,\boldsymbol{q}_k)_{k\in\mathbb{N}}$ via

$$\sup_{k\in\mathbb{N}}\mathcal{E}(t_k,\boldsymbol{q}_k)<\infty\quad\text{and}\quad\boldsymbol{q}_k\in\mathcal{S}(t_k)\quad\text{for all }k\in\mathbb{N}.\tag{4.3}$$

A function $q:[0,T]\to\mathcal{Q}$ is called an *energetic solution* of $(\mathcal{Q},\mathcal{E},\mathcal{D})$, if $t\mapsto\partial_t\mathcal{E}(t,\boldsymbol{q}(t))$ is integrable and if for all $t\in[0,T]$ we have global stability (S) and energy balance (E):

$$\begin{aligned}&\text{(S)}\quad\boldsymbol{q}(t)\in\mathcal{S}(t);\\ &\text{(E)}\quad\mathcal{E}\big(t,\boldsymbol{q}(t)\big)+\mathrm{Diss}_{\mathcal{D}}\big(\boldsymbol{q},[0,t]\big)=\mathcal{E}\big(0,\boldsymbol{q}(0)\big)+\int_0^t\partial_s\mathcal{E}\big(s,\boldsymbol{q}(s)\big)\,\mathrm{d}s.\end{aligned}$$

For the proof of the following existence result, we refer to Francfort and Mielke (2006), Mielke et al. (2008) and remark that it is based on abstract versions of ideas developed in Dal Maso et al. (2005). The measurability result was first obtained in Mainik (2005). The proof is based on time-incremental minimization as indicated in Sect. 1, but to keep this paper short, we will not go into details here and refer to Mielke (2005, 2008) for surveys.

Theorem 4.2 *Let $\mathcal{E}$ and $\mathcal{D}$ satisfy conditions* (4.E) *and* (4.D). *Moreover, let the following compatibility condition hold*:

For all stable seq. $(t_j,\boldsymbol{q}_j)_{j\in\mathbb{N}}$ *with* $(t_j,\boldsymbol{q}_j)\rightharpoonup(t_*,\boldsymbol{q}_*)$:

$$\partial_t\mathcal{E}(t_*,\boldsymbol{q}_j)\to\partial_t\mathcal{E}(t_*,\boldsymbol{q}_*),\tag{4.C1}$$

$$\boldsymbol{q}_*\in\mathcal{S}(t_*).\tag{4.C2}$$

Then for each $\boldsymbol{q}_0\in\mathcal{S}(0)$, there exists a solution $\boldsymbol{q}:[0,T]\to\mathcal{Q}$ of the rate-independent energetic system $(\mathcal{Q},\mathcal{E},\mathcal{D})$ satisfying $\boldsymbol{q}(0)=\boldsymbol{q}_0$. Moreover, the solution can be chosen such that $\boldsymbol{q}:[0,T]\to\boldsymbol{Q}$ is measurable.

The following abstract results are sufficient to establish the compatibility conditions (4.C1) and (4.C2) for our application to elastoplasticity. The first result is implicitly contained in Mainik and Mielke (2005), but for the readers we provide a direct short proof. See Mielke et al. (2008) for a discussion of more general *joint recovery sequence conditions*.

Proposition 4.3 *Let $\mathcal{E}$ and $\mathcal{D}$ satisfy* (4.*E*) *and* (4.*D*), *respectively. Assume that*

$$\begin{aligned}&\text{for all stable seq. } (t_j,\boldsymbol{q}_j)_{j\in\mathbb{N}} \text{ with } (t_j,\boldsymbol{q}_j)\rightharpoonup(t_*,\boldsymbol{q}_*)\ \forall\widehat{\boldsymbol{q}}\in\mathcal{Q}\\ &\text{there exists a joint recovery sequence } (\widehat{\boldsymbol{q}}_j)_{j\in\mathbb{N}}:\\ &\limsup_{j\to\infty}\big(\mathcal{E}(t_j,\widehat{\boldsymbol{q}}_j)+\mathcal{D}(\boldsymbol{q}_j,\widehat{\boldsymbol{q}}_j)\big)\ \le\ \mathcal{E}(t_*,\widehat{\boldsymbol{q}})+\mathcal{D}(\boldsymbol{q}_*,\widehat{\boldsymbol{q}}),\end{aligned}\tag{4.4}$$

then $\mathcal{E}$ is weakly continuous along stable sequences and (4.C2) *holds.*

Proof Let $(t_j,\boldsymbol{q}_j)_{j\in\mathbb{N}}$ be any stable sequence with $(t_j,\boldsymbol{q}_j)\rightharpoonup(t_*,\boldsymbol{q}_*)$. First take $\widehat{\boldsymbol{q}}=\boldsymbol{q}_*$ in (4.4) and obtain a sequence $(\widehat{\boldsymbol{q}}_j)_j$. Using stability of $\boldsymbol{q}_j$, we find

$$\limsup_{j\to\infty}\mathcal{E}(t_j,\boldsymbol{q}_j)\le\limsup_{j\to\infty}\mathcal{E}(t_j,\widehat{\boldsymbol{q}}_j)+\mathcal{D}(\boldsymbol{q}_j,\widehat{\boldsymbol{q}}_j)\le\mathcal{E}(t_*,\widehat{\boldsymbol{q}})+\mathcal{D}(\boldsymbol{q}_*,\widehat{\boldsymbol{q}})=\mathcal{E}(t_*,\boldsymbol{q}_*).$$

Using $\sup_j\mathcal{E}(t_j,\boldsymbol{q}_j)\le C_0$ and (4.1), we find

$$\begin{aligned}\big|\mathcal{E}(t_j,\boldsymbol{q}_j)-\mathcal{E}(t_*,\boldsymbol{q}_j)\big|&\le\big(\mathrm{e}^{c_1^E|t_j-t_j|}-1\big)\mathrm{e}^{c_1^E|t_j-t_j|}\big(\mathcal{E}(t_j,\boldsymbol{q}_j)+c_0^E\big)\\ &\le\big(\mathrm{e}^{c_1^E|t_j-t_j|}-1\big)\mathrm{e}^{c_1^E T}\big(C_0+c_0^E\big).\end{aligned}$$

Since $\mathcal{E}$ is lower semicontinuous, we find

$$\liminf_{j\to\infty}\mathcal{E}(t_j,\boldsymbol{q}_j)=\lim_{j\to\infty}\ \mathcal{E}(t_j,\boldsymbol{q}_j)-\mathcal{E}(t_*,\boldsymbol{q}_j)+\liminf_{j\to\infty}\mathcal{E}(t_*,\boldsymbol{q}_j)\ge0+\mathcal{E}(t_*,\boldsymbol{q}_*).$$

Together with the lim sup-estimate from above, we have $\mathcal{E}(t_j,\boldsymbol{q}_j)\to\mathcal{E}(t_*,\boldsymbol{q}_*)$, as desired.

To prove $\boldsymbol{q}_*\in\mathcal{S}(t_*)$, we now choose $\widehat{\boldsymbol{q}}$ in (4.4) arbitrary and take $(\widehat{\boldsymbol{q}}_j)_j$ as stated there. Using stability of $\boldsymbol{q}_j$, we obtain

$$\mathcal{E}(t_*,\boldsymbol{q}_*)=\lim_{j\to\infty}\ \mathcal{E}(t_j,\boldsymbol{q}_j)\le\liminf_{j\to\infty}\mathcal{E}(t_j,\widehat{\boldsymbol{q}}_j)+\mathcal{D}(\boldsymbol{q}_j,\widehat{\boldsymbol{q}}_j)\le\mathcal{E}(t_*,\widehat{\boldsymbol{q}})+\mathcal{D}(\boldsymbol{q}_*,\widehat{\boldsymbol{q}}),$$

which is the desired stability. ☐

The following Proposition 4.4, which is proved in Francfort and Mielke (2006), shows that condition (4.C1) on the continuity of the power can be deduced from Proposition 4.3. Thus, to apply the abstract existence result, it suffices to satisfy (4.E), (4.D), (4.4), and the uniform continuity of the power as stated in (4.5).

Proposition 4.4 *Let $\mathcal{E}$ satisfy conditions* (4.E) *and*

$$\begin{aligned}&\forall\varepsilon>0\ \forall E\in\mathbb{R}\ \exists\delta>0\ \forall t_1,t_2,\boldsymbol{q}:\\ &\mathcal{E}(t_1,\boldsymbol{q})\le E \text{ and } |t_1-t_2|<\delta\ \Longrightarrow\ \big|\partial_t\mathcal{E}(t_1,\boldsymbol{q})-\partial_t\mathcal{E}(t_2,\boldsymbol{q})\big|<\varepsilon.\end{aligned}\tag{4.5}$$

Then we have the following implication:

$$\left.\begin{array}{c} t_j \to t_*,\ q_j \rightharpoonup q_*, \\ \mathcal{E}(t_j,\boldsymbol{q}_j) \to \mathcal{E}(t_*,\boldsymbol{q}_*) < \infty \end{array}\right\} \quad \Longrightarrow \quad \partial_t\mathcal{E}(t_*,\boldsymbol{q}_j) \to \partial_t\mathcal{E}(t_*,\boldsymbol{q}_*).$$

5 Coercivity and Lower Semicontinuity

The aim of this section is to show that the assumptions in Sect. 3 for the elastoplastic problem are sufficient to establish the abstract assumption (4.E) for the stored-energy functional $\mathcal{E}$, (4.D) for the dissipation distance $\mathcal{D}$, and the compatibility conditions (4.C). Having done this, the existence Theorem 3.1 for the elastoplastic problem is a direct consequence of the abstract existence result of Sect. 4.

5.1 Stored Energy Potential

To establish the coercivity of $\mathcal{E}$, we note that we always use the matrix norm $|F| \overset{\text{def}}{=} (F{:}F)^{1/2}$, where the matrix scalar product is defined as $A{:}B \overset{\text{def}}{=} \operatorname{tr}(A^\mathsf{T}B) = \sum_{i,j=1}^d A_{ij}B_{ij}$. In particular, we have $|AB| \le |A|\,|B|$, which implies

$$\left|FP^{-1}\right| \ge |F|/|P| \ge \varrho\delta^{(\varrho-1)/\varrho}|F|^{1/\varrho} - (\varrho-1)\delta|P|^{1/(\varrho-1)} \quad \text{for } \det P \ne 0,$$

where $\delta > 0$ and $\varrho > 1$ are arbitrary. Using $\frac{1}{q_{\boldsymbol{Y}}} = \frac{1}{q_{\boldsymbol{F}}} + \frac{1}{q_{\boldsymbol{P}}}$ Hölder's inequality applied to $P \in \mathrm{L}^{q_{\boldsymbol{P}}}(\Omega;\mathbb{R}^{d\times d})$ and $AP^{-1} \in \mathrm{L}^{q_{\boldsymbol{F}}}(\Omega;\mathbb{R}^{d\times d})$ gives

$$\begin{aligned} \left\|AP^{-1}\right\|^{q_{\boldsymbol{F}}}_{\mathrm{L}^{q_{\boldsymbol{F}}}(\Omega;\mathbb{R}^{d\times d})} &\ge \|A\|^{q_{\boldsymbol{F}}}_{\mathrm{L}^{q_{\boldsymbol{Y}}}(\Omega;\mathbb{R}^{d\times d})}/\|P\|^{q_{\boldsymbol{F}}}_{\mathrm{L}^{q_{\boldsymbol{P}}}(\Omega;\mathbb{R}^{d\times d})} \\ &\ge \varrho\delta^{(\varrho-1)/\varrho}\|A\|^{q_{\boldsymbol{Y}}}_{\mathrm{L}^{q_{\boldsymbol{Y}}}(\Omega;\mathbb{R}^{d\times d})} - (\varrho-1)\delta\|P\|^{q_{\boldsymbol{P}}}_{\mathrm{L}^{q_{\boldsymbol{P}}}(\Omega;\mathbb{R}^{d\times d})}, \end{aligned} \tag{5.1}$$

where now $\varrho = q_{\boldsymbol{F}}/q_{\boldsymbol{Y}}$. Integrating the coercivity assumption (3.3b) over Ω and exploiting the bound on $\nabla g_{\mathrm{Dir}}^{-1}$ in (3.1) and (5.1) with $\delta > 0$ sufficiently small, we obtain

$$\begin{aligned} \mathcal{E}(t,y,P,p) &\ge \int_\Omega h\,\mathrm{d}x + c\Big(\left\|\nabla g_{\mathrm{Dir}}\nabla y P^{-1}\right\|^{q_{\boldsymbol{F}}}_{\mathrm{L}^{q_{\boldsymbol{F}}}} + \|P\|^{q_{\boldsymbol{P}}}_{\mathrm{L}^{q_{\boldsymbol{P}}}} + \|p\|^{q_{\boldsymbol{p}}}_{\mathrm{L}^{q_{\boldsymbol{p}}}} \\ &\qquad + \left\|(\nabla P,\nabla p)\right\|^{q_r}_{\mathrm{L}^r}\Big) \\ &\ge \widetilde{c}\big(\|\nabla g_{\mathrm{Dir}}\nabla y\|^{q_{\boldsymbol{Y}}}_{\mathrm{L}^{q_{\boldsymbol{Y}}}} + \|(P,p)\|^{q_Z}_Z\big) - C \\ &\ge \widehat{c}\Big(\|\nabla y\|^{q_{\boldsymbol{Y}}}_{\mathrm{L}^{q_{\boldsymbol{Y}}}}/\left\|\nabla g_{\mathrm{Dir}}^{-1}\right\|^{q_{\boldsymbol{Y}}}_{\mathrm{L}^\infty} + \left\|(P,p)\right\|^{q_Z}_Z\Big) - C, \end{aligned} \tag{5.2}$$

where $q_Z = \min\{q_{\boldsymbol{P}}, q_{\boldsymbol{p}}, r\}$. This shows that $\mathcal{E}(t,\boldsymbol{q}_k) \to \infty$ whenever $\|\boldsymbol{q}_k\|_{\boldsymbol{Q}} \to \infty$. Hence, all sublevels of $\mathcal{E}(t,\cdot)$ are bounded uniformly in $t \in [0,T]$.

Next, we establish the lower semicontinuity of $\mathcal{E}(t,\cdot)$. For this, we use the following result that relies on the weak continuity of the minors of gradients, cf. Reshetnyak (1967), Ball (1976).

Proposition 5.1 (Convergence of minors) *The three sequences* $(G_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ *and* $(P_k)_{k\in\mathbb{N}}$ *satisfy*

$$\begin{gathered} G_k \to G \quad \text{in } \mathrm{L}^\infty\big(\varOmega;\mathbb{R}^{d\times d}\big), \qquad y_k \rightharpoonup y \quad \text{in } \mathrm{W}^{1,q_Y}\big(\varOmega;\mathbb{R}^d\big), \\ P_k \to P \quad \text{in } \mathrm{L}^{\widehat{q}}\big(\varOmega;\mathbb{R}^{d\times d}\big) \qquad \text{and} \quad \det P_k \equiv 1. \end{gathered}$$

If $q_Y > d$, $\widehat{q} \geq 1$, *and* $\frac{1}{q_Y} + \frac{d-1}{\widehat{q}} \leq 1$, *then all minors of the product* $G_k \nabla y_k P_k^{-1}$ *converge weakly, i.e.*,

$$\mathbb{M}\big(G_k \nabla y_k P_k^{-1}\big) \rightharpoonup \mathbb{M}\big(G \nabla y P^{-1}\big) \quad \text{in } \mathrm{L}^1\big(\varOmega;\mathbb{R}^{\mu_d}\big).$$

Proof For a matrix $F\in\mathbb{R}^{d\times d}$, we introduce the matrix $\mathbb{M}_s(F)\in\mathbb{R}^{\binom{d}{s}\times\binom{d}{s}}$ consisting of all minors of order s. Then the weak continuity of minors of gradients gives

$$\mathbb{M}_s(\nabla y_k) \rightharpoonup \mathbb{M}_s(\nabla y) \quad \text{in } \mathrm{L}^{q_Y/s}\big(\varOmega;\mathbb{R}^{\binom{d}{s}\times\binom{d}{s}}\big). \tag{5.3}$$

Since $\mathbb{M}_s$ is a homogeneous polynomial of degree s, strong convergence $H_k\to H$ in $\mathrm{L}^{q_H}(\varOmega;\mathbb{R}^{d\times d})$ and $1\leq s\leq q_H$ imply strong convergence of the minors, i.e., $\mathbb{M}_s(H_k)\to\mathbb{M}_s(H)$ in $\mathrm{L}^{q_H/s}(\varOmega)$.

We prove the statement for $d\in\{1,2,3\}$ first and then the general case. For $d=1$, the result is trivial as the product of a weakly convergent sequence times several strongly convergent sequences is again weakly convergent.

For $d=2$, the result is again trivial for $s=1$ as $\mathbb{M}_1(F)=F$. Since $\mathbb{M}_2(F)=\det F$ and $\det P_k\equiv 1$, we have $\mathbb{M}_2(G_k\nabla y_k P_k^{-1})=\det G_k\,\det\nabla y_k$ and the result follows again.

For $d=3$, we have $\mathbb{M}_1(F)=F$ and $\mathbb{M}_3(F)=\det F$, and we may identify $\mathbb{M}_2(F)$ with the cofactor matrix $\operatorname{cof} F$, which satisfies $\operatorname{cof} F=(\det F)\,F^{-\mathsf{T}}$ for invertible F. Using $\det P\equiv 1$, we have $P^{-1}=(\operatorname{cof} P)^{\mathsf{T}}$. Thus, we have

$$\begin{gathered} \mathbb{M}_1\big(G_k\nabla y_k P_k^{-1}\big)=G_k\nabla y_k\,(\operatorname{cof} P_k)^{\mathsf{T}}, \qquad \operatorname{cof}\big(G_k\nabla y_k P_k^{-1}\big)=\operatorname{cof} G_k\,\operatorname{cof}\nabla y_k\,P_k^{\mathsf{T}}, \\ \det\big(G_k\nabla y_k P_k^{-1}\big)=\det G_k\,\det\nabla y_k. \end{gathered}$$

We again see that in all cases $s\in\{1,2,3\}$ we have the desired weak convergence in $\mathrm{L}^{\sigma_s}>1$ where $\frac{1}{\sigma_s}=\frac{s}{q_Y}+\frac{d-s}{\widehat{q}}$.

For $d\geq 4$, one needs the general definitions of the minor matrix $\mathbb{M}_s$ and the cofactor matrix $\mathbb{K}_s$; see Šilhavý (2002, Appendix A) or Mielke and Müller (2006, Lemma 2.4). In particular, we have

$$\mathbb{M}_s(AB)=\mathbb{M}_s(A)\mathbb{M}_s(B) \quad \text{and} \quad \mathbb{K}_s(P)=\det P\,\mathbb{M}_{d-s}\big(P^{-1}\big)^{\mathsf{T}}, \tag{5.4}$$

if $\det P\neq 0$. Again, $\mathbb{K}_s$ is a homogeneous polynomial of degree s. Thus, we obtain the desired convergence as above from

$$\mathbb{M}_s\big(G_k\nabla y_k P_k^{-1}\big)=\mathbb{M}_s(G_k)\mathbb{M}_s(\nabla y_k)\mathbb{K}_{d-s}(P)^{\mathsf{T}}$$

and the weak and strong convergence properties. □

Theorem 5.2 (Weak lower semicontinuity) *The assumptions* (3.1), (3.3a), (3.3b), *and* (3.6) *hold. Then,* $\mathcal{E}(t,\cdot):\mathcal{Q}\to\mathbb{R}_\infty$ *is weakly lower semicontinuous with respect to the topology of* $\boldsymbol{Q}=\boldsymbol{Y}\times\boldsymbol{Z}$.

Proof We take a sequence $\boldsymbol{q}_k=(y_k,P_k,p_k)\rightharpoonup(y,P,p)$ in $\boldsymbol{Q}$. The weak convergence of P_k in $\mathrm{L}^{q_{\boldsymbol{P}}}\cap\mathrm{W}^{1,r}$ implies by the compact embedding of $\mathrm{W}^{1,r}$ into L^r strong convergence in L^r. As weak convergence in $\mathrm{L}^{q_{\boldsymbol{P}}}$ implies boundedness, the classical interpolation yields strong convergence in L^q for all $q\in[1,q_{\boldsymbol{P}}[$. Similarly, p_k strongly convergences in L^σ for all $\sigma\in[1,q_{\boldsymbol{p}}[$.

Using $y_k\rightharpoonup y$ in $\mathrm{W}^{1,q_Y}(\Omega)$ and $q_{\boldsymbol{Y}}>d$, we have $y_k\to y$ in $\mathrm{C}^0(\overline{\Omega};\mathbb{R}^d)$. Hence, for $G_k\overset{\mathrm{def}}{=}\nabla g(t,y_k(x))$, assumption (3.1) gives $G_k\to G$ in $\mathrm{C}^0(\overline{\Omega};\mathbb{R}^d)$. Since from $q_{\boldsymbol{Y}}>d$ and (3.6), we have $q_{\boldsymbol{P}}>d$; we can apply Proposition 5.1 by choosing $\widehat{q}\in[d,q_{\boldsymbol{P}}[$.

Now, we use that $(P_k,p_k)\to(P,p)$ strongly in $\mathrm{L}^d\times\mathrm{L}^r$ and that

$$\begin{aligned}&\big(\mathbb{M}\big(\nabla g_{\mathrm{Dir}}\big(t,y_k(\cdot)\big)\nabla y_k\,P_k^{-1}\big),\nabla P_k,\nabla p_k\big)\\&\quad\rightharpoonup\big(\mathbb{M}\big(\nabla g_{\mathrm{Dir}}(t,y(\cdot))\nabla y\,P^{-1}\big),\nabla P_k,\nabla p_k\big).\end{aligned}$$

Property (3.3a) states that the integrand has the form $W:(x,F,z,A)\mapsto\mathbb{W}(x,\mathbb{M}(F),z,A)$ where $\mathbb{W}$ is a normal integrand that is convex in $(\mathbb{M},A)$. Hence, together with the lower bound (3.3b), the classical lower semicontinuity results (cf., e.g. Eisen 1979 or Struwe 1990, Chap. I.1) give the desired result.

Since the last mentioned references only treat the case that $\mathbb{W}$ is a Carathéodory function, we may obtain our more general result for normal integrands as follows. For $\varepsilon>0$, define the Yosida–Moreau regularization

$$\mathbb{W}^\varepsilon(x,M,z,A)=\inf\left\{\mathbb{W}(x,U)+\frac{1}{\varepsilon}\big|(M,z,A)-U\big|^2\;\middle|\;U\in\mathbb{R}^{\mu_d}\times\mathbb{R}^m\times L^{(m,d)}\right\}.$$

Now $\mathbb{W}^\varepsilon$ is a Carathéodory function, and it approximates $\mathbb{W}$ pointwise, monotonically from below. Moreover, the convexity with respect to (M,A) is maintained. Thus, we may define functionals $\mathcal{I}_\varepsilon:\mathcal{Q}\to\mathbb{R}_\infty$ by replacing $\mathbb{W}$ in $\mathcal{E}(t,\cdot)$ by $\mathbb{W}^\varepsilon$. Each $\mathcal{I}_\varepsilon$ is weakly lower semicontinuous and $\mathcal{I}_\varepsilon(\boldsymbol{q})$ is nondecreasing in ε. Using the monotone convergence lemma of Levi, we find $\mathcal{I}_\varepsilon(\boldsymbol{q})\to\mathcal{E}(t,\boldsymbol{q})$. Thus, for $\boldsymbol{q}_k\rightharpoonup\boldsymbol{q}_*$ and each $\varepsilon>0$, we have

$$\mathcal{I}_\varepsilon(\boldsymbol{q}_*)\le\liminf_{k\to\infty}\mathcal{I}_\varepsilon(\boldsymbol{q}_k)\le\liminf_{k\to\infty}\mathcal{E}(t,\boldsymbol{q}_k)\overset{\mathrm{def}}{=}\alpha.$$

In the limit $\varepsilon\to0^+$, we find $\mathcal{E}(t,\boldsymbol{q}_*)\le\alpha$ as desired. □

Combining the coercivity estimate (5.2) with this weak lower semicontinuity result, we have established the abstract condition (4.E1).

Finally, we investigate the differentiability of $\mathcal{E}(t,\boldsymbol{q})$ with respect to time. For this, we fix $\boldsymbol{q}=(y,P,p)\in\mathcal{Q}$ such that $\mathcal{E}(0,\boldsymbol{q})<\infty$ and introduce the Kirchhoff tensor

$$K_{\boldsymbol{q}}(x,F)\overset{\mathrm{def}}{=}\partial_F W\big(x,FP(x)^{-1},P(x),p(x),\nabla P(x),\nabla p(x)\big)\big(FP^{-1}\big)^{\mathsf{T}}\in\mathbb{R}^{d\times d}.$$

Theorem 5.3 (Power of the boundary conditions) *If assumption* (3.1) *and* (3.3) *hold, then $\mathcal{E}$ satisfies* (4.E2) *and* (4.5), *i.e., there exist constants $c_0^E \in \mathbb{R}$ and $c_1^E > 0$ and a modulus of continuity ω such that the following holds*:

For $\boldsymbol{q} \in \mathcal{Q}$ with $\mathcal{E}(0, \boldsymbol{q}) < \infty$, we have $\mathcal{E}(\cdot, \boldsymbol{q}) \in \mathrm{C}^1([0, T])$ with

$$\partial_t \mathcal{E}(t, \boldsymbol{q}) = \int_\Omega K_{\boldsymbol{q}}\big(x, \nabla g_{\mathrm{Dir}}\big(t, y(x)\big)\nabla y(x)\big){:}V\big(t, y(x)\big)\,\mathrm{d}x,$$

$$\text{where } V(t, y) = \big(\nabla g_{\mathrm{Dir}}(t, y)\big)^{-1} \frac{\partial}{\partial t} \nabla g_{\mathrm{Dir}}(t, y), \tag{5.5a}$$

$$\big|\partial_t \mathcal{E}(t, \boldsymbol{q})\big| \le c_1^{\mathcal{E}}\big(\mathcal{E}(t, \boldsymbol{q}) + c_0^E\big), \quad \text{and} \tag{5.5b}$$

$$\big|\partial_t \mathcal{E}(t_1, \boldsymbol{q}) - \partial_t \mathcal{E}(t_2, \boldsymbol{q})\big| \le \omega\big(|t_2 - t_1|\big)\big(\mathcal{E}(t_1, \boldsymbol{q}) + c_0^E\big). \tag{5.5c}$$

Proof First, observe that (i) in (3.3c) provides $\delta > 0$ and $C > 1$ such that

$$\begin{aligned}&\forall (x, F, z, A) \in \Omega \times \mathbb{R}^{d\times d} \times \mathrm{SL}(d) \times \mathbb{R}^m \times L^{(m,d)}\ \forall N \in \mathcal{N}_\delta :\\ &\quad \big(W(x, NF, z, A) + c_0^W\big) + \big|\partial_F W(x, NF, z, A)F^{\mathsf{T}}\big| \le C\big(W(x, F, z, A) + c_0^W\big),\end{aligned} \tag{5.6}$$

see (Ball 2002, Lemma 2.5). We fix $(t_*, \boldsymbol{q}) \in [0, T] \times \mathcal{Q}$ with $\mathcal{E}(t_*, \boldsymbol{q}) < \infty$. Hence, the function

$$\begin{aligned}w(t, \cdot) : \Omega \to \mathbb{R}_\infty; x \mapsto W\big(&x, \nabla g_{\mathrm{Dir}}\big(t, y(x)\big)\nabla y(x)P(x)^{-1}, P(x), p(x),\\ &\nabla P(x), \nabla p(x)\big)\end{aligned}$$

is finite for $t = t_*$ and $x \in \widetilde{\Omega}$, where $\Omega \backslash \widetilde{\Omega}$ has measure 0. For $x \in \widetilde{\Omega}$, (5.6) and (3.1) shows that $t \mapsto w(t, x)$ is differentiable near t_* with derivative

$$\dot{w}(t, x) = K_{\boldsymbol{q}}\big(x, \nabla g_{\mathrm{Dir}}\big(t, y(x)\big)\nabla y(x)P(x)^{-1}\big){:}V\big(t, y(x)\big).$$

Because of (3.1) V is uniformly bounded on $[0, T] \times \Omega$ and by (5.6), we conclude that $\mathcal{E}(t, \boldsymbol{q})$ is differentiable for t near t_*, see, e.g. Roubíček (2005, Theorem 1.29).

Using (3.3c)(i), we find the estimate

$$\begin{aligned}\big|\partial_t \mathcal{E}(t, \boldsymbol{q})\big| &\le \int_\Omega \big|K_{\boldsymbol{q}}\big(x, \nabla g_{\mathrm{Dir}}\big(t, y(x)\big)\nabla y(x)P(x)^{-1}\big)\big|\,\big|V\big(t, y(x)\big)\big|\,\mathrm{d}x\\ &\le \int_\Omega c_1^W\big(W\big(x, \nabla g_{\mathrm{Dir}}\big(t, y(x)\big)\nabla y(x)P(x)^{-1}, z, A\big) + c_0^W\big)\,\mathrm{d}x\ \mathbb{V}\\ &\le c_1^E\big(\mathcal{E}(t, \boldsymbol{q}) + c_0^E\big),\end{aligned}$$

where $\mathbb{V} = \|V(\cdot, \cdot)\|_{\mathrm{L}^\infty([0,T]\times\Omega)}$, $c_1^E = \mathbb{V}c_1^W$, and $c_0^E = \mathbb{V}c_1^W c_0^W|\Omega|$. As this estimate is independent of t, a simple Gronwall estimate shows that $\mathcal{E}(t, \boldsymbol{q})$ is finite for all $t \in [0, T]$ if it is finite for one t, cf. (4.1). Thus, we have proved (5.5b).

To show (5.5c), we use formula (5.5a) and that the sublevel $L_{0,E} = \{\boldsymbol{q} \mid \mathcal{E}(0, \boldsymbol{q}) \le E\}$ is bounded in $\boldsymbol{Q}$. In particular, there exists R_E such that all $\boldsymbol{q} = (y, z) \in L_{0,E}$ satisfy $\|y\|_{\mathrm{L}^\infty} \le R_E$. We set $B_E = \{\widehat{y} \mid |\widehat{y}| \le R_E\} \subset \mathbb{R}^d$ and denote by ω_V the modulus of continuity of the mapping $V : [0,T] \to \mathrm{L}^\infty(B_E; \mathbb{R}^{d\times d})$. Moreover, (3.1) guarantees that there is a modulus of continuity ω_G such that

$$\big\| \nabla g_{\mathrm{Dir}}\big(t_2, y(\cdot)\big) \nabla g_{\mathrm{Dir}}\big(t_1, y(\cdot)\big)^{-1} - \mathbf{1} \big\|_{\mathrm{L}^\infty(B_E;\mathbb{R}^{d\times d})} \le \omega_G\big(|t_2 - t_1|\big).$$

For $t_1, t_2 \in [0,T]$ and $\boldsymbol{q} \in L_{0,E}$, we now estimate

$$\begin{aligned}
&\big|\partial_t \mathcal{E}(t_1, \boldsymbol{q}) - \partial_t \mathcal{E}(t_2, \boldsymbol{q})\big| \\
&\quad \le \int_\Omega \big| K_{\boldsymbol{q}}\big(x, \nabla g_{\mathrm{Dir}}\big(t_1, y(\cdot)\big)\nabla y P^{-1}\big) \\
&\qquad - K_{\boldsymbol{q}}\big(x, \nabla g_{\mathrm{Dir}}\big(t_2, y(\cdot)\big)\nabla y P^{-1}\big)\big| \big|V\big(t_1, y(\cdot)\big)\big| \,\mathrm{d}x \\
&\qquad + \int_\Omega \big|K_{\boldsymbol{q}}\big(x, \nabla g_{\mathrm{Dir}}\big(t_2, y(\cdot)\big)\nabla y P^{-1}\big)\big| \big|V\big(t_1, y(\cdot)\big) - V\big(t_2, y(\cdot)\big)\big| \,\mathrm{d}x \\
&\quad \le \int_\Omega \omega\big(\big|\nabla g_{\mathrm{Dir}}\big(t_2, y(\cdot)\big)\nabla g_{\mathrm{Dir}}\big(t_1, y(\cdot)\big)^{-1} - \mathbf{1}\big|\big)\big(W_{\boldsymbol{q}} + c_0^W\big) \,\mathrm{d}x \, \mathbb{V} \\
&\qquad + \int_\Omega c_1^W \big(W_{\boldsymbol{q}} + c_0^W\big)\omega_V\big(|t_2 - t_1|\big) \,\mathrm{d}x \\
&\quad \le \big(\mathcal{E}(t_1, \boldsymbol{q}) + c_0^W |\Omega|\big)\big(\mathbb{V}\, \omega\big(\omega_G\big(|t_2 - t_1|\big)\big) + c_1^W \omega_V\big(|t_2 - t_1|\big)\big),
\end{aligned}$$

where ω is defined in (3.3c)(ii). This is the desired result. □

5.2 Dissipation Potential

The dissipation distance $\mathcal{D}$ on $\mathcal{Z}$ is defined via $D(x, z_0, z_1)$. Condition (3.2a) implies that $\mathcal{D}$ is well defined and the positivity (4.D1)(i) follows from (3.2b). Integrating the pointwise triangle inequality (3.2c), we see that (4.D1)(ii) holds.

Using again that $z_k \rightharpoonup z$ in $\boldsymbol{Z}$ implies $z_k \to z$ in $\mathrm{L}^r(\Omega)$ and that D is nonnegative and lower semicontinuous in both z-variables, the classical lower semicontinuity theory implies the lower semicontinuity of $\mathcal{D}$, namely (4.D2).

5.3 Compatibility Conditions (4.C)

The compatibility conditions (4.C) are derived via Proposition 4.3. Note that (4.5) is already established as part of Theorem 5.3, such that Proposition 4.4 can be applied to obtain (4.C2) as soon as Proposition 4.3 holds. Hence, it remains to show that (4.4) can be derived from the alternative conditions (3.4) or (3.5).

Case (3.4) is conceptually simpler than the other one. Since D is a Carathéodory function, it is continuous in the variables (z_0, z_1). Moreover, the upper bounds on D imposed in (3.4b) implies that $\mathcal{D}$ maps $\mathcal{Z} \times \mathcal{Z}$ into $[0, \infty[$. Since weak convergence of $z_k = (P_k, p_k)$ in Z implies strong convergence of P_k in $\mathrm{L}^{q_1}(\Omega; \mathbb{R}^{d\times d})$ and of p_k in $\mathrm{L}^{q_2}(\Omega; \mathbb{R}^m)$, a classical argument shows that $\mathcal{D}$ is weakly continuous:

$$z_k \rightharpoonup z,\ \widehat{z}_k \rightharpoonup \widehat{z} \quad \Longrightarrow \quad \mathcal{D}(z_k, \widehat{z}_k) \to \mathcal{D}(z, \widehat{z}).$$

Now (4.4) is obviously satisfied by letting $\widehat{\boldsymbol{q}}_j = \widehat{\boldsymbol{q}}$.

Case (3.5) is more involved. We consider a weakly convergent stable sequence $(t_k, \boldsymbol{q}_k) \rightharpoonup (t_*, \boldsymbol{q}_*)$ and arbitrary test state $\widehat{\boldsymbol{q}} \in \mathcal{Q}$. If $\mathcal{E}(t_*, \widehat{\boldsymbol{q}}) = \infty$ or $\mathcal{D}(\boldsymbol{q}_*, \widehat{\boldsymbol{q}}) = \infty$, there is nothing to show as we may take any sequence $\widehat{\boldsymbol{q}}_j$. Hence, we assume $\mathcal{D}(\boldsymbol{q}_*, \widehat{\boldsymbol{q}}) < \infty$ from now on. From $r > d$, we know that $\boldsymbol{Z}$ embeds into $\mathrm{C}^0(\overline{\Omega})$; hence there exists $R > 0$ such that z_j, z_*, and $\widehat{z}$ lie in the ball of radius $R - 1$ in $\mathbb{R}^{d\times d} \times \mathbb{R}^m$. From condition (ii) of (3.5d), we obtain a function $a :]0, 1[\to]0, \delta_0[$ with $a(\rho) \to 0$ for $\rho \to 0^+$, such that for $\rho \in]0, 1[$ estimate (ii) in (3.5d) holds for this R and $\delta \geq a(\rho)$.

Now, we set $\rho_k = \|P_k - P_*\|_{\mathrm{L}^\infty} + \|p_k - p_*\|_{\mathrm{L}^\infty}$, $\delta_k = a(\rho_k)$, and $\widehat{\boldsymbol{q}}_k = (\widehat{y}, \widehat{P}, \widehat{p} + \delta_k v^*)$. Using $r > d$ and $\boldsymbol{q}_k \rightharpoonup \boldsymbol{q}_*$, we find $\rho_k, \delta_k \to 0$, and by construction we have $(z_k(x), \widehat{z}_k(x)) \in \mathbb{D}_R(x)$ on Ω. Hence, the continuity of D on $\mathbb{D}_R$ (cf. (3.5b)) gives $D(x, z_k(x), \widehat{z}_k(x)) \to D(x, z_*(x), \widehat{z}(x))$ pointwise. Exploiting the uniform bound (3.5c) we find (b) in the following statement:

$$\text{(a)} \quad \mathcal{E}(t_k, \widehat{\boldsymbol{q}}_k) \to \mathcal{E}(t_*, \widehat{\boldsymbol{q}}_*), \qquad \text{(b)} \quad \mathcal{D}(z_k, \widehat{z}_k) \to \mathcal{D}(z_*, \widehat{z}). \tag{5.7}$$

If (a) holds, too, then (4.4) holds with "lim sup" replaced by "lim" and with equality.

To establish (a), first note that as above we may consider $t_k = t_*$ by the uniform Lipschitz continuity on sublevels of $\mathcal{E}$, cf. (4.1). Since $\widehat{\boldsymbol{q}}$ and $\widehat{\boldsymbol{q}}_k$ differ only by the term $(0, 0, \delta_k v^*)$, we can employ part (i) of (3.5d) to obtain

$$\big|\mathcal{E}(t_*, \widehat{\boldsymbol{q}}_k) - \mathcal{E}(t_*, \widehat{\boldsymbol{q}})\big| \leq \int_\Omega \omega(\delta_k)\big(W_{\widehat{\boldsymbol{q}}} + c_0^W\big)\,\mathrm{d}x \leq \omega(\delta_k)\big(\mathcal{E}(t_*, \widehat{\boldsymbol{q}}) + c_0^W|\Omega|\big),$$

which is the desired convergence (a).

6 Generalizations and Discussion

The conditions (3.4) and (3.5) are given to fit the Examples 3.2 and 3.3, respectively. Of course, these conditions can be modified to match other constitutive assumptions. The essential point for the mathematical analysis is that the stored-energy density W is coercive in the gradients of the internal variables $z = (P, p)$ while the dissipation distance D only depends on the point values of z. Thus, it is easily possible to include models of crystal plasticity as discussed in Ortiz and Repetto (1999), Gurtin (2000), Svendsen (2002) and formulated in the present framework in Hackl et al. (2003), Mielke (2003) (see Example 3.3 and Sect. 3.4.4 in the latter work).

Of course, the regularization via the gradient $\nabla z = (\nabla P, \nabla p)$ could be replaced by coercivity in a weaker norm that still guarantees a compact embedding into $\mathrm{L}^q(\Omega)$. On the one hand, we may use the physically more desirable growth rate $r = 1$ (cf. Conti and Ortiz 2005) by using the space $\mathrm{BV}(\Omega)$ instead of $\mathrm{W}^{1,r}(\Omega)$. Exploiting the compact embedding of $\mathrm{BV}(\Omega)$ into $\mathrm{L}^q(\Omega)$ for each $q \in [1, d/(d-1)[$, the proof of Theorem 3.1 still works for the case that (3.4) holds. On the other hand we may use a regularizing term like

$$\int_\Omega \int_\Omega \kappa \frac{|\nabla z(x) - \nabla z(\widetilde{x})|^r}{|x - \widetilde{x}|^{d+rs}}\,\mathrm{d}x\,\mathrm{d}\widetilde{x},$$

where $\kappa > 0$, $s \in \,]0,1[$, and $r > 1$. Then we have coercivity in the Sobolev–Slobodetsky space $\mathrm{W}^{s,r}(\Omega)$. Theorem 3.1 still holds with either (3.4) or (3.5), if (3.5a) is strengthened to $r > d/s$, which provides the compact embedding into $\mathrm{C}(\overline{\Omega})$.

However, it remains an open problem to generalize our result to the case treated in Mielke and Müller (2006), where only the term $G = (\operatorname{curl} P)P^{\mathsf{T}}$ is used for regularization. At the moment, the best we can do in this direction is to use a regularizing term in the form

$$\int_\Omega \kappa_1 \big|(\operatorname{curl} P)P^{\mathsf{T}}\big|^r + \kappa_0 |\nabla P|^r \,\mathrm{d}x \quad \text{with } 0 < \kappa_0 \ll \kappa_1.$$

To treat the more interesting case $\kappa_0 = 0$, new ideas have to be developed.

From the general theory of energetic solutions for rate-independent systems (cf. Mainik and Mielke 2005; Francfort and Mielke 2006), it is clear that the Lie-group structure of $\mathfrak{P}$ is not essential at all. The only importance is that the dissipation distance D is a extended quasi-distance (i.e., it satisfies (3.2)). Some engineering models do not take the plastic spin into account and assume that P represents a "plastic metric" taken from

$$\mathfrak{S}(d) \overset{\text{def}}{=} \big\{G \in \mathbb{R}^{d\times d} \,\big|\, G = G^{\mathsf{T}},\ \det G > 0\big\} \subset \mathrm{GL}^+(d),$$

which may be considered as a symmetric space, but not a Lie group. Introducing a dissipation potential $R(G,\dot G) = \widehat{R}(G^{-1/2}\dot G G^{-1/2})$ for some convex and 1-homogeneous functional $\widehat{R} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,\infty[$, the dissipation distance reads $D(G_1,G_2) = \widehat{R}(\log(G_1^{-1/2}G_2G_1^{-1/2}))$ and our theory is again applicable.

In cases of single-crystal plasticity with infinite latent hardening, the set of plastic tensors does not even have a manifold structure. Let $S_a = s_a \otimes m_a$, $a = 1,\dots,N$, be the N glide systems with $s_a, m_a \in \mathbb{R}^d$ and $s_a \cdot m_a = 0$. Then we choose

$$P \in \boldsymbol{S} \overset{\text{def}}{=} \bigcup_{a=1}^{N} \{\mathbf{1} + \gamma_a S_a \mid \gamma_a \ge 0\},$$

and the dissipation distance $D : \boldsymbol{S} \times \boldsymbol{S} \to [0,\infty]$ with

$$D(\mathbf{1} + \gamma S_a, \mathbf{1} + \widetilde{\gamma} S_b) = \begin{cases} \kappa_b(\widetilde{\gamma} - \gamma) & \text{for } a = b \text{ and } \widetilde{\gamma} \ge \gamma, \\ \infty & \text{elsewhere.} \end{cases}$$

Our theory is again applicable, since the set $\mathcal{Z} = \{P \in \mathrm{L}^{q_P}(\Omega) \cap \mathrm{W}^{1,r}(\Omega) \mid P \in \boldsymbol{S} \text{ a.e.}\}$ is weakly closed.

Finally, we address the question of self-interpenetration. The property $W(F,\cdots) = \infty$ for $\det F \le 0$ implies $\det \nabla y(t,x) > 0$ a.e. in Ω, which means that there is no local self-interpenetration. Following Ciarlet and Nečas (1987) (see also Mariano and Modica 2007 for a similar approach using currents), we may define the "non-self-interpenetration" version of the space $\mathcal{Y}$ of admissible deformations via

$$\mathcal{Y}_{\mathrm{nsi}} \overset{\mathrm{def}}{=} \Big\{ y \in \mathrm{W}^{1,q_Y}\big(\Omega;\mathbb{R}^d\big) \;\big|\; y|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id},\ \det\nabla y \geq 0 \text{ a.e. in } \Omega, \\ \int_\Omega \det\nabla y \,\mathrm{d}x \leq \mathrm{vol}\big(y(\Omega)\big) \Big\}.$$

In Ciarlet and Nečas (1987), it is shown that $\mathcal{Y}_{\mathrm{nsi}}$ is weakly closed in $\mathrm{W}^{1,q_Y}(\Omega;\mathbb{R}^d)$ if $q_Y > d$ (see also Ball 2002 for further discussion), and hence our theory works exactly the same way if $\mathcal{Y}$ is replaced by $\mathcal{Y}_{\mathrm{nsi}}$.

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