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THERMALLY DRIVEN PHASE TRANSFORMATION IN SHAPE-MEMORY ALLOYS

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Abstract. This paper analyzes a model for phase transformation in shape-memory alloys induced by temperature changes and by mechanical loading. We assume that the temperature is prescribed and formulate the problem within the framework of the energetic theory of rate-independent processes. Existence and uniqueness results are proved.

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1 Introduction

Shape-memory alloys (SMA) have some surprising thermo-mechanical behavior; one can observe that severely deformed alloys recover their original shape after a thermal cycle (*shape-memory effect*). The exploitation in innovative and commercially valuable applications stimulates the interest in the development of mathematical models for shape-memory materials. Many underlying one-dimensional models are available in the literature but multi-dimensional models allowing for multi-axial loadings and anisotropies are rare. For the isothermal setting such models are discussed in [MTL02, GMH02, KMR05].

In this paper, we are concerned with quasi-static evolution of shape-memory materials in a small-strain regime under non-isothermal conditions. More precisely, we study a macroscopic phenomenological model for shape-memory polycrystalline materials undergoing phase transformation driven by stress or temperature changes. This model was originally proposed by Souza et al. [SMZ98] and later addressed and extended by Auricchio et al. [AuP02, AuP04, AMS07]. We follow the mathematical study of a temperature-driven phase transformation as proposed by Mielke in [Mie07].

The temperature θ *is given a priori* as an applied load and we write $\theta = \theta_{\mathrm{appl}}(t, x)$. This assumption is used in engineering models and it is acceptable if the body is small in at least one direction. Then, the excessive or missing heat can be balanced through the environment.

Our model is described by a stored-energy density $W(e, z, \theta_{\mathrm{appl}})$, where $e = e(u) = \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}})$ and z is a macroscopic internal variable being the effective transformation strain, thus keeping track of the effective phase distribution. The potential energy takes the following form

$$\mathcal{E}(t, u, z) \stackrel{\mathrm{def}}{=} \int_\Omega W(e(u), z, \theta_{\mathrm{appl}}(t, \cdot)) + \frac{\sigma}{2}|\nabla z|^2 \,\mathrm{d}x - \langle l(t), u\rangle, \quad \sigma > 0,$$

where l denotes the time-dependent applied loading. Moreover we specify a dissipation potential by

$$\mathcal{R}(\dot{z}) \stackrel{\mathrm{def}}{=} \int_\Omega \rho|\dot{z}| \,\mathrm{d}x = \rho\|\dot{z}\|_{L^1(\Omega)}, \quad \rho > 0,$$

where $z \in \mathcal{Z} \stackrel{\mathrm{def}}{=} W^{1,2}(\Omega)$. The set $\mathcal{F}$ of admissible displacements is specified as those functions $u \in W^{1,2}(\Omega)$ satisfying the Dirichlet data at the part $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$. Then, our problem can be posed like the *energetic formulation* for rate-independent problems. For a given initial value $(u(0), z(0)) = (u_0, z_0) \in \mathcal{F} \times \mathcal{Z}$, we have to find a function $(u, z) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ such that for all $t \in [0, T]$, the *global stability condition* (S) and the *global energy conservation* (E) are satisfied, i.e.

$$\text{(S)}\ \ \forall (u, z) \in \mathcal{F} \times \mathcal{Z} : \ \mathcal{E}(t, u(t), z(t)) \le \mathcal{E}(t, \bar{u}, \bar{z}) + \mathcal{R}(\bar{z} - z(t)),$$

$$\text{(E)}\ \ \mathcal{E}(t, u(t), z(t)) + \int_0^t \mathcal{R}(\dot{z}(s)) \,\mathrm{d}s = \mathcal{E}(0, u_0, z_0) + \int_0^t \partial_s \mathcal{E}(s, u(s), z(s)) \,\mathrm{d}s.$$

The paper is organized as follows. In Section 2, the mathematical formulation of the problem within the framework of the energetic theory of rate-independent processes is presented. In Section 3, we specify the exact assumptions, and then some helpful estimates on the constitutive function W are obtained. These estimates imply that the partial derivative $\partial_t\mathcal{E}(t,u,z)$ is defined whenever $\mathcal{E}(t,u,z)<\infty$. Then, with standard arguments we show that for all stable initial data (u_0,z_0) an energetic solution exists. Finally, in Section 6 using uniform convexity of $\mathcal{E}(t,\cdot,\cdot)$ and the temporal smoothness of solutions obtained in Section 5, we prove the uniqueness result.

The model discussed here is much simpler than the ones treated in [CHM02, BC*04, KMR05], since it is essentially restricted to isotropic behavior in polycrystals. However, this allows us to go much further in the mathematical analysis. For the more elaborate models only existence result are known, while here we are able to derive Lipschitz dependence of the solutions on the initial data.

2 Mathematical formulation

We consider a body with reference configuration $\Omega\subset\mathbb{R}^d$. This body may undergo phase transformation and elastic displacements $u:\Omega\to\mathbb{R}^d$. The phase transformation will be characterized by the internal variable $z:\Omega\to\mathbb{R}^{d\times d}_{\mathrm{dev}}$ denoting the mesoscopic transformation strain where $\mathbb{R}^{d\times d}_{\mathrm{dev}}$ is the space of symmetric $d\times d$ tensors with vanishing trace. We will denote by $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ the space of symmetric $d\times d$ tensors endowed with the scalar product $v{:}w \stackrel{\mathrm{def}}{=} \mathrm{tr}(v^{\mathsf{T}}w)$ and the corresponding norm is given by $|v|^2\stackrel{\mathrm{def}}{=} v{:}v$ for all $v,w\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$. Here $(\cdot)^{\mathsf{T}}$ and $\mathrm{tr}(\cdot)$ denote the transpose and the trace of the matrix $(\cdot)$, respectively.

The set of admissible displacements $\mathcal{F}\stackrel{\mathrm{def}}{=}\{u\in W^{1,2}(\Omega;\mathbb{R}^d)\,|\,u|_{\Gamma_{\mathrm{Dir}}}=0\}$ is chosen as a suitable subspace of $W^{1,2}(\Omega;\mathbb{R}^d)$ by describing Dirichlet data at the part Γ_{Dir} of $\partial\Omega$. The internal variable z lies in $\mathcal{Z}=W^{1,2}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$. We will denote the states by $q\stackrel{\mathrm{def}}{=}(u,z)$ and the norm and the scalar product in $\mathcal{Q}\stackrel{\mathrm{def}}{=}\mathcal{F}\times\mathcal{Z}$ by $\|\cdot\|_{\mathcal{Q}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{Q}}$, respectively.

The material behavior will depend on the temperature θ, which will be considered as a given time dependent field. Hence, we will not solve an associated heat equation, but we will treat θ as an applied load and denote it by $\theta_{\mathrm{appl}}:[0,T]\times\Omega\to[\theta_{\min},\theta_{\max}]$. This approximation for the temperature is used in engineering models and we may justify it in the case where the changes of the loading are slow and the body is small in at least one direction such that excess of heat can be transported very fast to the surface and then radiated into the environment.

The linearized strain tensor $e=e(u)$ is given by $e(u)\stackrel{\mathrm{def}}{=}\frac{1}{2}(\nabla u+\nabla u^{\mathsf{T}})\in\mathbb{R}^{d\times d}_{\mathrm{dev}}$. We assume that Ω is such that there exists $c_\Omega>0$ such that Korn's inequality holds, i.e.

$$v\in W^{1,2}(\Omega;\mathbb{R}^d):\;\; c_\Omega\|v\|^2_{W^{1,2}}\le\|v\|^2_{L^2}+\|e(v)\|^2_{L^2}.$$

Moreover, $\Gamma_{\mathrm{Dir}}\subset\partial\Omega$ is assumed to be big enough such that there exists $c_{\mathrm{Korn}}>0$ with

$$\forall u\in\mathcal{F}:\;\; \|e(u)\|^2_{L^2}\ge c_{\mathrm{Korn}}\|u\|^2_{W^{1,2}}. \tag{2.1}$$

For more details on Korn's inequality and its consequences, we refer to [DuL76].

The potential energy takes then the following form

$$\widehat{\mathcal{E}}(t,q,\theta)\stackrel{\mathrm{def}}{=}\int_\Omega W(e(u),z,\theta)+\frac{\sigma}{2}|\nabla z|^2\,\mathrm{d}x-\langle l(t),u\rangle, \tag{2.2}$$

where $W : \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{dev}} \times [\theta_{\min}, \theta_{\max}] \to \mathbb{R}$ takes the form

$$W(e(u), z, \theta) \stackrel{\mathrm{def}}{=} \frac{1}{2}(e(u){-}z){:}\mathbb{C}(\theta){:}(e(u){-}z) + h(z, \theta)$$

with $h(\cdot, \theta) : \mathbb{R}^{d\times d}_{\mathrm{dev}} \to \mathbb{R}$ convex. Here σ is a positive coefficient that accounts for nonlocal interaction effects for the internal variable z, $\mathbb{C}(\theta)$ is the elasticity tensor which depends on the temperature θ, and $l(t)$ denotes the applied mechanical loading in the form

$$\langle l(t), u\rangle \stackrel{\mathrm{def}}{=} \int_\Omega f_{\mathrm{appl}}(t, x){\cdot}u(x)\,\mathrm{d}x + \int_{\partial\Omega} g_{\mathrm{appl}}(t, x){\cdot}u(x)\,\mathrm{d}\gamma.$$

Since θ is given we denote the potential energy by $\mathcal{E}(t, q) \stackrel{\mathrm{def}}{=} \widehat{\mathcal{E}}(t, q, \theta_{\mathrm{appl}}(t))$.

The dissipation potential is defined by

$$\mathcal{R}(\dot z) \stackrel{\mathrm{def}}{=} \int_\Omega \rho|\dot z|\,\mathrm{d}x = \rho\|\dot z\|_{L^1(\Omega)}, \quad \rho > 0. \tag{2.3}$$

As usual, the notation $\dot{(\,)}$ denotes the time derivative $\frac{\mathrm{d}}{\mathrm{d}t}$. One can prove that $\mathcal{R} : \mathcal{Z} \to \mathbb{R}$ is convex, lower semicontinuous and positively homogeneous of degree 1, i.e. for all $\gamma \geq 0$ and $v \in \mathcal{Z}$, $\mathcal{R}(\gamma v) = \gamma\mathcal{R}(v)$. Its subdifferential is defined by

$$\partial\mathcal{R}(v) = \Big\{\sigma \in \mathcal{Z}^* \,\Big|\, \forall w \in \mathcal{Z} : \mathcal{R}(w) \geq \mathcal{R}(v) + \int_\Omega \sigma{:}(w{-}v)\,\mathrm{d}x\Big\}.$$

The evolution of smooth processes $q : [0, T] \to \mathcal{Q}$ is governed by the following *doubly nonlinear subdifferential inclusion* (cf. [CoV90, Col92])

$$\begin{pmatrix} 0 \\ \partial\mathcal{R}(\dot z) \end{pmatrix} + \begin{pmatrix} \partial_u\mathcal{E}(t, q) \\ \partial_z\mathcal{E}(t, q) \end{pmatrix} \ni \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.4}$$

where $\partial_u\mathcal{E}(t, q) = -\mathrm{div}\,(\mathbb{C}(\theta){:}(e(u){-}z)) - l(t)$ and $\partial_z\mathcal{E}(t, q) = -\mathbb{C}(\theta){:}(e(u){-}z) + \partial_z h(z, \theta) - \sigma\Delta z$. Using $\mathrm{D}_q\mathcal{E}(t, q) \stackrel{\mathrm{def}}{=} (\partial_u\mathcal{E}(t, q), \partial_z\mathcal{E}(t, q))^{\mathsf{T}}$ and the definition of the subdifferential $\partial\mathcal{R}(\dot z)$ leads to the *variational inequality*

$$\forall v \in \mathcal{Q} : \ \langle \mathrm{D}_q\mathcal{E}(t, q), v{-}\dot q\rangle_{\mathcal{Q}} + \mathcal{R}(v) - \mathcal{R}(\dot z) \geq 0. \tag{2.5}$$

It can be easily seen that (2.5) is equivalent to two local conditions:

$$\begin{aligned} &(\mathrm{S})_{\mathrm{loc}} \ \forall v \in \mathcal{Q} : \ \langle \mathrm{D}_q\mathcal{E}(t, q), v\rangle_{\mathcal{Q}} + \mathcal{R}(v) \geq 0, \\ &(\mathrm{E})_{\mathrm{loc}} \ \langle \mathrm{D}_q\mathcal{E}(t, q), \dot q\rangle_{\mathcal{Q}} + \mathcal{R}(\dot z) \leq 0. \end{aligned}$$

Since $\mathcal{E}(t, \cdot) : \mathcal{Q} \to \mathbb{R}$ is convex, our problem has an equivalent *energetic formulation* in the sense of rate-independent processes, for the details the reader is referred to [MiT04, MTL02, MaM05, FrM06, Mie05]. A function $q : [0, T] \to \mathcal{Q}$ is called an *energetic solution* of the rate-independent problem associated with $\mathcal{E}$ and $\mathcal{R}$ if for all $t \in [0, T]$ the *global stability condition* (S) and the *global energy balance* (E) are satisfied, i.e.

$$\begin{aligned} &(\mathrm{S}) \ \forall \bar q = (\bar u, \bar z) \in \mathcal{Q} : \ \mathcal{E}(t, q(t)) \leq \mathcal{E}(t, \bar q) + \mathcal{R}(\bar z{-}z(t)), \\ &(\mathrm{E}) \ \mathcal{E}(t, q(t)) + \int_0^t \mathcal{R}(\dot z(s))\,\mathrm{d}s = \mathcal{E}(0, q(0)) + \int_0^t \partial_s\mathcal{E}(s, q(s))\,\mathrm{d}s. \end{aligned}$$

Following SOUZA ET AL. [SMZ98] and AURICCHIO ET AL. [Aur01, AuP04], we are particularly interested in $h = h_{\rm SA}$ with

$$h_{\rm SA}(z,\theta) \stackrel{\rm def}{=} c_1(\theta)\sqrt{\delta^2+|z|^2} + c_2(\theta)|z|^2 + \frac{1}{\delta}(|z|-c_3(\theta))_+^3, \tag{2.6}$$

where $\delta > 0$, and $c_i(\theta) > 0$, $i = 1, 2, 3$, are given in term of the temperature θ. Observe that $c_1(\theta)$ is an activation threshold for initiation of martensitic phase transformations, $c_2(\theta)$ measures the occurrence of hardening with respect to the internal variable z, and $c_3(\theta)$ represents the maximum modulus of transformation strain that can be obtained by alignment of martensitic variants. We set

$$W_{\rm SA}(e,z,\theta) \stackrel{\rm def}{=} \frac{1}{2}(e-z):\mathbb{C}(\theta):(e-z) + h_{\rm SA}(z,\theta).$$

The original model is obtained in the limit $\delta \to 0$ and $\sigma \to 0$, namely

$$h_{\rm org}(z,\theta) \stackrel{\rm def}{=} c_1(\theta)|z| + c_2(\theta)|z|^2 + \chi(z),$$

where $\chi : \mathbb{R}^{d\times d}_{\rm dev} \to [0,+\infty]$ is the indicator function of the ball $\{z \in \mathbb{R}^{d\times d}_{\rm dev} : |z| \le c_3(\theta)\}$. For mathematical purposes we need to stay with fixed $\delta, \sigma > 0$. For recent further development we refer to [AuS04, AuS05, AMS07].

3 The mathematical assumptions

We clarify now the assumptions and establish some preliminary results that we will use in the next sections. In the Appendix we will show that $h(z,\theta) = h_{\rm SA}(z,\theta)$ fulfills these assumptions.

Assumptions on *h*. There exist positive constants C^h, $C^h_{\theta_j}$, $c^h_{\theta_j}$, $C^h_{z\theta}$, C^h_z and $\gamma_d \in [3,\infty)$ with $\gamma_d \le \frac{2d}{d-2}$ if $d \ge 3$ such that for all $t \in [0,T]$, $\theta \in [\theta_{\rm min}, \theta_{\rm max}]$ and $z, \widehat{z} \in \mathbb{R}^{d\times d}_{\rm dev}$, we have

$$h \in C^3(\mathbb{R}^{d\times d}_{\rm dev}; C^2([\theta_{\rm min},\theta_{\rm max}])), \tag{3.1a}$$

$$h(z,\theta) \ge C^h(|z|^2-1), \tag{3.1b}$$

$$\forall j = 1,2 : |\partial^j_\theta h(z,\theta)| \le C^h_{\theta_j}(h(z,\theta)+c^h_{\theta_j}), \tag{3.1c}$$

$$\forall i = 0,1,2\ \forall j = 0,1 : |\partial^j_\theta\partial^i_z h(z,\theta)-\partial^j_\theta\partial^i_z h(\widehat{z},\theta)| \le C^h_{z\theta}(1+|z|+|\widehat{z}|)^{\gamma_d-i-1}|z-\widehat{z}|, \tag{3.1d}$$

$$\forall i = 1,2,3\ \forall j = 0,1 :\ |\partial^j_\theta\partial^i_z h(z,\theta)| \le C^h_z(1+|z|)^{\gamma_d-i-j}. \tag{3.1e}$$

Assumptions on $\mathbb{C}$. The elasticity tensor $\mathbb{C}(\theta) : \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}^{d\times d}_{\rm sym}$ is a symmetric positive definite map such that

$$\mathbb{C} \in C^1([\theta_{\rm min},\theta_{\rm max}]; {\rm Lin}(\mathbb{R}^{d\times d}_{\rm sym};\mathbb{R}^{d\times d}_{\rm sym})), \tag{3.2a}$$

$$\exists \alpha > 0\ \forall e \in \mathbb{R}^{d\times d}_{\rm sym}\ \forall \theta \in [\theta_{\rm min},\theta_{\rm max}] :\ e:\mathbb{C}(\theta):e \ge \alpha|e|^2. \tag{3.2b}$$

For later use, we define

$$C^{\mathbb{C}} \stackrel{\rm def}{=} \sup\big\{|\mathbb{C}(\theta)|\,\big|\,\theta \in [\theta_{\rm min},\theta_{\rm max}]\big\} \text{ and } C^{\mathbb{C}}_\theta \stackrel{\rm def}{=} \sup\big\{|\partial_\theta\mathbb{C}(\theta)|\,\big|\,\theta \in [\theta_{\rm min},\theta_{\rm max}]\big\}. \tag{3.3}$$

In particular, if $\lambda, \mu \in C^1([\theta_{\min}, \theta_{\max}])$ with $\lambda(\theta) \geq 0$ and $\mu(\theta) \geq \alpha$ then $\mathbb{C}(\theta){:}z \stackrel{\text{def}}{=} \lambda(\theta)\text{tr}(z)\mathbf{1} + 2\mu(\theta)z$ satisfies the assumptions given above. Here $\lambda(\theta)$ and $\mu(\theta)$ are temperature dependent Lamé coefficients, and $\mathbf{1}$ denotes the identity matrix. The latter decomposition is not exploited in our analysis but it is clearly suggested by the mechanical application. From (3.1b) and (3.2b) we may deduce that there exist $c, C > 0$ such that

$$\forall e \in \mathbb{R}^{d\times d}_{\text{sym}}\ \forall z \in \mathbb{R}^{d\times d}_{\text{dev}} : W(e, z, \theta) \geq c(|e|^2+|z|^2) - C. \tag{3.4}$$

The applied temperature will insert or extract energy according to $\partial_\theta W(e(u), z, \theta_{\text{appl}})\dot{\theta}_{\text{appl}}$. To control this term we assume that θ_{appl} is smooth enough (cf. Section 4) and we prove in the following lemma that the derivatives $\partial^j_\theta W$ exist for $j = 1, 2$ and they can be estimated linearly via W.

Lemma 3.1 *Assume that $\mathbb{C}$ satisfies above assumptions and h satisfies* (3.1c)*. Then there exist $C^W_0, C^W_1 > 0$ such that for all $j = 1, 2$,*

$$|\partial^j_\theta W(e, z, \theta)| \leq C^W_1 (W(e, z, \theta)+C^W_0). \tag{3.5}$$

Proof. These estimates are obtained using (3.1c) and (3.2). □

Lemma 3.2 *Under the assumptions of Lemma 3.1, for all $\theta_1 \in [\theta_{\min}, \theta_{\max}]$, we have*

$$W(e, z, \theta_1) + C^W_0 \leq \exp(C^W_1|\theta_1-\theta|)(W(e, z, \theta)+C^W_0). \tag{3.6}$$

Proof. We consider (e, z) to be fixed and define $w(\theta) = W(e, z, \theta)+C^W_0$. Lemma 3.1 provides exactly $|w'(\theta)| \leq C^W_1 w(\theta)$, and Gronwall's lemma yields the desired result $w(\theta_1) \leq \exp(C^W_1|\theta_1 - \theta|)w(\theta)$ for all $\theta, \theta_1 \in [\theta_{\min}, \theta_{\max}]$. □

4 The existence result

For a given temperature profile θ_{appl} and a given external loading l with

$$\theta_{\text{appl}} \in C^1([0, T]; L^\infty(\Omega; [\theta_{\min}, \theta_{\max}])), \tag{4.1a}$$
$$l \in C^1([0, T]; W^{1,2}(\Omega; \mathbb{R}^d)^*), \tag{4.1b}$$

we now study the potential energy $\mathcal{E}$ as defined in (2.2).

Proposition 4.1 *Under the above assumptions the following holds:*

(i) If for some $(t_*, q) \in [0, T]\times \mathcal{Q}$ we have $\mathcal{E}(t_*, q) < \infty$, then $\mathcal{E}(\cdot, q)$ lies in $C^1([0, T])$ and

$$\partial_t\mathcal{E}(t, q) = \int_\Omega \partial_\theta W(e(u), z, \theta_{\text{appl}}(t))\dot{\theta}_{\text{appl}}(t)\,\mathrm{d}x - \langle \dot{l}(t), u\rangle. \tag{4.2}$$

(ii) There exist $C^E_0, C^E_1 > 0$ such that $\mathcal{E}(t, q) < \infty$ implies $|\partial_t\mathcal{E}(t, q)| \leq C^E_1(\mathcal{E}(t, q)+C^E_0)$.

(iii) For each $\varepsilon > 0$ and $E \in \mathbb{R}$ there exists $\delta > 0$ such that $\mathcal{E}(t_1, q) \leq E$ and $|t_1 - t_2| < \delta$ imply $|\partial_t \mathcal{E}(t_1, q) - \partial_t \mathcal{E}(t_2, q)| \leq \varepsilon$.

Proof. Observe that (3.4) and Korn's inequality (2.1) lead to

$$\mathcal{E}(t,q) \geq \frac{cc_{\text{Korn}}}{2}\|u\|^2_{W^{1,2}} + \min\Big(c, \frac{\sigma}{2}\Big)\|z\|^2_{W^{1,2}} - C|\Omega| - \frac{1}{2cc_{\text{Korn}}}\|l(t)\|^2_{(W^{1,2})'},$$

which implies that there exist $c_0, C_0 > 0$ such that

$$\mathcal{E}(t,q) \geq c_0\|q\|^2_{\mathcal{Q}} - C_0. \tag{4.3}$$

We show now the differentiability of $\mathcal{E}(t,q)$ with respect to t using Lemma 3.1 and assumption (4.1). For all $h \neq 0$ and $t_* + h \in [0,T]$ the mean-value theorem provides some $s \in (0,1)$ such that

$$\begin{aligned}&\frac{1}{h}(\mathcal{E}(t_*{+}h,q){-}\mathcal{E}(t_*,q))\\ &= \int_\Omega \partial_\theta W(e(u),z,\theta_{\text{appl}})\dot{\theta}_{\text{appl}}(t_*{+}sh)\,\mathrm{d}x - \langle l(t_*{+}h){-}l(t_*),u\rangle.\end{aligned} \tag{4.4}$$

We observe that Lemma 3.2 leads to

$$\sup_{\theta_1\in[\theta_{\min},\theta_{\max}]} W(e,z,\theta_1) \leq \exp\big(c_1^W(\theta_{\max}{-}\theta_{\min})\big)\big(W(e,z,\theta_{\text{appl}}){+}C_0^W\big) - C_0^W. \tag{4.5}$$

Since $\mathcal{E}(t_*,q) < \infty$ we have $0 \leq W(e(u),z,\theta_{\text{appl}}(t)) \in L^1(\Omega)$ which implies that the right-hand side of (4.5) belongs to $L^1(\Omega)$. On the other hand, (4.1a) gives $\dot{\theta}_{\text{appl}} \in C^0([0,T];L^\infty(\Omega))$ and we may pass to the limit $h \to 0$ in (4.4) using Lebesgue's theorem. This proves (4.2).

For part *(ii)*, one can see that assumptions (4.1) lead to the following estimate

$$|\partial_t\mathcal{E}(t,q)| \leq \Theta\int_\Omega |\partial_\theta W(e(u),z,\theta_{\text{appl}})|\,\mathrm{d}x + \|\dot{l}(t)\|_{(W^{1,2})'}\|u\|_{W^{1,2}}, \tag{4.6}$$

where $\Theta \stackrel{\text{def}}{=} \|\dot{\theta}_{\text{appl}}\|_{L^\infty}$. Carrying (3.5) for $j=1$ into (4.6) and using Cauchy-Schwarz's inequality, we have

$$|\partial_t\mathcal{E}(t,q)| \leq \Theta\int_\Omega C_1^W\big(W(e(u),z,\theta_{\text{appl}}){+}C_0^W\big)\,\mathrm{d}x + \frac{1}{2}\|\dot{l}(t)\|^2_{(W^{1,2})'} + \frac{1}{2}\|u\|^2_{W^{1,2}},$$

which implies that

$$\begin{aligned}|\partial_t\mathcal{E}(t,q)| \leq{}& C_1^W\Theta\mathcal{E}(t,q) + \frac{1}{2}\big(1{+}C_1^W\Theta\big)\|u\|^2_{W^{1,2}} + C_0^W C_1^W|\Omega|\Theta\\ &+ \frac{1}{2}\Big(C_1^W\|l\|^2_{(W^{1,2})'}\|\dot{\theta}_{\text{appl}}\|_{W^{1,2}}{+}\|\dot{l}\|^2_{(W^{1,2})'}\Big).\end{aligned} \tag{4.7}$$

Using (4.3) in (4.7), the desired result *(ii)* follows immediately.

For part *(iii)*, we use

$$
\begin{aligned}
|\partial_t\mathcal{E}(t_1,q)-\partial_t\mathcal{E}(t_2,q)| \le\ &\Theta\int_\Omega |\partial_\theta W(e(u),z,\theta_{\rm appl}(t_1))-\partial_\theta W(e(u),z,\theta_{\rm appl}(t_2))|\,{\rm d}x\\
&+\int_\Omega |\partial_\theta W(e(u),z,\theta_{\rm appl}(t_1))|\,{\rm d}x\,\|\dot\theta_{\rm appl}(t_1)-\dot\theta_{\rm appl}(t_2)\|_{L^\infty}\\
&+\|\dot l(t_1)-\dot l(t_2)\|_{(W^{1,2})'}\|u\|_{W^{1,2}}.
\end{aligned}
\tag{4.8}
$$

The mean-value theorem provides some $s\in(0,1)$ such that

$$
\begin{aligned}
&|\partial_\theta W(e(u),z,\theta_{\rm appl}(t_1))-\partial_\theta W(e(u),z,\theta_{\rm appl}(t_2))|\\
&=|\partial_\theta^2 W(e(u),z,\theta_{\rm appl}(t_1+s(t_2-t_1)))||\theta_{\rm appl}(t_1)-\theta_{\rm appl}(t_2)|.
\end{aligned}
\tag{4.9}
$$

Introducing (3.5) in (4.9) and then using (3.6), we obtain

$$
\begin{aligned}
&|\partial_\theta W(e(u),z,\theta_{\rm appl}(t_1))-\partial_\theta W(e(u),z,\theta_{\rm appl}(t_2))|\\
&\le K_1^W(W(e(u),z,\theta_{\rm appl}(t_1))+C_0^W)|\theta_{\rm appl}(t_1)-\theta_{\rm appl}(t_2)|,
\end{aligned}
\tag{4.10}
$$

where $K_1^W \overset{\rm def}{=} C_1^W\exp(C_1^W|\theta_{\max}-\theta_{\min}|)$. One can see that (3.5) and (3.6) lead to

$$
|\partial_\theta W(e(u),z,\theta_{\rm appl}(t_1))|\le K_1^W(W(e(u),z,\theta_{\rm appl})+C_0^W). \tag{4.11}
$$

Carrying (4.10) and (4.11) into (4.8), we obtain

$$
\begin{aligned}
&|\partial_t\mathcal{E}(t_1,q)-\partial_t\mathcal{E}(t_2,q)|\\
&\le\int_\Omega K_1^W(W(e(u),z,\theta_{\rm appl}(t_1))+C_0^W)|\theta_{\rm appl}(t_1)-\theta_{\rm appl}(t_2)|\,{\rm d}x\,\|\dot\theta_{\rm appl}(t_1)\|_{L^\infty}\\
&\quad+\int_\Omega K_1^W(W(e(u),z,\theta_{\rm appl}(t_1))+C_0^W)\,{\rm d}x\,\|\dot\theta_{\rm appl}(t_1)-\dot\theta_{\rm appl}(t_2)\|_{L^\infty}\\
&\quad+\|\dot l(t_1)-\dot l(t_2)\|_{(W^{1,2})'}\|u\|_{W^{1,2}},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&|\partial_t\mathcal{E}(t_1,q)-\partial_t\mathcal{E}(t_2,q)|\\
&\le\int_\Omega K_1^W(\mathcal{E}(t_1,q)+C_0^W))\,{\rm d}x\|\theta_{\rm appl}(t_1)-\theta_{\rm appl}(t_2)\|_{L^\infty}\Theta\\
&\quad+\int_\Omega K_1^W(\mathcal{E}(t_1,q)+C_0^W)\,{\rm d}x\|\dot\theta_{\rm appl}(t_1)-\dot\theta_{\rm appl}(t_2)\|_{L^\infty}\\
&\quad+K_1^W\|l(t_1)\|_{(W^{1,2})'}\|u\|_{W^{1,2}}\big(\|\theta_{\rm appl}(t_1)-\theta_{\rm appl}(t_2)\|_{L^\infty}\|\dot\theta_{\rm appl}(t_1)\|_{L^\infty}\\
&\quad+\|\dot\theta(t_1)-\dot\theta(t_2)\|_{L^\infty}\big)+\|\dot l(t_1)-\dot l(t_2)\|_{(W^{1,2})'}\|u\|_{W^{1,2}}.
\end{aligned}
\tag{4.12}
$$

One can observe that (4.1) leads to the following estimate

$$
\|\dot l(t_1)-\dot l(t_2)\|_{(W^{1,2})'}+\|\dot\theta_{\rm appl}(t_1)-\dot\theta_{\rm appl}(t_2)\|_{L^\infty}\le\omega(|t_1-t_2|). \tag{4.13}
$$

Here ω:$[0,\infty) \to [0,\infty)$ is a modulus of continuity, i.e. ω is nondecreasing and $\omega(\tau) \to 0$ for $\tau \searrow 0$. Since $\mathcal{E}(t_1, q) \le E$, (4.3) and (4.13) hold, there exists $\bar{c} > 0$ such that

$$|\partial_t \mathcal{E}(t_1, q) - \partial_t \mathcal{E}(t_2, q)| \le \bar{c}\big(\omega(|t_1 - t_2|) + |t_1 - t_2|\big).$$

Thus, the proposition is established. □

We prove now that the *energetic formulation* (S) and (E) has at least one solution $q : [0,T] \to \mathcal{Q}$ for a given stable initial datum $q_0 \in \mathcal{Q}$, i.e. q_0 satisfies the *global stability condition* (S) at $t = 0$. The existence theory was developed in [MaM05, Mie05, FrM06] and it is based on incremental minimization problems. More precisely, for a given partition $\Pi = \{0 = t < t_1 < \ldots < t_N = T\}$, we define the incremental problem as follows:

$$(\mathrm{IP})_\Pi \quad \begin{cases} \text{For } k = 1, \ldots, N \text{ find} \\ q_k \overset{\text{def}}{=} (u_k, z_k) \in \operatorname{Argmin}\{\mathcal{E}(t_k, \widetilde{q}) + \mathcal{R}(\widetilde{z} - z_k) : \widetilde{q} = (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}\}. \end{cases}$$

Define the piecewise constant interpolant $q^\Pi : [0,T] \to \mathcal{Q}$ with $q^\Pi(t) = q_{j-1}$ for $t \in [t_{j-1}, t_j)$ for $j = 0, \ldots, N$. Then, one shows that a subsequence has a limit and this limit function satisfies the *energetic formulation* (S) and (E).

Theorem 4.2 *Assume that $\mathcal{E}$ and $\mathcal{R}$ satisfy the assumptions* (2.1), (3.1), (3.2), (4.1). *Let $q_0 \in \mathcal{Q}$ be stable for $t = 0$, i.e. $\mathcal{E}(0, q_0) \le \mathcal{E}(0, q) + \mathcal{R}(z - z_0)$ for all $q = (u, z) \in \mathcal{Q}$. Then there exists an energetic solution $q = (u, z) : [0,T] \to \mathcal{Q}$ with $q(0) = q_0$ and*

$$u \in L^\infty([0,T]; W^{1,2}(\Omega; \mathbb{R}^d)),$$
$$z \in L^\infty([0,T]; W^{1,2}(\Omega; \mathbb{R}^{d\times d}_{\text{dev}})) \cap BV([0,T]; L^1(\Omega; \mathbb{R}^{d\times d}_{\text{dev}})).$$

Moreover, let $\Pi_k = \{0 = t_0^k < t_1^k < \ldots < t_{N_k}^k = T\}$, $k \in \mathbb{N}$, be a sequence of partitions with fineness $\Delta(\Pi_k) \overset{\text{def}}{=} \max\{t_j^k - t_{j-1}^k : j = 1, \ldots, N_k\}$ tending to 0 for $k \to \infty$. Let $q^{\Pi_k} \overset{\text{def}}{=} (u^{\Pi_k}, z^{\Pi_k}) : [0,T] \to \mathcal{Q}$ be the piecewise constant interpolants associated with the incremental problem $(\mathrm{IP})_{\Pi_k}$, then there exist a subsequence $(\bar{u}_n, \bar{z}_n) \overset{\text{def}}{=} (u^{\Pi_{k_n}}, z^{\Pi_{k_n}})$ and an energetic solution $\widetilde{q} \overset{\text{def}}{=} (\widetilde{u}, \widetilde{z}) : [0,T] \to \mathcal{Q}$ such that for all $t \in [0,T]$ the following holds

$$\bar{z}_n(t) \rightharpoonup \widetilde{z}(t) \ \text{in } \mathcal{Z},$$
$$\mathcal{E}(t, \bar{q}_n(t)) \to \mathcal{E}(t, \widetilde{q}(t)),$$
$$\int_0^t \mathcal{R}(\dot{\bar{z}}_n(s))\,\mathrm{d}s \to \int_0^t \mathcal{R}(\dot{\widetilde{z}}(s))\,\mathrm{d}s.$$

Proof. We use the abstract result of [FrM06] which relies on the following abstract assumptions (H1)—(H5), where $\mathcal{F}$ and $\mathcal{Z}$ are considered as topological spaces carrying the weak topology of $W^{1,2}(\Omega)$. All topological notions are to be understood in the "sequential" sense.

(H1) $\forall z_1, z_2, z_3 \in \mathcal{Z}$: $\mathcal{R}(z_1) = 0 \Leftrightarrow z_1 = 0$ and $\mathcal{R}(z_1 - z_3) \le \mathcal{R}(z_1 - z_2) + \mathcal{R}(z_2 - z_3)$,

(H2) $\mathcal{R} : \mathcal{Z} \to [0,\infty]$ is continuous, (H3) $\forall t \in [0,T]$: $\mathcal{E}(t,\cdot) : \mathcal{Q} \to [0,\infty)$ has compact sublevels,

(H4) there exist $C_0^E, C_1^E > 0$ such that for all $(t_*, q) \in [0,T] \times \mathcal{Q}$:

$$\mathcal{E}(t_*, q) < \infty \implies \begin{cases} \partial_t \mathcal{E}(\cdot, q) : [0,T] \to \mathbb{R} \text{ is continuous and} \\ |\partial_t \mathcal{E}(t,q)| \le C_1^E(\mathcal{E}(t,q) + C_0^E), \end{cases}$$

(H5) $\forall E > 0\ \forall \varepsilon > 0\ \exists \delta > 0$: $\big(\mathcal{E}(t_1,q) \le E, |t_1 - t_2| \le \delta\big) \Rightarrow |\partial_t \mathcal{E}(t_1,q) - \partial_t \mathcal{E}(t_2,q)| < \varepsilon$.

(H1) follows from the definition (2.3) of the dissipation potential $\mathcal{R}$. Since $\mathcal{R}$ is strongly continuous in $L^1(\Omega)$, the compact embedding of $W^{1,2}(\Omega)$ into $L^1(\Omega)$ provides (H2).

On the one hand, $\mathcal{E}(t,\cdot)$ is coercive because of (3.1b), (3.2b) and (2.1). On the other hand, by $\mathcal{E}(t,\cdot)$ is weakly lower semi-continuous, as the integrand is convex in $(\nabla u, \nabla z)$ and continuous in z. This provides (H3). Finally, (H4) and (H5) were already obtained in Proposition 4.1.

Since the assumptions (H1)—(H5) are fulfilled, the abstract theory is applicable, and the theorem is proved. ☐

The above result does not need any convexity assumption on $h(\cdot,\theta)$, hence solutions may have jumps in general and uniqueness can not be expected.

For the original Souza-Auricchio model, i.e. $\delta = 0$ in (2.6), but still $\sigma > 0$, it is also possible to obtain existence of solutions. For the case $\sigma = 0$ the question of existence is still open, even in the isothermal case, see [AMS07].

5 Temporal regularity via uniform convexity

We assume that $h(\cdot,\theta)$ is α_h-uniformly convex on $\mathbb{R}^{d\times d}_{\text{dev}}$, namely there exists a modulus of convexity $\alpha_h > 0$ such that for all $z_0, z_1 \in \mathbb{R}^{d\times d}_{\text{dev}}$ and $\lambda \in [0,1]$ we have

$$h(z_\lambda, \theta) \le (1-\lambda) h(z_0,\theta) + \lambda h(z_1,\theta) - \frac{\alpha_h}{2}\lambda(1-\lambda)|z_1 - z_0|^2, \tag{5.1}$$

where $z_\lambda \overset{\text{def}}{=} (1-\lambda)z_0 + \lambda z_1$. By (3.2b) the expression $(e-z){:}\mathbb{C}(\theta){:}(e-z)$ is α-uniformly convex in $(e-z)$. With $q_\lambda \overset{\text{def}}{=} (1-\lambda)q_0 + \lambda q_1$, we conclude

$$\begin{aligned} &\exists \widehat{\kappa} > 0\ \forall q_0, q_1 \in \mathcal{Q}\ \forall t \in [0,T]\ \forall \lambda \in [0,1] : \\ &\mathcal{E}(t, q_\lambda) \le (1-\lambda)\mathcal{E}(t,q_0) + \lambda \mathcal{E}(t,q_1) - \frac{\widehat{\kappa}}{2}\lambda(1-\lambda)\|q_1 - q_0\|_{\mathcal{B}}^2, \end{aligned} \tag{5.2}$$

where $\widehat{\kappa} \overset{\text{def}}{=} \min(\alpha, \alpha_h, \sigma)$ and $\|q\|_{\mathcal{B}}^2 \overset{\text{def}}{=} \|e(u) - z\|_{L^2}^2 + \|z\|_{W^{1,2}}^2$. Using Korn's inequality (2.1), we find $\|q\|_{\mathcal{B}} \ge c_0 \|q\|_{\mathcal{Q}}$. Hence, we deduce from (5.2) that

$$\begin{aligned} &\forall q_0, q_1 \in \mathcal{Q}\ \forall t \in [0,T]\ \forall \lambda \in [0,1] : \\ &\mathcal{E}(t, q_\lambda) \le (1-\lambda)\mathcal{E}(t,q_0) + \lambda \mathcal{E}(t,q_1) - \frac{\kappa}{2}\lambda(1-\lambda)\|q_1 - q_0\|_{\mathcal{Q}}^2, \end{aligned} \tag{5.3}$$

where $\kappa = \widehat{\kappa} c_0$. In other words, we have proved that $\mathcal{E}(t,q)$ is κ-uniformly convex in the variable q. Observe that (5.3) implies that

$$\forall q, \widehat{q} \in \mathcal{Q}: \ \mathcal{E}(t,\widehat{q}) \geq \mathcal{E}(t,q) + \langle \mathrm{D}_q \mathcal{E}(t,q), \widehat{q}-q \rangle_{\mathcal{Q}} + \frac{\kappa}{2}\|q-\widehat{q}\|^2_{\mathcal{Q}}. \tag{5.4}$$

We establish now an estimate that is crucial to prove the temporal regularity result given in Theorem 5.2.

Lemma 5.1 *Let assumption* (3.2) *on* $\mathbb{C}$ *and assumption* (4.1) *on the loadings be satisfied. Then for all* $R > 0$*, there exists* $C_R > 0$ *such that*

$$\forall t \in [0,T] \ \forall q, \widehat{q} \in \mathcal{Q} \ \text{ with } \|q\|_{\mathcal{Q}}, \|\widehat{q}\|_{\mathcal{Q}} \leq R: \ |\partial_t \mathcal{E}(t,q) - \partial_t \mathcal{E}(t,\widehat{q})| \leq C_R \|q-\widehat{q}\|_{\mathcal{Q}}. \tag{5.5}$$

Proof. We denote by $w(u,z,\theta) \stackrel{\text{def}}{=} \frac{1}{2}(e(u)-z){:}\partial_\theta \mathbb{C}(\theta){:}(e(u)-z)$. First, we point out that

$$\partial_t \mathcal{E}(t,q) = \int_\Omega \partial_\theta W(e(u), z, \theta_{\text{appl}}) \dot{\theta}_{\text{appl}} \, \mathrm{d}x - \langle \dot{l}(t), u \rangle, \tag{5.6}$$

where $\partial_\theta W(e(u),z,\theta) = w(u,z,\theta) + \partial_\theta h(z,\theta)$. Then, we deduce from (5.6) and (4.1b) that

$$|\partial_t \mathcal{E}(t,q) - \partial_t \mathcal{E}(t,\widehat{q})| \leq I \|\dot{\theta}_{\text{appl}}\|_{L^\infty} + \|\dot{l}(t)\|_{(W^{1,2})'} \|u-\widehat{u}\|_{W^{1,2}}, \tag{5.7}$$

where $I \stackrel{\text{def}}{=} \int_\Omega |\partial_\theta W(e(u),z,\theta_{\text{appl}}) - \partial_\theta W(e(\widehat{u}),\widehat{z},\theta_{\text{appl}})| \, \mathrm{d}x$. This gives

$$I \leq \int_\Omega |w(u,z,\theta_{\text{appl}}) - w(\widehat{u},\widehat{z},\theta_{\text{appl}})| \, \mathrm{d}x + \int_\Omega |\partial_\theta h(z,\theta_{\text{appl}}) - \partial_\theta h(\widehat{z},\theta_{\text{appl}})| \, \mathrm{d}x.$$

The first integral is estimated by using (3.3) and Cauchy-Schwarz's inequality. The second one by (3.1d) with $i = 0$ and $j = 1$ and Hölder's inequality to give

$$I \leq C^{\mathbb{C}}_\theta \big(\|q\|_{\mathcal{Q}} + \|\widehat{q}\|_{\mathcal{Q}}\big) \|q-\widehat{q}\|_{\mathcal{Q}} + C^h_{z\theta} \|1+|z|+|\widehat{z}|\|^{\gamma_d-1}_{L^{\gamma_d}} \|z-\widehat{z}\|_{L^{\gamma_d}}. \tag{5.8}$$

Since $W^{1,2}(\Omega) \subset L^{\gamma_d}(\Omega)$ (recall that $\gamma_d \leq \frac{2d}{d-2}$ if $d \geq 3$), then the last term on the right-hand side of (5.8) is estimated by $C C^h_{z\theta} \big(1 + \|z\|_{W^{1,2}} + \|\widehat{z}\|_{W^{1,2}}\big)^{\gamma_d-1} \|z-\widehat{z}\|_{W^{1,2}}$ where $C > 0$. Then, we deduce that

$$I \leq \max(C^{\mathbb{C}}_\theta, C C^h_{z\theta}) \big(1 + \|q\|_{\mathcal{Q}} + \|\widehat{q}\|_{\mathcal{Q}}\big)^{\gamma_d-1} \|q-\widehat{q}\|_{\mathcal{Q}}. \tag{5.9}$$

Introducing (5.9) in (5.7) the assertion (5.5) follows. □

Theorem 5.2 (Lipschitz continuity). *Assume that* (3.1a), (3.1c) *with j=1,* (3.3), (4.1), *and* (5.4) *hold. Then any energetic solution* q *is Lipschitz continuous. In fact, let* $R \stackrel{\text{def}}{=} \|q\|_{L^\infty([0,T];\mathcal{Q})}$ *and* $C_R > 0$ *given by Lemma 5.1, then* $\|\dot{q}(t)\|_{\mathcal{Q}} \leq \frac{C_R}{\kappa}$ *for a.e.* $t \in [0,T]$ *where* κ *is defined in* (5.3).

Proof. Considering (5.4) for $t \overset{\text{def}}{=} s$ and $q \overset{\text{def}}{=} q(s)$ we have

$$\forall \widehat{q} \in \mathcal{Q}: \ \mathcal{E}(s,\widehat{q}) \geq \mathcal{E}(s,q(s)) + \langle \mathrm{D}_q\mathcal{E}(s,q(s)), \widehat{q}-q(s)\rangle_{\mathcal{Q}} + \frac{\kappa}{2}\,\|\widehat{q}-q(s)\|^2_{\mathcal{Q}}. \tag{5.10}$$

For arbitrary $s \in [0,T]$ we know that $q(s)$ fulfills $(\mathrm{S})_{\mathrm{loc}}$. Choosing $v \overset{\text{def}}{=} \widehat{q}-q(s)$ in $(\mathrm{S})_{\mathrm{loc}}$ we deduce from (5.10) that for all $\widehat{q} \in \mathcal{Q}$ we have

$$\frac{\kappa}{2}\,\|\widehat{q}-q(s)\|^2_{\mathcal{Q}} \leq \mathcal{E}(s,\widehat{q}) - \mathcal{E}(s,q(s)) + \mathcal{R}(\widehat{z}-z(s)). \tag{5.11}$$

Then, for all $t \in [0,T]$ and $s \in [0,t]$, we have

$$\begin{aligned}
&\frac{\kappa}{2}\,\|q(t)-q(s)\|^2_{\mathcal{Q}} \\
&\leq \mathcal{E}(s,q(t)) - \mathcal{E}(s,q(s)) + \mathcal{R}(z(t)-z(s)) \ \leq\ \mathcal{E}(s,q(t)) - \mathcal{E}(s,q(s)) + \int_s^t \mathcal{R}(\dot{z}(r))\,\mathrm{d}r \\
&= -\int_s^t \partial_r\mathcal{E}(r,q(t))\,\mathrm{d}r + \int_s^t \partial_r\mathcal{E}(r,q(r))\,\mathrm{d}r \ \leq\ C_R\int_s^t \|q(r)-q(t)\|_{\mathcal{Q}}\,\mathrm{d}r.
\end{aligned}$$

The first inequality is obtained by choosing $\widehat{q} \overset{\text{def}}{=} q(t)$ in (5.10), the second one comes from the convexity of $\mathcal{R}(\cdot)$, the third identity follows from the energy identity (E), and the last one results from (5.5). Note that this estimate is exactly the assumptions of the following Lemma 5.3, hence the result follows. ☐

Lemma 5.3 *Let $q \in L^\infty([0,T];\mathcal{Q})$ and $C > 0$ be given such that for all $t \in [0,T]$ and $s \in [0,t]$ we have*

$$\frac{\kappa}{2}\,\|q(t)-q(s)\|^2_{\mathcal{Q}} \leq C\int_s^t \|q(r)-q(t)\|_{\mathcal{Q}}\,\mathrm{d}r.$$

Then, $q \in C^{\mathrm{Lip}}([0,T];\mathcal{Q})$ with $\|\dot{q}(t)\|_{\mathcal{Q}} \leq \frac{C}{\kappa}$ for a.e. $t \in [0,T]$.

Proof. The proof is obtained using the same techniques detailed in the proof of Theorem 7.5 in [MiT04]. Since it is quite a routine to adapt this proof to our case, we let the verification to the reader. ☐

6 Uniqueness result

Uniqueness results in rate-independent hysteresis models are rather exceptional, as usually one needs strong assumptions on the nonlinearities, see [MiT04, MiR07]. To show uniqueness we consider two solutions q_0 and q_1 and prove our result using the techniques developed in [MiT04]. For this we introduce now some convenient notations. For $i = 0,1$ and $j = 0,1,2$, let

$$w_i \overset{\text{def}}{=} e(u_i) - z_i \ \text{ and } \ \mathrm{D}^j_q\mathcal{E}_i \overset{\text{def}}{=} \mathrm{D}^j_q\mathcal{E}(t,q_i(t)).$$

We denote by $\mathcal{Q}_R \overset{\text{def}}{=} \{q \in \mathcal{Q}: \ \|q\|_{\mathcal{Q}} \leq R\}$.

Proposition 6.1 *Assume that* (3.1d), (3.1e), (3.2a) *and* (4.1) *hold. Then*

$$\mathrm{D}_q\mathcal{E}(t,q)[\widehat{q}] = \int_\Omega \big(\widehat{w}{:}\mathbb{C}(\theta_{\mathrm{appl}}){:}w + \partial_z h(z,\theta_{\mathrm{appl}}){:}\widehat{z} + \sigma\nabla z{:}\nabla\widehat{z}\big)\,\mathrm{d}x - \langle l(t),\widehat{u}\rangle, \quad (6.1\mathrm{a})$$

$$\partial_t\mathrm{D}_q\mathcal{E}(t,q)[\widehat{q}] = \int_\Omega \dot{\theta}_{\mathrm{appl}}\big(\widehat{w}{:}\partial_\theta\mathbb{C}(\theta_{\mathrm{appl}}){:}w + \partial_\theta\partial_z h(z,\theta_{\mathrm{appl}}){:}\widehat{z}\big)\,\mathrm{d}x - \langle \dot{l}(t),\widehat{u}\rangle, \quad (6.1\mathrm{b})$$

$$\mathrm{D}_q^2\mathcal{E}(t,q)[\widehat{q},\widehat{q}] = \int_\Omega \big(\widehat{w}{:}\mathbb{C}(\theta_{\mathrm{appl}}){:}\widehat{w} + \widehat{z}{:}\partial_z^2 h(z,\theta_{\mathrm{appl}}){:}\widehat{z} + \sigma|\nabla\widehat{z}|^2\big)\,\mathrm{d}x, \quad (6.1\mathrm{c})$$

where $w \overset{\mathrm{def}}{=} e(u)-z$ *and* $\widehat{w} \overset{\mathrm{def}}{=} e(\widehat{u})-\widehat{z}$*. For* $R>0$*, we have* $\mathcal{E}(t,\cdot)\in C^{2,\mathrm{Lip}}(\mathcal{Q}_R;\mathbb{R})$*.*

Proof. First, using (3.1e) and (3.2a), one can deduce from Lebesgue's theorem that $\mathrm{D}_q\mathcal{E}(t,q), \partial_t\mathrm{D}_q\mathcal{E}(t,q)$ and $\mathrm{D}_q^2\mathcal{E}(t,q)$ exist and (6.1) holds. On the other hand, one observes using (3.3) and (3.1d) with $i=2$ and $j=0$ that

$$|\mathrm{D}_q^2\mathcal{E}(t,q)[\widehat{q},\widehat{q}]| \le C_\theta^{\mathbb{C}}\|\widehat{w}\|_{L^2}^2 + C_z^h\|1{+}|z|\|_{L^{\gamma_d}}^{\gamma_d-2}\|\widehat{z}\|_{L^{\gamma_d}}^2 + \sigma\|\nabla\widehat{z}\|_{L^2}^2. \quad (6.2)$$

Since $W^{1,2}(\Omega)\subset L^{\gamma_d}(\Omega)$ with $\gamma_d \le \frac{2d}{d-2}$, the second term on the right side of (6.2) is estimated by $C_1C_z^h(1+\|z\|_{W^{1,2}})^{\gamma_d-2}\|\widehat{z}\|_{W^{1,2}}^2$ where $C_1>0$ and it follows that

$$|\mathrm{D}_q^2\mathcal{E}(t,q)[\widehat{q},\widehat{q}]| \le C\big(1{+}\|q\|_{\mathcal{Q}}\big)^{\gamma_d-2}\|\widehat{q}\|_{\mathcal{Q}}^2, \quad (6.3)$$

where $C \overset{\mathrm{def}}{=} \max(2C_\theta^{\mathbb{C}}, C_1C_z^h,\sigma)$. Observe now that (3.1d) with $i=2$ and $j=0$ and Hölder's inequality give

$$|\mathrm{D}_q^2\mathcal{E}_1[\widehat{q},\widehat{q}]{-}\mathrm{D}_q^2\mathcal{E}_0[\widehat{q},\widehat{q}]| \le C_{z\theta}^h\|1{+}|z_0|{+}|z_1|\|_{L^{\gamma_d}}^{\gamma_d-3}\|\widehat{z}\|_{L^{\gamma_d}}^2\|z_1{-}z_0\|_{L^{\gamma_d}}.$$

Since $W^{1,2}(\Omega)\subset L^{\gamma_d}(\Omega)$ with $\gamma_d\le\frac{2d}{d-2}$ then the latter estimate implies that there exists $C_2>0$ such that

$$|\mathrm{D}_q^2\mathcal{E}_1[\widehat{q},\widehat{q}]{-}\mathrm{D}_q^2\mathcal{E}_0[\widehat{q},\widehat{q}]| \le C_2C_{z\theta}^h\big(1{+}\|q_0\|_{\mathcal{Q}}{+}\|q_1\|_{\mathcal{Q}}\big)^{\gamma_d-3}\|\widehat{q}\|_{\mathcal{Q}}^2\|q_1{-}q_0\|_{\mathcal{Q}}. \quad (6.4)$$

Hence, $\mathcal{E}(t,\cdot)\in C^{2,\mathrm{Lip}}(\mathcal{Q}_R;\mathbb{R})$ for every $R>0$. □

In the following lemma we establish some estimates that are crucial to obtain the uniqueness result given in Theorem 6.3.

Lemma 6.2 *Assume that* (3.1d), (3.1e), (3.2a) *and* (4.1) *hold. Then for each* $R>0$*, there exist* $C_1,C_2>0$ *such that for all* $q_0,q_1\in\mathcal{Q}_R$*, we have*

$$\|\partial_t\mathrm{D}_q\mathcal{E}_1{-}\partial_t\mathrm{D}_q\mathcal{E}_0\|_{\mathcal{Q}'} \le C_1\|q_1{-}q_0\|_{\mathcal{Q}}, \quad (6.5\mathrm{a})$$

$$\|\mathrm{D}_q\mathcal{E}_{1-i}{-}\mathrm{D}_q\mathcal{E}_i{+}\mathrm{D}_q^2\mathcal{E}_i[q_i{-}q_{1-i}]\|_{\mathcal{Q}'} \le C_2\|q_1{-}q_0\|_{\mathcal{Q}}^2. \quad (6.5\mathrm{b})$$

Proof. Defining $W_i \overset{\mathrm{def}}{=} W(e(u_i),z_i,\theta_{\mathrm{appl}})$ for $i=0,1$, we observe that

$$\|\partial_t\mathrm{D}_q\mathcal{E}_1{-}\partial_t\mathrm{D}_q\mathcal{E}_0\|_{\mathcal{Q}'} = \sup_{\|\widehat{q}\|_{\mathcal{Q}}\le 1}\Big|\int_\Omega \dot{\theta}_{\mathrm{appl}}\big(\partial_\theta\mathrm{D}_qW_1{-}\partial_\theta\mathrm{D}_qW_0\big){:}\widehat{q}\,\mathrm{d}x\Big|,$$

where $\widehat{q} = (\widehat{u}, \widehat{z})^\mathsf{T} \in \mathcal{Q}$. With $\widehat{w} = e(\widehat{u}) - \widehat{z}$ and $w_j = e(u_j) + z_j$ it follows

$$\|\partial_t \mathrm{D}_q \mathcal{E}_1 - \partial_t \mathrm{D}_q \mathcal{E}_0\|_{\mathcal{Q}'} \le \Theta \sup_{\|(\widehat{u},\widehat{z})\|_{W^{1,2}} \le 1} \Bigg(\int_\Omega |\partial_\theta \partial_z h(z_1, \theta_{\mathrm{appl}}) - \partial_\theta \partial_z h(z_0, \theta_{\mathrm{appl}})||\widehat{z}| \mathrm{d}x + \int_\Omega |(w_1 - w_0) : \partial_\theta \mathbb{C}(\theta_{\mathrm{appl}}) : \widehat{w}| \mathrm{d}x \Bigg), \tag{6.6}$$

where $\Theta \stackrel{\text{def}}{=} \|\dot{\theta}_{\mathrm{appl}}\|_{L^\infty} < \infty$ due to (4.1a). On the one hand, using (3.1d), with $i = 1$ and $j = 1$, and Cauchy-Schwarz's inequality one has

$$\int_\Omega |\partial_\theta \partial_z h(z_1, \theta_{\mathrm{appl}}) - \partial_\theta \partial_z h(z_0, \theta_{\mathrm{appl}})||\widehat{z}| \, \mathrm{d}x \le C^h_{z\theta} \|1 + |z_0| + |z_1|\|^{\gamma_d - 2}_{L^{\gamma_d}} \|z_1 - z_0\|_{L^{\gamma_d}} \|\widehat{z}\|_{L^{\gamma_d}}.$$

Since $W^{1,2}(\Omega) \subset L^{\gamma_d}(\Omega)$ with $\gamma_d \le \frac{2d}{d-2}$, we deduce from the latter inequality that there exists $C > 0$ such that

$$\begin{aligned} &\int_\Omega |\partial_\theta \partial_z h(z_1, \theta_{\mathrm{appl}}) - \partial_\theta \partial_z h(z_0, \theta_{\mathrm{appl}})||\widehat{z}| \, \mathrm{d}x \\ &\qquad \le C C^h_{z\theta} \big(1 + \|z_0\|_{W^{1,2}} + \|z_1\|_{W^{1,2}}\big)^{\gamma_d - 2} \|z_1 - z_0\|_{W^{1,2}} \|\widehat{z}\|_{W^{1,2}}. \end{aligned} \tag{6.7}$$

On the other hand, using (3.3) and Cauchy-Schwarz's inequality, we obtain

$$\int_\Omega |\partial_\theta \mathbb{C}(\theta_{\mathrm{appl}})||w_1 - w_0||\widehat{w}| \, \mathrm{d}x \le C^{\mathbb{C}}_{z\theta} \big(\|u_1 - u_0\|_{W^{1,2}} + \|z_1 - z_0\|_{L^2}\big) \|\widehat{q}\|_{W^{1,2}}. \tag{6.8}$$

Introducing (6.7) and (6.8) in (6.6), we obtain (6.5a).

For $i = 0, 1$, let us evaluate now

$$\|\mathrm{D}_q \mathcal{E}_{1-i} - \mathrm{D}_q \mathcal{E}_i + \mathrm{D}^2_q \mathcal{E}_i [q_i - q_{1-i}]\|_{\mathcal{Q}'} = \left\| \int_0^1 \big(\mathrm{D}_q \mathcal{E}(t, q_i + \rho(q_{1-i} - q_i)) + \mathrm{D}^2_q \mathcal{E}_i\big)[q_i - q_{1-i}] \, \mathrm{d}\rho \right\|_{\mathcal{Q}'},$$

which implies by using (6.4) that there exists $C_R > 0$ such that

$$\|\mathrm{D}_q \mathcal{E}_{1-i} - \mathrm{D}_q \mathcal{E}_i + \mathrm{D}^2_q \mathcal{E}_i [q_i - q_{1-i}]\|_{\mathcal{Q}'} \le \int_0^1 C_R \rho \|q_{1-i} - q_i\|^2_{\mathcal{Q}} \, \mathrm{d}\rho = \frac{C_R}{2} \|q_0 - q_1\|^2_{\mathcal{Q}}. \tag{6.9}$$

Thus, (6.5b) follows. □

Theorem 6.3 *Assume that* (3.1d)*,* (3.1e)*,* (3.2a) *and* (4.1) *hold. Then for each stable initial condition q_0, there exists a unique energetic solution q. In particular, for each $R > 0$ there exist constants $C, c > 0$ such that for all stable initial conditions $q_0(0), q_1(0) \in \mathcal{Q}_R$, the solutions q_0 and q_1 satisfy*

$$\|q_1(t) - q_0(t)\|_{\mathcal{Q}} \le C \exp(ct) \|q_1(0) - q_0(0)\|_{\mathcal{Q}} \ \textit{for all } t \in [0, T].$$

Proof. The uniqueness result will follow from the estimates obtained in the Lemma 6.2 and Gronwall's lemma. Given two solutions q_0 and q_1, there exists $R > 0$ such that $\|q_j\|_{C^0([0,T],\mathcal{Q})} \le R$ for $j = 0, 1$. Define

$$\gamma(t) \stackrel{\text{def}}{=} \langle \mathrm{D}_q\mathcal{E}_1 - \mathrm{D}_q\mathcal{E}_0, q_1 - q_0\rangle.$$

Then, by κ-uniform convexity (see (5.4)), we have

$$\|q_1(t) - q_0(t)\|_{\mathcal{Q}}^2 \le \frac{\gamma(t)}{\kappa}. \tag{6.10}$$

On the other hand, the derivative of $\gamma(t)$ denoted by $\dot{\gamma}(t)$ is given by

$$\begin{aligned}\dot{\gamma}(t) = {} & 2\langle \mathrm{D}_q\mathcal{E}_1 - \mathrm{D}_q\mathcal{E}_0, \dot{q}_1 - \dot{q}_0\rangle + \langle \partial_t\mathrm{D}_q\mathcal{E}_1 - \partial_t\mathrm{D}_q\mathcal{E}_0, q_1 - q_0\rangle \\ & + \sum_{i=0}^{1}\langle \mathrm{D}_q\mathcal{E}_{1-i} - \mathrm{D}_q\mathcal{E}_i + \mathrm{D}_q^2\mathcal{E}_i[q_i - q_{i-1}], \dot{q}_i\rangle.\end{aligned} \tag{6.11}$$

Taking the test functions $v \stackrel{\text{def}}{=} \dot{q}_{1-i}$ in (2.5) for $i = 0, 1$ and then adding the both inequalities, we obtain

$$\langle \mathrm{D}_q\mathcal{E}_1 - \mathrm{D}_q\mathcal{E}_0, \dot{q}_1 - \dot{q}_0\rangle \le 0. \tag{6.12}$$

Using (6.12) in (6.11), we have

$$\begin{aligned}\dot{\gamma}(t) & \le \langle \partial_t\mathrm{D}_q\mathcal{E}_1 - \partial_t\mathrm{D}_q\mathcal{E}_0, q_1 - q_0\rangle + \sum_{i=0}^{1}\langle \mathrm{D}_q\mathcal{E}_{1-i} - \mathrm{D}_q\mathcal{E}_i + \mathrm{D}_q^2\mathcal{E}_i[q_i - q_{i-1}], \dot{q}_i\rangle \\ & \le \|\partial_t\mathrm{D}_q\mathcal{E}_1 - \partial_t\mathrm{D}_q\mathcal{E}_0\|_{\mathcal{Q}'}\|q_1 - q_0\|_{\mathcal{Q}} + \sum_{i=0}^{1}\|\mathrm{D}_q\mathcal{E}_{1-i} - \mathrm{D}_q\mathcal{E}_i + \mathrm{D}_q^2\mathcal{E}_i[q_i - q_{i-1}]\|_{\mathcal{Q}'}\|\dot{q}_i\|_{\mathcal{Q}}.\end{aligned}$$

Then, Theorem 5.2 and Lemma 6.2 enable us to deduce

$$\dot{\gamma}(t) \le C_3\|q_1(t) - q_0(t)\|_{\mathcal{Q}}^2 \le \frac{C_3}{\kappa}\gamma(t) \ \text{ with } \ C_3 = 2(C_1 + C_2C_R).$$

Hence, the classical Gronwall's lemma and (6.10) lead to

$$\|q_1(t) - q_0(t)\|_{\mathcal{Q}}^2 \le \frac{1}{\kappa}\exp\left(\frac{C_3}{\kappa}t\right)\gamma(0). \tag{6.13}$$

Using $q_0(0), q_1(0) \in \mathcal{Q}_R$ and $\mathcal{E}(0,\cdot) \in C^{1,\mathrm{Lip}}(\mathcal{Q}_R;\mathbb{R})$ we have $\gamma(0) \le C\|q_0(0) - q_1(0)\|_{\mathcal{Q}}^2$ with $C > 0$, and the result follows. □

A Appendix: On the Souza-Auricchio model

We prove now that the assumptions on h introduced in Section 3 are satisfied for h_{SA} given in (2.6). More precisely, we establish the following lemma:

Lemma A.1 *Assume that $c_i \in C^2([\theta_{\min}, \theta_{\max}])$, $i = 1, 2, 3$, for all $\theta \in [\theta_{\min}, \theta_{\max}]$ and $c_i(\theta) > 0$. Then there exist positive constants C^h, $C^h_{\theta_j}$, $c^h_{\theta_j}$, C^h_θ, $C^h_{z\theta}$, C^h_z such that for all $t \in [0, T]$, $\theta \in [\theta_{\min}, \theta_{\max}]$, $z, \widehat{z} \in \mathbb{R}^{d\times d}_{\text{dev}}$, we have,*

$$h_{SA}(\cdot,\theta) \text{ is } \alpha_{h_{SA}}\text{-uniformly convex and belongs to } C^3(\mathbb{R}^{d\times d}_{\text{dev}}) \text{ with} \quad \text{(A.1a)}$$
$$\alpha_{h_{SA}} \stackrel{\text{def}}{=} \min\{c_1(\theta) : \theta \in [\theta_{\min}, \theta_{\max}]\},$$
$$h_{SA}(z,\theta) \geq C^h(|z|^2 - 1), \quad \text{(A.1b)}$$
$$\forall j = 1, 2: \ |\partial^j_\theta h_{SA}(z,\theta)| \leq C^h_{\theta_j}(h_{SA}(z,\theta) + c^h_{\theta_j}), \quad \text{(A.1c)}$$
$$\forall i = 0, 1: \ |\partial_\theta \partial^i_z h_{SA}(z,\theta) - \partial_\theta \partial^i_z h_{SA}(\widehat{z},\theta)| \leq C^h_{z\theta}(1 + |z| + |\widehat{z}|)|z - \widehat{z}|, \quad \text{(A.1d)}$$
$$\forall i = 0, 1, 2: \ |\partial^i_z h_{SA}(z,\theta) - \partial^i_z h_{SA}(\widehat{z},\theta)| \leq C^h_{z\theta}(1 + |z| + |\widehat{z}|)^{3-i}|z - \widehat{z}|, \quad \text{(A.1e)}$$
$$\forall i = 1, 2, 3 \ \forall j = 0, 1: \ |\partial^j_\theta \partial^i_z h_{SA}(z,\theta)| \leq C^h_z(1 + |z|)^{3-i-j}. \quad \text{(A.1f)}$$

Proof. Note that $h_{\text{SA}}(\cdot,\theta)$ is a sum of three non-negative convex and belonging to $C^3(\mathbb{R}^{d\times d}_{\text{dev}})$:

$$h_1(z,\theta) \stackrel{\text{def}}{=} c_1(\theta)\sqrt{\delta^2 + |z|^2}, \ h_2(z,\theta) \stackrel{\text{def}}{=} c_2(\theta)|z|^2, \ h_3(z,\theta) \stackrel{\text{def}}{=} \frac{1}{\delta}(|z| - c_3(\theta))^3_+.$$

Moreover, the quadratic term is α-uniformly convex and coercive. Hence (A.1a) and (A.1b) hold. Define for $i = 1, 2, 3$, $c'_i(\theta) \stackrel{\text{def}}{=} \partial_\theta c_i(\theta)$ and $c''_i(\theta) \stackrel{\text{def}}{=} \partial^2_\theta c_i(\theta)$. The estimates (A.1c) will result from the application of Young's inequality. First, we differentiate $h_{\text{SA}}(z,\theta)$ with respect to θ and we obtain easily the following inequality

$$|\partial_\theta h_{\text{SA}}(z,\theta)| \leq |c'_1(\theta)|\sqrt{\delta^2 + |z|^2} + |c'_2(\theta)||z|^2 + \frac{3}{\delta}|c'_3(\theta)(|z|^2 - c_3(\theta))^2_+|. \quad \text{(A.2)}$$

The last term on the right-hand side of (A.2) is estimated by Young's inequality and then (A.1c) for $j = 1$ follows.

We differentiate $\partial_\theta h_{\text{SA}}(z,\theta)$ with respect to θ and obtain the estimate

$$\begin{aligned} |\partial^2_\theta h_{\text{SA}}(z,\theta)| \leq\ & |c''_1(\theta)|\sqrt{\delta^2 + |z|^2} + |c''_2(\theta)||z|^2 + \frac{3}{\delta}|2c'_3(\theta)|(|z|^2 - c_3(\theta))_+ \\ & + \frac{3}{\delta}|c''_3(\theta)|(|z|^2 - c_3(\theta))^2_+. \end{aligned} \quad \text{(A.3)}$$

Once again, we use Young's inequality to estimate the last two terms on the right-hand side of (A.3) and obtain (A.1c) for $j = 2$.

We note that (A.1d) will be obtained by a simple calculus explained below and since it is quite a routine, we let the details to the reader. First, we define $\mu_i \stackrel{\text{def}}{=} \sup_{\theta\in[\theta_{\min},\theta_{\max}]}|c_i(\theta)|$ and $\eta_i \stackrel{\text{def}}{=} \sup_{\theta\in[\theta_{\min},\theta_{\max}]}|c'_i(\theta)|$, $i = 1, 2, 3$. On the one hand, using the previous notation, we obtain

$$\begin{aligned} |\partial_\theta h_{\text{SA}}(z,\theta) - \partial_\theta h_{\text{SA}}(\widehat{z},\theta)| \leq\ & \eta_1\left|\sqrt{\delta^2 + |z|^2} - \sqrt{\delta^2 + |\widehat{z}|^2}\right| + \eta_2\left||z|^2 - |\widehat{z}|^2\right| \\ & + \frac{3\eta_3}{\delta}\left|(|z| - c_3(\theta))^2_+ - (|\widehat{z}| - c_3(\theta))^2_+\right|, \end{aligned}$$

which implies that

$$|\partial_\theta h_{\mathrm{SA}}(z,\theta)-\partial_\theta h_{\mathrm{SA}}(\widehat{z},\theta)| \leq |z-\widehat{z}| \Big[\Big(\frac{\eta_1}{\delta^2}+\eta_2\Big)(|z|+|\widehat{z}|)+(2\eta_3+|z|+|\widehat{z}|)\Big]. \tag{A.4}$$

Then, the desired inequality (A.1d) for $i=0$ follows from (A.4). On the other hand, one can observe that

$$\begin{aligned}|\partial_\theta\partial_z h_{\mathrm{SA}}(z,\theta)-\partial_\theta\partial_z h_{\mathrm{SA}}(\widehat{z},\theta)| \leq \eta_1 &\left|\frac{z}{\sqrt{\delta^2+|z|^2}}-\frac{\widehat{z}}{\sqrt{\delta^2+|\widehat{z}|^2}}\right|+2\eta_2|z-\widehat{z}| \\ &+\frac{12\eta_3}{\delta}\big|z(|z|-c_3(\theta))_+-\widehat{z}(|\widehat{z}|-c_3(\theta))_+\big|,\end{aligned}$$

which leads to

$$|\partial_\theta\partial_z h_{\mathrm{SA}}(z,\theta)-\partial_\theta\partial_z h_{\mathrm{SA}}(\widehat{z},\theta)| \leq |z-\widehat{z}|\Big[\frac{\eta_1}{\delta}\Big(1+\frac{1}{\delta}\Big)(1+|z|+|\widehat{z}|)+2\eta_2+\frac{12\eta_3}{\delta}(1+|z|+|\widehat{z}|)\Big].$$

Then, one easily deduces (A.1d) for $i=1$ by using the latter inequality.

One can show easily that

$$|h_{\mathrm{SA}}(z,\theta)-h_{\mathrm{SA}}(\widehat{z},\theta)| \leq \Big(\mu_1+\frac{\mu_2}{2\delta}\Big)(|z|+|\widehat{z}|)|z-\widehat{z}|+(|z|+|\widehat{z}|)^2|z-\widehat{z}|,$$

which gives (A.1e) for $i=0$. We may also observe that

$$|\partial_z h_{\mathrm{SA}}(z,\theta)-\partial_z h_{\mathrm{SA}}(\widehat{z},\theta)| \leq c_\delta\mu_1(1+|z|+|\widehat{z}|)|z-\widehat{z}|+\mu_2|z-\widehat{z}|+\frac{6}{\delta}(1+|z|+|\widehat{z}|)|z-\widehat{z}|,$$

where $c_\delta \overset{\mathrm{def}}{=} \max\big(\frac{1}{\delta},\frac{1}{\delta^2}\big)$. Then (A.1e) for $i=1$ follows from the latter estimate. Let us remark now that there exists $C>0$ such that

$$|\partial_z^2 h_{\mathrm{SA}}(z,\theta)-\partial_z^2 h_{\mathrm{SA}}(\widehat{z},\theta)| \leq Cc_\delta\big(\mu_1(|z|+|\widehat{z}|)|z-\widehat{z}|+(1+|z|+|\widehat{z}|)|z-\widehat{z}|\big),$$

which implies (A.1e) for $i=2$.

Finally, one can easily obtain (A.1f), the verification is left to the reader. □

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