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Infinite-Dimensional Hyperbolic Sets and Spatio-Temporal Chaos in Reaction Diffusion Systems in $\mathbb{R}^n$*

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The paper is devoted to a rigorous construction of a parabolic system of partial differential equations which displays space–time chaotic behavior in its global attractor. The construction starts from a periodic array of identical copies of a temporally chaotic reaction-diffusion system (RDS) on a bounded domain with Dirichlet boundary conditions. We start with the case without coupling where space–time chaos, defined via embedding of multidimensional Bernoulli schemes, is easily obtained. We introduce small coupling by replacing the Dirichlet boundary conditions by strong absorption between the active islands. Using hyperbolicity and delicate PDE estimates we prove persistence of the embedded Bernoulli scheme. Furthermore we smoothen the nonlinearity and obtain a RDS which has polynomial interaction terms with space and time-periodic coefficients and which has a hyperbolic invariant set on which the dynamics displays spatio-temporal chaos. Finally we show that such a system can be embedded in a bigger system which is autonomous and homogeneous and still contains space–time chaos. Obviously, hyperbolicity is lost in this step.

KEY WORDS: Hyperbolicity; Bernoulli shifts; space–time chaos; global attractor; weighted Sobolev spaces.

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0. INTRODUCTION

Even relatively simple dynamical systems generated by ordinary differential equations (ODEs) can generate rather complicated chaotic dynamics, see e.g. [20] and references therein. This dynamics becomes much more complicated in the case of dynamical systems extended in space, e.g., generated by dissipative partial differential equations (PDEs) in large and unbounded domains, due to the formation of spatially chaotic patterns. More generally, such systems may display interactions between spatially and temporally chaotic modes which leads to the so-called spatio-temporal chaos. One of the most challenging problems in this field is the one of turbulence which displays statistical behavior in temporal and spatial directions, whose correlations decay with distance in space and time, see e.g. [11, 19, 25, 26, 40].

However, despite the fact, that there are many statistical approaches to turbulence and spatio-temporal chaos, there seem to be very few mathematically rigorous results concerning the nature of spatio-temporal chaos in deterministic systems. Indeed, one of the few known mathematical descriptions of that phenomenon was suggested in [12], see also [3, 35] and references therein. There, a spatially discrete system is constructed such that it admits an infinite-dimensional hyperbolic, invariant subset Γ of its phase space. This set is homeomorphic to the multidimensional Bernoulli scheme $\{0,1\}^{\mathbb{Z}^{n+1}}$ and the Bernoulli shifts are conjugated to the spatio-temporal shifts on the hyperbolic set Γ. Unfortunately, the existence of such a hyperbolic set was rigorously verified only for some very special classes of *lattice* dynamical systems and its existence for the dynamical systems in continuous media was a long-standing open question.

In the present paper, we give a positive answer on this question. To be more precise, we restrict ourselves to consider extended dynamical systems generated by the systems of reaction-diffusion equations, or shortly reaction-diffusion systems (RDSs) in the full unbounded, physical space $\Omega = \mathbb{R}^n$:

$$\partial_t u = \gamma \Delta_x u - f(u) \quad \text{in } \mathbb{R}^n, \quad \text{and } u\big|_{t=0} = u_0. \tag{0.1}$$

Here $u(t,x) = (u^1(t,x), \dots, u^k(t,x))$ is an unknown vector-valued function, $\gamma > 0$ is a fixed diffusion coefficient, Δ_x is a Laplacian with respect to $x = (x_1, \dots, x_n) \in \Omega$ and $f: \mathbb{R}^k \to \mathbb{R}^k$ is a given smooth, nonlinear interaction function.

It is well-known that, under natural assumptions on the nonlinearity f Eq. (0.1) has a unique global solution in an appropriate phase space Φ (usually, $\Phi = L^\infty(\mathbb{R}^n)$ or the so-called uniformly local spaces $\Phi = L^2_b(\mathbb{R}^n)$, see Section 1 for the definitions) and, consequently, it generates a (dissipative) semigroup $\{S_t\colon t\geq 0\}$ in Φ via

$$S_t u_0 = u(t), \quad u \ \text{ solves (0.1) with } \ u(0) = u_0. \tag{0.2}$$

It is also well-known that, in many cases, the asymptotic behavior of the trajectories of that semigroup as $t\to\infty$ can be described in terms of the so-called global attractor $\mathcal{A}\subset\Phi$ which is, by definition, a compact (in the appropriate local topology) invariant set which attracts as $t\to\infty$ the images of all bounded subsets of Φ. Thus, on the one hand the attractor $\mathcal{A}$ (if it exists) captures all of the nontrivial dynamics of the system considered and, on the other hand, it is usually essentially smaller than the initial phase space Φ (see [7,8,16,32,40,47,48], and the references therein).

In particular, in the case of *bounded* domains Ω, the above attractor $\mathcal{A}$ is usually finite-dimensional (in the sense of Hausdorff and fractal dimension). Therefore, in spite of the infinite-dimensionality of the initial phase space, the reduced limit dynamics on the attractor is finite-dimensional and can be effectively studied using the methods of the classical theory of dynamical systems. Thus, the infinite-dimensionality of the initial phase space plays here the role of (maybe essential) technical difficulty which, however cannot produce new types of dynamical complexity which are not observed in the finite-dimensional theory.

The situation changes drastically when the domain Ω becomes unbounded (e.g. $\Omega=\mathbb{R}^n$). In this case, the attractor is usually infinite-dimensional, see [8,9,16,14,33]. Thus, we do not have any finite-dimensional reduction and truely infinite-dimensional dynamics of a much "higher level of complexity" can appear. Another principal difference to the case of bounded domains is that system (0.1) has now not only the temporal "unbounded direction", but also the spatial one which may lead to complicated *spatial* structures, namely to so-called spatial chaos, see [2, 6,22,27,28,34,38]. However, only nontrivial interaction of temporal chaos and spatial chaos will be called *spatio-temporal* chaos, and this is the topic of the present work.

To make this phenomenon more precise, we introduce the group of spatial shifts $\{T_h\colon h\in\mathbb{R}^n\}$ acting on the phase space Φ via $(T_h u_0)(x) := u_0(x+h)$. Then, since the model equation (0.1) is spatially homogeneous this group acts on the attractor $\mathcal{A}$ as well and its action obviously commutes with the semigroup S_t associated with temporal evolution. Therefore, an extended $(n+1)$-parametrical semigroup $\{\mathbb{S}_{(t,h)}\colon t\in\mathbb{R}_+,\ h\in\mathbb{R}^n\}$ acts on the attractor $\mathcal{A}$:

$$\mathbb{S}_{(t,h)}\colon \mathcal{A}\to\mathcal{A}, \quad \mathbb{S}_{(t,h)} := S_t\circ T_h. \tag{0.3}$$

Thus, following [45,48], the extended spatio-temporal semigroup (0.3) can be considered as a dynamical system with multidimensional "time" (t,h) on the attractor which is responsible for all spatio-temporal dynamical effects arising in system (0.1). Consequently, the study of the spatio-temporal chaos in (0.1) is equivalent to the study of the dynamical properties of that semigroup restricted to $\mathcal{A}$.

However we note that the study of the dynamics of (0.3) is a highly nontrivial problem; and at present its complete description is not available even for the simplest examples of (0.1) (such as e.g. Chafee-Infante equation in $\mathbb{R}^1$: $u_t = u_{xx} + u - u^3$). Nevertheless, a number of rather essential results concerning the general properties of that system has been recently obtained. In particular, a natural generalization of the finite-dimensionality of global attractors to the case of unbounded domains were formulated in terms of Kolmogorov's ϵ-entropy which allows us to measure the "size" of infinite-dimensional sets in functional spaces, see [13,14,42,43,46] and Appendix B below. The obtained estimates of the ε-entropy of the global attractor $\mathcal{A}$ allowed, for instance, to verify (see [48]) that the *topological* entropy of the action of $\mathbb{S}_{(t,h)}$ on the attractor $\mathcal{A}$ is finite:

$$h_{\mathrm{top}}(\mathbb{S}_{(t,h)}, \mathcal{A}) < \infty, \tag{0.4}$$

see Appendix B. On the other hand, for the particular case of gradient nonlinearity f, i.e., $f = \mathrm{D}_u F(u)$ for some $F\colon \mathbb{R}^k \to \mathbb{R}$, this topological entropy is known to equal 0, see [48]. We recall that, in contrast to the case of bounded domains, the gradient structure does not give a global Lyapunov function for equation (0.1), so $h_{\mathrm{top}}(\mathbb{S}_{(t,h)}, \mathcal{A}) = 0$ is not immediate.

It is also worth to note that the extended dynamical system (0.3) possesses a natural family of subsemigroups $\mathbb{S}^{V_k}_{(t,h)}$ generated by restrictions of the argument (t,h) of (0.3) to various k-dimensional hyperplanes V_k of the space–time $\mathbb{R}^{n+1}$ whose dynamical investigation can be essentially simpler than for the initial "whole" semigroup (0.3). The most studied is the case $V_n := \mathbb{R}^n_x$ associated with the *spatial* dynamical system $\{T_h\colon h \in \mathbb{R}^n\}$ and spatial chaos, see [1,2,6,9,15,21,22,28,34,38,47] and references therein. This means that even the restriction of spatial dynamical system to the set $\mathcal{E}$ of all (bounded) equilibria of problem (0.1) which obviously solve the *elliptic* equation in $\mathbb{R}^n$

$$\gamma \Delta_x u_0 - f(u_0) = 0 \tag{0.5}$$

possesses a very reach and nontrivial chaotic dynamics. So, a family of multibump solutions of has been constructed in [38] by variational methods starting from a single hyperbolic bump solution of (0.5) and using small space-periodic perturbation in order to kill the neutral foliation. In a fact this family gives a homeomorphic embedding of the multidimensional Bernoulli scheme $\mathcal{M}^n := \{0, 1\}^{\mathbb{Z}^n}$ to the (discrete) spatial dynamical system of the perturbed spatially periodic equation. Analogous embeddings for the attractor of spatially periodic RDSs were obtained in [9] under weaker assumptions on the nonlinearity.

We now recall that, in contrast to the classical dynamical systems generated by ODEs, the action of the spatial dynamical system $\{T_h : h \in \mathbb{R}^n\}$ on the attractor $\mathcal{A}$ usually has *infinite* topological entropy (this fact has been established in [47] under very weak assumption that (0.1) possesses at least one spatially homogeneous exponentially unstable equilibrium) although the topological entropy of Bernoulli scheme $\mathcal{M}^n$ is finite. So, Bernoulli schemes with finite number of symbols are not sufficient for modeling of spatial chaos. Therefore, a Bernoulli scheme $\mathcal{M}^n_\infty := [0, 1]^{\mathbb{Z}^n}$ with infinite number of symbols (here $[0, 1]$ is a segment of $\mathbb{R}^1$ in contrast to the two-point set $\{0, 1\}$ involved into the definition of the standard Bernoulli scheme $\mathcal{M}^n$) were used in [47] in order to clarify the nature of chaos in spatial dynamical systems. Moreover, it is shown there that there exists a topological invariant (the so-called mean topological dimension) which is always finite for spatial dynamics and strictly positive for the Bernoulli scheme $\mathcal{M}^n_\infty$. This description has been extended in [48] from spatial dynamical system $\{T_h : h \in \mathbb{R}^n\}$ to all n-parametrical semigroups $\mathbb{S}^{V_n}_{(t,h)}$ generated by *n-dimensional* hyperplanes V_n of the space–time. This result shows, in particular, that the topological entropy of purely temporal evolution semigroup $\{S_t : t \in \mathbb{R}_+\}$ is also usually infinite

$$h_{\text{top}}(S_t, \mathcal{A}) = \infty.$$

It is however worth to emphasize that the embeddings of the Bernoulli schemes $\mathcal{M}^n_\infty$ mentioned above are based on the infinite-dimensional unstable manifolds technique (in a fact, the image of $\mathcal{M}^n_\infty$ in the attractor $\mathcal{A}$ belongs to the unstable manifold of a spatially homogeneous equilibrium where the direction orthogonal to the hyperplane V_n is interpreted as "time"). Thus, that approach gives an adequate model only for "n-directional" space–time chaos (for all directions belonging to the fixed hyperplane V_n in space–time) in n-parametrical subsemigroup $\mathbb{S}^{V_n}_{(t,h)}$ and is not applicable for clarifying the nature of the "complete" $(n+1)$-directional space–time chaos arising in (0.3). Moreover, to the best of our knowledge, there were no reasonable models for that $(n+1)$-directional space–time chaos for the case of dynamical systems in *continuous* media generated by PDEs. In particular, it was not known whether or not the topological entropy of the extended semigroup $\mathbb{S}_{(t,h)}$ can be strictly positive.

In contrast to that, for the case of *discrete* media, more or less adequate model for the "complete" space–time chaos has been suggested by Sinai and Bunimovich. Roughly speaking, this model consists of infinitely many chaotic oscillators situated at every node of a grid $\mathbb{Z}^n$ coupled by sufficiently weak interaction. If every chaotic oscillator contains a hyperbolic set Γ_0, then without interaction the whole system has a hyperbolic set $\Gamma := (\Gamma_0)^{\mathbb{Z}^n}$ and, according to the structure stability theorem, this set preserves under sufficiently small coupling. In particular, if the initial hyperbolic set Γ_0 is a one-dimensional Bernoulli scheme $\mathcal{M}^1 = \{0, 1\}^{\mathbb{Z}}$, then the obtained hyperbolic set is $(n+1)$-dimensional Bernoulli scheme $\Gamma = \mathcal{M}^{n+1} = \{0, 1\}^{\mathbb{Z}^{n+1}}$ and Bernoulli shifts on it is naturally conjugated with spatio-temporal dynamics. Thus, according to this model, the spatio-temporal chaos is illustrated by Bernoulli shifts on the $(n+1)$-dimensional Bernoulli scheme with a finite number of symbols. Associated invariant measures may also be introduced, see [3, 12, 36].

The main goal of the present paper is to extend the Sinai–Bunimovich construction to the case of *continuous* media and obtain the analogous description of the space–time chaos in the RDS (0.1) and associated extended semigroup (0.3) acting on its attractor $\mathcal{A}$. In order to do so, we start from the special *space–time periodic* RDS:

$$\partial_t u = \gamma \Delta_x u - f_\lambda(t, x, u) \quad \text{in } \in \mathbb{R}^n, \tag{0.6}$$

where the nonlinearity f_λ has the following structure: there exists a smooth bounded domain $\Omega_0 \Subset (0, 1)^n$ such that, for every $x \in [0, 1]^n$ we have

$$f_\lambda(t, x, u) := \begin{cases} f(t, u) & \text{for } x \in \Omega_0, \\ \lambda u & \text{for } x \in [0, 1]^n \backslash \Omega_0, \end{cases} \tag{0.7}$$

where $f(t, u)$ is a given function (which is assumed 1-periodic with respect to t) and $\lambda \gg 1$ is a large parameter. Then we extend function (0.7) space-periodically from $[0, 1]^n$ to all $x \in \mathbb{R}^n$. Thus, we have a periodic grid of "islands" $\Omega_l := l + \Omega_0$, $l \in \mathbb{Z}^n$ where our nonlinearity f_λ coincides with $f(t, u)$ and can generate nontrivial dynamics. These islands are separated from each other by the "ocean" $\Omega_- := \mathbb{R}^n \backslash (\cup_{l \in \mathbb{Z}^n} \Omega_l)$ where we have the strong absorption provided by the nonlinearity $f_\lambda(t, x, u) \equiv \lambda u$.

It is intuitively clear that, for sufficiently large absorption coefficient λ, the solutions u of equation (0.6) should be small in the absorption domain Ω_- and, consequently, the interaction between the islands is also expected to be small and the dynamics inside of the islands will be "almost-independent". Thus, if the RDS in Ω_0

$$\partial_t v = \gamma \Delta_x v - f(t, v) \quad \text{in } \Omega_0, \quad v = 0 \quad \text{on } \partial\Omega_0, \tag{0.8}$$

which describes the limit dynamics inside of one “island” as $\lambda = \infty$, possesses a hyperbolic set Γ_0, then, according to the structural-stability principle, the whole system (0.6) should have a hyperbolic set homeomorphic to $(\Gamma_0)^{\mathbb{Z}^n}$ if the absorption parameter λ is large enough. Moreover, if, in addition, the initial hyperbolic set Γ_0 is homeomorphic to the Bernoulli scheme $\{0,1\}^{\mathbb{Z}}$, then system (0.6) will contain an $(n+1)$-dimensional Bernoulli scheme $\{0,1\}^{\mathbb{Z}^{n+1}} \sim (\{0,1\}^{\mathbb{Z}})^{\mathbb{Z}^n}$ in complete analogy with the Sinai-Bunimovich lattice model.

In turns, in order to construct an example of system (0.8) with the required hyperbolic set, we transform it as follows:

$$\partial_t v = \gamma(\Delta_x v - \lambda_1 v) - \tilde{f}(t, v),$$

where $\gamma \gg 1$ is a large parameter, $\lambda_1 < 0$ is the first eigenvalue of the Laplacian in Ω_0 with Dirichlet boundary conditions, and $\tilde{f}$ is independent of γ. Then, for sufficiently large γ, this system possesses a Lyapunov–Schmidt reduction to the finite-dimensional subspace generated by the first eigenvector of the Laplacian, see Appendix A. Thus, for sufficiently large γ the associated dynamics will be close to some reduced ODE in that space. Moreover, we show that every ODE with polynomial nonlinearity can be obtained in such way. Since the existence of a hyperbolic sets in ODEs with polynomial nonlinearities is well-known, this gives the required example, see Appendix A for details.

The following theorem, which gives a mathematical justification of the above heuristic scheme, is the main result of the paper.

Theorem 0.1. *Let the time-periodic function* $f\,(f(t+1, v) \equiv f(t, v))$ *satisfy some regularity and dissipativity assumptions which guarantee the existence of a global attractor* $\mathcal{A}_0$ *for equation* (0.8). *Assume also that there exists a hyperbolic set* Γ_0 *of* (0.8) *that is homeomorphic to the Bernoulli scheme* $\{0,1\}^{\mathbb{Z}}$. *Then, there exists a* $\lambda_0 \gg 1$ *such that, for every* $\lambda > \lambda_0$, *there exists a homeomorphic embedding* κ_λ *of the* $(n+1)$*-dimensional Bernoulli scheme* $\mathcal{M}^{n+1} = \{0,1\}^{\mathbb{Z}^{n+1}}$ *to the attractor* $\mathcal{A}_{per}$ *of problem* (0.6) *such that*

$$\mathbb{S}_{(l_0,l')} \circ \kappa_\lambda = \kappa_\lambda \circ \mathcal{T}_{(l_0,l')}, \quad l_0 \in \mathbb{N}, \;\; l' \in \mathbb{Z}^n, \tag{0.9}$$

where $\mathcal{T}_{(l_0,l')}$ *is the* $(n+1)$*-parametrical group of Bernoulli shifts on* $\mathcal{M}^{n+1}$ *and* $\{\mathbb{S}_{(l_0,l')} : l_0 \in \mathbb{N},\ l' \in \mathbb{Z}^n\}$ *is a discrete analog of the extended spatio-temporal DS* (0.3) *acting on the attractor* $\mathcal{A}_{per}$. *In particular, the topological entropy of that system is strictly positive, viz.,*

$$h_{\text{top}}(\mathbb{S}_{(l_0,l')}, \mathcal{A}) \ge h_0 > 0. \tag{0.10}$$

It is worth to emphasize that, although the nonlinearity f_λ in the initial equation (0.6) is discontinuous with respect to x, approximating this nonlinearity by trigonometrical polynomials with respect to (t, x) and using again the structural-stability principle, we obtain the analog of Theorem 0.1 with the nonlinearity $\widetilde{f}$ *analytic* with respect to all variables, see Proposition 5.4. Moreover, embedding the attractor $\mathcal{A}_{per}$ of the space–time periodic RDS to the attractor $\mathcal{A}$ of a larger RDS of the form (0.1), we construct the embedding of the Bernoulli scheme $\{0, 1\}^{\mathbb{Z}^{n+1}}$ to the spatio-temporal DS acting on the attractor of autonomous and spatially homogeneous RDS of the form (0.1).

Corollary 0.2. *There exists a RDS of the form* (0.1) *with polynomial nonlinearity f such that its attractor possesses a homeomorphic embedding $\bar{\kappa}$ of the Bernoulli scheme $\{0, 1\}^{\mathbb{Z}^{n+1}}$ satisfying the conjugacy relations* (0.9).

To conclude we note that, passing from a periodic to an autonomous system, we lose the hyperbolicity (in contrast to Theorem 0.1, now the image $\bar{\kappa}(\{0, 1\}^{\mathbb{Z}^{n+1}})$ is no longer a hyperbolic subset of the attractor $\mathcal{A}$). Nevertheless, Corollary 0.2 shows that $(n + 1)$-dimensional Bernoulli schemes with a finite number of symbols can be used in order to clarify the nature of "true spatio-temporal chaos" arising in spatially homogeneous media as well. In particular, Corollary 0.2 provides an example of a RDS of the form (0.1) with strictly positive spatio-temporal topological entropy.

The structure of the paper is as follows. In Section 1 we introduce some classes of weighted functional spaces and formulate several regularity results for the boundary value problems for the heat equations in those spaces which are needed for the subsequent structural-stability analysis. In Section 2 we recall the definitions of hyperbolic sets adopted to the *infinite-dimensional* case and obtain some preliminary results for the system (0.6) with $\lambda = \infty$. In Section 3 we establish several auxiliary results which allow us to verify that system (0.6) for $\lambda \gg 1$ is indeed close in the appropriate functional space to the uncoupled $\mathbb{Z}^n$-array of Eq. (0.8). Based on these results we establish, in Section 4, Theorem 0.1 using a suitable modification of the structural-stability theory for hyperbolic sets. The example of an *autonomous and spatially homogeneous* RDS announced in Corollary 0.2 is constructed in Section 5. Finally, for the convenience of the reader, we explain in Appendix A how to construct a RDS (0.8) in a bounded domain Ω_0 with Dirichlet boundary conditions which contains a hyperbolic set homeomorphic to $\{0, 1\}^{\mathbb{Z}}$. In Appendix B we recall the definitions and main results concerning Kolmogorov's ϵ-entropy and the topological entropy of attractors of dissipative systems in unbounded domains.

After this work was finished another method for the construction of space–time chaos has been developed. It relies on the construction of an infinitedimensional center manifold that describes the weak interaction of exponentially decaying pulses, see [49].

1. FUNCTIONAL SPACES AND LINEAR PARABOLIC REGULARITY THEOREMS

In this section we introduce some classes of weighted Sobolev spaces and formulate the corresponding parabolic regularity theorems which will be used throughout the paper. Our final goal is to construct examples of RDS in $\mathbb{R}^n$ with spatio-temporal chaotic behavior but for this construction we will essentially use the corresponding equations in unbounded domains $\Omega \neq \mathbb{R}^n$. That is why we start our consideration from the class of admissible (uniformly regular) unbounded domains in $\mathbb{R}^n$ (see, e.g., [10, 16]).

Here and below B_x^R denotes the open ball in $\mathbb{R}^n$ with center x and radius R.

Definition 1.1. A domain $\Omega \subset \mathbb{R}^n$ is called C^N-*regular*, if there exist radii $0 < R_0 < R_1 < R_2$ and a constant K such that the following holds:

For each $x_0 \in \Omega$ there exists a domain $V_{x_0} \subset \Omega$ with

$$(B_{x_0}^{R_0} \cap \Omega) \subset V_{x_0} \subset (B_{x_0}^{R_1} \cap \Omega) \tag{1.1}$$

and a C^N-diffeomorphism θ_{x_0}: $B_0^2 \to B_0^{R_2}$ such that $x_0 + \theta_{x_0}(B_0^1) = V_{x_0}$ and

$$\|\theta_{x_0}\|_{C^N} + \|\theta_{x_0}^{-1}\|_{C^N} \leqslant K. \tag{1.2}$$

The constant K is called the C^N-regularity constant of the domain Ω.

For all results formulated below C^2-regularity will be sufficient, so, for simplicity, we will write in the sequel "regular domain" instead of C^2-regular domain and "regularity constant" instead of C^2-regularity constant.

For bounded domains Ω the conditions (1.1) and (1.2) are equivalent to the condition that the boundary $\partial\Omega$ is a smooth manifold. But for unbounded domains smoothness of the boundary is not sufficient to obtain the regular structure of Ω when $|x| \to \infty$ since uniformity with respect to $x_0 \in \Omega$ of the smoothness conditions is required.

Now we introduce the class of admissible weight functions.

Definition 1.2. A function $\phi \in C_{\text{loc}}(\mathbb{R}^n)$ is called a weight function with the (exponential) growth rate $\mu \geqslant 0$, if there exists $C_\phi > 0$ such that

$$\phi(x+y) \leqslant C_\phi e^{\mu|x|}\phi(y), \quad \phi(x) > 0, \text{ for every } x, y \in \mathbb{R}^n. \tag{1.3}$$

Remark 1.3. It is not difficult to deduce from (1.3) that

$$\phi(x+y) \geqslant C_\phi^{-1} e^{-\mu|x|}\phi(y) \tag{1.4}$$

is also satisfied for every $x, y \in \mathbb{R}^n$. The estimates (1.1) and (1.3) imply particularly that

$$C_\phi^{-1} e^{-\mu R}\phi(x) \leqslant \sup_{|y|\leqslant R} \phi(x+y) \leqslant C_\phi e^{\mu R}\phi(x). \tag{1.5}$$

The typical examples of that weight functions are the following:

$$\phi_{\varepsilon,x_0}(x) = e^{-\varepsilon|x-x_0|}, \quad \varepsilon \in \mathbb{R}, \ \ x_0 \in \mathbb{R}^n. \tag{1.6}$$

Evidently these weights have the growth rate $|\varepsilon|$ and satisfy (1.3) uniformly with respect to $x_0 \in \mathbb{R}^n$ (i.e., the constant $C_{\phi_{\varepsilon,x_0}}$ in (1.3) is independent of x_0). We will mainly use below exponentially decaying weight functions (1.6) with sufficiently small positive ε (or their smooth analogs (1.20)), although the exponentially growing weights (with negative ε) will be also usefull, see e.g. Corollary 1.16.

Now we are in a position to introduce several classes of weighted Sobolev spaces in unbounded domains Ω.

Definition 1.4. Let $\Omega \subset \mathbb{R}^n$ be a regular (unbounded) domain in $\mathbb{R}^n$ and let ϕ be a weight function with the growth rate μ. For $p \in [1, \infty)$ define the space

$$\mathrm{L}^p_\phi(\Omega) := \{ u \in \mathrm{L}^r_{\mathrm{loc}}(\Omega) \colon \|u\|^p_{\mathrm{L}^p_\phi(\Omega)} := \int_\Omega \phi(x)|u(x)|^p\,dx < \infty \}.$$

Analogously the weighted Sobolev space $\mathrm{W}^{l,p}_\phi(\Omega)$, $l \in \mathbb{N}$, is defined as the space of functions $u \in \mathrm{L}^p_\phi(\Omega)$ whose distributional derivatives up to order l inclusively belong to $\mathrm{L}^p_\phi(\Omega)$.

We define also another class of weighted Sobolev spaces

$$\mathrm{W}^{l,p}_{\mathrm{b},\phi}(\Omega) := \{ u \in \mathrm{W}^{l,p}_{\mathrm{loc}}(\Omega) \colon \|u\|^p_{\mathrm{W}^{l,p}_{\mathrm{b},\phi}(\Omega)} := \sup_{x_0\in\Omega} \phi(x_0)\|u\|^p_{\mathrm{W}^{l,p}(\Omega\cap B^1_{x_0})} < \infty \}.$$

Here b stands for "bounded", and for $\phi \equiv 1$ we write $\mathrm{W}^{l,p}_{\mathrm{b}}$ instead of $\mathrm{W}^{l,p}_{\mathrm{b},1}$.

Let us recall shortly several important properties of the introduced spaces, see, e.g., [16] or [47] for details.

Proposition 1.5. *Let ϕ be a weight function with the growth rate $\mu \geqslant 0$.*

1. Then for every $\varepsilon > \mu$ *there exists a constant* C *(just depending on* ε, μ *and* C_ϕ *from* (1.3)*) such that for every domain* $\Omega\subset\mathbb{R}^n$ *every* $q\in[1,\infty]$ *and every* $u\in L^p_\phi(\Omega)$ *the following estimate is valid:*

$$\left(\int_\Omega \phi(x_0)^q \left(\int_\Omega e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\right)^q dx_0\right)^{1/q} \leqslant C\int_\Omega \phi(x)|u(x)|^p\,dx. \quad (1.7)$$

2. On $L^\infty_\phi(\Omega)$ *the following analog of the estimate* (1.7) *is valid:*

$$\sup_{x_0\in\Omega}\left\{\phi(x_0)\sup_{x\in\Omega}\{e^{-\varepsilon|x-x_0|}|u(x)|\}\right\}\leqslant C\sup_{x\in\Omega}\{\phi(x)|u(x)|\}. \quad (1.8)$$

The proof of this proposition can be found in [16] or [46].

Proposition 1.6. *Let* Ω *be a regular domain, let* ϕ *be the weight function with exponential growth rate, and let* R *be a positive number. Then, there exist constants* $0<c_1<C_1$ *(depending on the regularity constant* $K(\Omega)$ *and* μ *and* C_ϕ *from* (1.3)*) such that for all* $u\in L^P_\phi(\Omega)$ *the following estimates are valid:*

$$c_1\int_\Omega \phi(x)|u(x)|^p\,dx\leqslant \int_\Omega\phi(x_0)\int_{\Omega\cap B^R_{x_0}}|u(x)|^p\,dx\,dx_0\leqslant C_1\int_\Omega\phi(x)|u(x)|^p\,dx. \quad (1.9)$$

The proof of this Proposition is given in [16].

Corollary 1.7. *Let the assumptions of Proposition 1.6 hold. Then, for* $R>0$ *an equivalent norm in the weighted Sobolev space* $W^{l,p}_\phi(\Omega)$ *is given by the following expression:*

$$\|u\|_{W^{l,p}_\phi(\Omega)}:=\left(\int_\Omega\phi(x_0)\|u,\Omega\cap B^R_{x_0}\|^p_{l,p}\,dx_0\right)^{1/p}. \quad (1.10)$$

Particularly, the norms (1.10) *are equivalent for different* $R>0$.

In the sequel we need also weighted Sobolev spaces with fractional derivatives $s\in\mathbb{R}_+$ (not only $s\in\mathbb{Z}_+$). Recall (see [41] for details) that the norm in the space $W^{s,p}(V)$, $s=[s]+\sigma$, $0<\sigma<1$, $[s]\in\mathbb{Z}_+$ can be given by the following expression

$$\|u\|^p_{W^{s,p}(V)}=\|u\|^p_{W^{[s],p}(V)}+\sum_{|\alpha|=[s]}\int_{x\in V}\int_{y\in V}\frac{|D^\alpha u(x)-D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}}\,dxdy. \quad (1.11)$$

Moreover, the space $W^{-s,p}(V)$, $1<p<\infty$, is usually defined as a dual space for $W^{s,q}_0(V)$, where $\frac{1}{p}+\frac{1}{q}=1$, i.e. the norm in this space is given via

$$\|u\|_{W^{-s,p}(V)}:=\sup_{\psi\in C^\infty_c(V)}\frac{\langle u,\psi\rangle}{\|\psi\|_{W^{s,p}(V)}}\quad\text{where }\langle u,\psi\rangle:=\int_V u(x)\psi(x)\,dx. \quad (1.12)$$

It is not difficult to prove that for any regular V and for any $s\in\mathbb{R}$, $1<p<\infty$,

$$c_1\|u\|^p_{W^{s,p}(V)}\leqslant\int_{x_0\in V}\|u\|^p_{W^{s,p}(V\cap B^R_{x_0})}\,dx_0\leqslant C_1\|u\|^p_{W^{s,p}(V)}. \tag{1.13}$$

This justifies the following definition.

Definition 1.8. For $s\in\mathbb{R}$ define the space $W^{s,p}_\phi(\Omega)$ by the norm (1.10) where the integer l is replaced by s.

It is not difficult to check that these norms are also equivalent for different $R>0$ and consequently the definition makes sense.

The following proposition admits to estimate the $W^{s,p}_b$-norm via the corresponding weighted Sobolev norms.

Proposition 1.9. Let $s\in\mathbb{R}$, $1<p<\infty$, and let ϕ be a weight function with the growth rate $0\leqslant\mu<\varepsilon$. Then, there exist constants $0\leqslant c_1\leqslant C_1$ such that

$$\begin{aligned}c_1\|u\|^p_{W^{s,p}_{b,\phi}(\Omega)}&\leqslant\sup_{x_0\in\Omega}\left\{\phi(x_0)\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|u\|^p_{W^{s,p}(\Omega\cap B^1_{x_0})}\,dx\right\}\\&\leqslant C_1\|u\|^p_{W^{s,p}_{b,\phi}(\Omega)}\quad\text{for all } u\in W^{s,p}_{b,\phi}(\Omega).\end{aligned} \tag{1.14}$$

For the proof of this corollary see [46].

In order to handle elliptic and parabolic boundary problems with nonhomogeneous boundary conditions we need also weighted Sobolev spaces on the boundary $\partial\Omega$ of regular unbounded domain Ω.

Definition 1.10. Analogously to the Definition 1.4, we define weighted Sobolev spaces of functions defined on the boundary $\partial\Omega$. For instance the weighted space $W^{s,p}_\phi(\partial\Omega)$, $s\in\mathbb{R}$, $1<p<\infty$, is defined by the following norm:

$$\|u_0\|^p_{W^{s,p}_\phi(\partial\Omega)}:=\int_{\partial\Omega}\phi(\gamma)\|u_0\|^p_{W^{s,p}(\partial\Omega\cap B^1_\gamma)}\,d\gamma,$$

where $\phi(\gamma)$ is a restriction of the weight function $\phi(x)$, $x\in\mathbb{R}^n$ to the boundary ($\gamma\in\partial\Omega$), and the spaces $W^{s,p}_{b,\phi}(\partial\Omega)$ are defined analogously.

It is known (see, e.g., [45]) that the assertions of Propositions 1.5–1.9 remain valid for the spaces of distributions on the boundary $\partial\Omega$ as well.

For parabolic systems we now introduce anisotropic Sobolev spaces of functions defined on $\mathbb{R}\times\Omega$ or $\mathbb{R}\times\partial\Omega$. Denote by $W^{(l_1,l_2),q}([T,T+1]\times V)$ the classical Sobolev–Slobodetskij space of functions which have the t-derivatives up to the order l_1 and x-derivatives up to the order l_2

belonging to L^q (see, e.g., [24]). Recall that for the case of integer $l_i \geq 0$ the norm in this space is defined by

$$\|u\|^q_{W^{(l_1,l_2),q}([T,T+1]\times V)} := \|\partial_t^{l_1} u\|^q_{L^q([T,T+1]\times V)} + \|D_x^{l_2} u\|^q_{L^q([T,T+1]\times V)} + \|u\|^q_{L^q([T,T+1]\times V)}$$

where $D_x^{l_2}$ means a collection of all x-derivatives of the order l_2, and for the case of noninteger l_i can be defined by the interpolation analogously to (1.11) and (1.12) (see [24] or [41]). Of course, interpolation shows that mixed derivatives exist in the corresponding Sobolev spaces, i.e., $u \in W^{(l_1,l_2),q}([T, T+1] \times V)$ implies $u \in W^{(1-\theta)l_1,q}([T, T+1], W^{\theta l_2,q}(V))$ for all $\theta \in [0, 1]$.

Definition 1.11. Define the anisotropic spaces $W_b^{(l_1,l_2),q}(\mathbb{R} \times \Omega)$ and $W_b^{(l_1,l_2),q}(\mathbb{R} \times \partial\Omega)$ in analogy to Definition 1.4 (with $\phi \equiv 1$). For instance, in the space $W_b^{(l_1,l_2),q}(\mathbb{R} \times \partial\Omega)$ the norm is defined by the following expression:

$$\|u\|_{W_b^{(l_1,l_2),q}(\mathbb{R}\times\partial\Omega)} := \sup_{T\in\mathbb{R},\, x_0\in\partial\Omega} \|u\|_{W^{(l_1,l_2),q}([T,T+1]\times(\partial\Omega\cap B^1_{x_0}))}$$

Moreover, let $\phi\colon \mathbb{R} \times \mathbb{R}^n \to (0, \infty)$ be a weight function in the variables (t, x) with the exponential growth rate μ (see Definition 1.2). Then one may define the spaces $W_\phi^{(l_1,l_2),q}(\mathbb{R}\times\Omega)$ and $W_\phi^{(l_1,l_2),q}(\mathbb{R}\times\partial\Omega)$ in a standard way. For instance,

$$\|u\|^q_{W_\phi^{(l_1,l_2),q}(\mathbb{R}\times\partial\Omega)} := \int_{(t,\gamma)\in\mathbb{R}\times\partial\Omega} \phi(t,\gamma)\|u\|^q_{W^{(l_1,l_2),q}([t,t+1]\times(\partial\Omega\cap B^1_\gamma))}\, d\gamma\, dt.$$

It is not difficult to show that the analogs of the assertions of Propositions 1.5, 1.6 and 1.9 remain valid for these anisotropic spaces as well.

Now we are in a position to recall the weighted L^2-regularity theory for the following parabolic equation in the regular unbounded domain $\Omega \subset \mathbb{R}^n$:

$$\partial_t u - \gamma \Delta_x u + u = h, \quad u\big|_{\mathbb{R}\times\partial\Omega} = u_0, \quad (t, x) \in \mathbb{R} \times \Omega. \tag{1.15}$$

The following result is formulated for general $s \in (1/2, 2]$, but we will mainly use below the case $s = 1$ which corresponds to the classical "energy formulation" of the boundary value problem for heat equations.

Proposition 1.12. *Let Ω be a regular unbounded domain and let $\frac{1}{2} < s \leqslant 2$ and $s \neq 3/2$. Then, for each*

$$h \in L^2(\mathbb{R}, W^{s-2,2}(\Omega)) \text{ and } u_0 \in W^{(s/2-1/4,s-1/2),2}(\mathbb{R}\times\partial\Omega) \tag{1.16}$$

problem (1.15) *has a unique solution u, which satisfies the estimate*

$$\|\partial_t u\|_{L^2(\mathbb{R},W^{s-2,2}(\Omega))}+\|u\|_{W^{(s/2,s),2}(\mathbb{R}\times\Omega))}$$
$$\leqslant C\left(\|h\|_{L^2(\mathbb{R},W^{s-2,2}(\Omega))}+\|u_0\|_{W^{(s/2-1/4,s-1/2),2}(\mathbb{R}\times\partial\Omega)}\right), \qquad (1.17)$$

where the constant C depends on the regularity constant K of the domain Ω.

The assertion of the proposition is more or less known, and can be derived using the analogous result for bounded domains and the standard localization technique (see, e.g., [5,16,24,41]). Therefore, we omit its rigorous proof here.

We also need the following variant of (1.17) in weighted Sobolev spaces.

Corollary 1.13. *Under the assumptions of Proposition 1.12 there exists an exponent* $\mu_0=\mu_0(K)>0$ *such that for every weight function* $\phi\colon\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ *of growth rate* $\mu\in(0,\mu_0)$ *and every*

$$h\in W_\phi^{(0,s-2),2}(\mathbb{R}\times\Omega) \quad \text{and} \quad u_0\in W_\phi^{(s/2-1/4,s-1/2),2}(\mathbb{R}\times\partial\Omega) \qquad (1.18)$$

Eq. (1.15) *has a unique solution u and the following analog of* (1.17) *holds:*

$$\|\partial_t u\|_{W_\phi^{(0,s-2),2}(\mathbb{R}\times\Omega))}+\|u\|_{W_\phi^{(s/2,s),2}(\mathbb{R}\times\Omega)}$$
$$\leqslant C\left(\|h\|_{W_\phi^{(0,s-2),2}(\mathbb{R}\times\Omega))}+\|u_0\|_{W_\phi^{(s/2-1/4,s-1/2),2}(\mathbb{R}\times\partial\Omega)}\right), \qquad (1.19)$$

where the constant C depends on $K(\Omega)$ *and on the constant* C_ϕ *introduced in* (1.3) *and is independent of the concrete choice of the weight* ϕ.

Our proofs will often use the space–time weight function $\varphi_{\varepsilon,y_0}$ defined via

$$\varphi_{\varepsilon,y_0}(t,x):=e^{-\varepsilon(1+|t-t_0|^2+|x-x_0|^2)^{1/2}}, \quad \text{where } y_0:=(t_0,x_0)\in\mathbb{R}^{n+1}. \qquad (1.20)$$

It satisfies the evident estimate

$$|\partial_t\varphi_{\varepsilon,y_0}(t,x)|+|\nabla_x\varphi_{\varepsilon,y_0}(t,x)|\leqslant(n+1)\varepsilon\varphi_{\varepsilon,y_0}(t,x) \text{ for all } (t_0,x_0),(t,x)\in\mathbb{R}^{n+1}. \qquad (1.21)$$

Indeed, the proof of (1.19) is based on a standard trick of substituting $u=\widetilde{u}/\varphi_{\varepsilon,y_0}$ into equation (1.15), multiplying by $\varphi_{\varepsilon,y_0}$ and solving the equation for $\widetilde{u}$ by a perturbation argument for ε small. This trick together with estimates (1.5) and (1.7) reduce the proof of estimate (1.19) to the case $\phi\equiv 1$ obtained in the Proposition 1.12 (see, e.g., [16,29,30,45] for details).

Corollary 1.14. *Let the assumptions of Proposition 1.12 hold, then for each*

$$h\in \mathrm{W}_{\mathrm{b}}^{(0,s-2),2}(\mathbb{R}\times\Omega)) \quad \textit{and} \quad u_0\in \mathrm{W}_{\mathrm{b}}^{(s/2-1/4,s-1/2),2}(\mathbb{R}\times\partial\Omega)$$

Eq. (1.15) *has a unique solution satisfying*

$$\begin{aligned}&\|\partial_t u\|_{\mathrm{W}_{\mathrm{b}}^{(0,s-2),2}(\mathbb{R}\times\Omega))}+\|u\|_{\mathrm{W}_{\mathrm{b}}^{(s/2,s),2}(\mathbb{R}\times\Omega)}\\&\quad\leqslant C\left(\|h\|_{\mathrm{W}_{\mathrm{b}}^{(0,s-2),2}(\mathbb{R}\times\Omega))}+\|u_0\|_{\mathrm{W}_{\mathrm{b}}^{(s/2-1/4,s-1/2),2}(\mathbb{R}\times\partial\Omega)}\right).\end{aligned}\tag{1.22}$$

Indeed, the function $\phi=\varphi_{\varepsilon,y_0}$ of (1.20) satisfies condition (1.3). Consequently estimates (1.19) with $\phi=\varphi_{\varepsilon,y_0}$ are uniformly valid with respect to $y_0\in\mathbb{R}\times\Omega$. Applying now the operator $\sup_{y_0\in\mathbb{R}\times\Omega}$ to both parts of them and using Proposition 1.12 we derive estimate (1.22).

At the end of this section we consider a family of equations of type (1.15) depending on a large parameter $\lambda\gg 1$:

$$\partial_t u=\gamma\Delta_x u-\lambda u, \quad u\big|_{\mathbb{R}\times\partial\Omega}=u_0, \quad (t,x)\in\mathbb{R}\times\Omega. \tag{1.23}$$

We study the behavior of several norms of u with respect to the parameter λ.

Proposition 1.15. *Let $C_*>1$ and the regularity constant $K_*>0$ be given and the associated growth rate $\mu_*=\mu_0(K_*)>0$. Then, there exist constants C_1 and C_2 such that for every regular domain Ω with $K(\Omega)\leqslant K_*$ and every weight function ϕ with growth $\mu\leqslant\mu_*$ and $C_\phi\leqslant C_*$, for all $\lambda\geqslant 1$ and all solutions u of* (1.23)*, which satisfy* (1.18) *with $s=1$ the following estimates are valid:*

$$\begin{aligned}&C_1\Big(\|u_0\|^2_{\mathrm{W}_\phi^{(1/4,1/2),2}(\mathbb{R}\times\partial\Omega)}+\lambda^{1/2}\|u_0\|^2_{\mathrm{L}^2_\phi(\mathbb{R}\times\partial\Omega)}\Big)\\&\quad\leqslant\|\nabla_x u\|^2_{\mathrm{L}^2_\phi(\mathbb{R}\times\partial\Omega)}+\lambda\|u\|^2_{\mathrm{L}^2_\phi(\mathbb{R}\times\partial\Omega)}\\&\quad\leqslant C_2\Big(\|u_0\|^2_{\mathrm{W}_\phi^{(1/4,1/2),2}(\mathbb{R}\times\partial\Omega)}+\lambda^{1/2}\|u_0\|^2_{\mathrm{L}^2_\phi(\mathbb{R}\times\partial\Omega)}\Big).\end{aligned}\tag{1.24}$$

Proof. As in the case of Corollaries 1.13 and 1.14 it is sufficient to prove (1.24) for $\phi\equiv 1$ only. To this end we make a rescaling

$$x:=\lambda^{-1/2}x', \quad t:=\lambda^{-1}t', \quad \widetilde{u}(t',x'):=u(t,x), \quad \Omega':=\lambda^{1/2}\Omega. \tag{1.25}$$

Note that the new domain Ω' is also regular in the sense of Definition 1.1 for every $\lambda > 1$. Moreover, it is not difficult to show that the regularity constant K can be chosen independently of $\lambda > 1$. Thus, the rescaled function $\widetilde{u}$ satisfies the equation

$$\partial_{t'}\widetilde{u} - \gamma\Delta_{x'}\widetilde{u} + \widetilde{u} = 0, \quad \widetilde{u}\big|_{\partial\Omega'} = \widetilde{u}_0, \quad (t', x') \in \mathbb{R} \times \Omega'. \tag{1.26}$$

It follows now from Proposition 1.12 and from the standard trace theorem that

$$c_1\|\widetilde{u}_0\|^2_{W^{(1/4,1/2),2}(\mathbb{R}\times\partial\Omega')} \leqslant \|\widetilde{u}\|^2_{L^2(\mathbb{R},W^{1,2}(\Omega'))} \leqslant C_1\|\widetilde{u}_0\|^2_{W^{(1/4,1/2),2}(\mathbb{R}\times\partial\Omega')}. \tag{1.27}$$

Indeed, the right-hand side of that inequality is an immediate corollary of (1.17). In order to verify the left inequality, we note that, due to trace theorems for anisotropic Sobolev spaces

$$\|\widetilde{u}_0\|_{W^{(1/4,1/2),2}(\mathbb{R}\times\partial\Omega')} \leqslant C\|\widetilde{u}\|_{W^{(1/2,1),2}(\mathbb{R}\times\Omega')}. \tag{1.28}$$

On the other hand, it follows from Eq. (1.26) that

$$\|\partial_t\widetilde{u}\|_{L^2(\mathbb{R},W^{-1,2}(\Omega'))} \leqslant C\|\widetilde{u}\|_{L^2(\mathbb{R},W^{1,2}(\Omega'))}$$

and extending the standard interpolation inequality

$$\|\widetilde{u}\|^2_{L^2(\Omega')} \leqslant C\|\widetilde{u}\|_{W^{1,2}(\Omega')}\|\widetilde{u}\|_{W^{-1,2}(\Omega')}$$

to time-dependent functions, we have

$$\begin{aligned}\|\widetilde{u}\|_{W^{(1/2,1),2}(\mathbb{R}\times\Omega')} &\leqslant C(\|\widetilde{u}\|_{W^{1,2}(\mathbb{R},W^{-1,2}(\Omega'))} + \|\widetilde{u}\|_{L^2(\mathbb{R},W^{1,2}(\Omega'))})\\ &\leqslant C_1\|\widetilde{u}\|_{L^2(\mathbb{R},W^{1,2}(\Omega'))}.\end{aligned} \tag{1.29}$$

Combining estimates (1.28) and (1.29), we deduce the desired left-hand side of inequality (1.27). Moreover, since the regularity constant K for Ω' can be chosen independent of λ, the constants c_1 and C_1 are also independent of λ. Applying the inverse of the rescaling (1.25) to estimate (1.27) we derive estimate (1.24) for $\phi \equiv 1$. Thus, Proposition 1.15 is proved. □

Corollary 1.16. *Define the Dirichlet–Neumann operator P_λ by*

$$P_\lambda u_0 := \partial_n u\big|_{\mathbb{R}\times\partial\Omega}, \quad \text{where } \partial_n u(t,x) = \nabla_x u(t,x)\cdot n(x) \tag{1.30}$$

and $u: \mathbb{R}\times\Omega \to \mathbb{R}$ is the unique solution of Eq. (1.23)*. Then, for every weight function ϕ on $\mathbb{R}\times\mathbb{R}^n$ with a sufficiently small growth rate $\mu < \mu_0(K)$ the operator P_λ maps $W_\phi^{(1/4,1/2),2}(\mathbb{R}\times\partial\Omega)$ into $W_\phi^{(-1/4,-1/2),2}(\mathbb{R}\times\partial\Omega)$ and*

$$\|P_\lambda u_0\|_{W_\phi^{(-1/4,-1/2),2}(\mathbb{R}\times\partial\Omega)} \leqslant C_\lambda\|u_0\|_{W_\phi^{(1/4,1/2),2}(\mathbb{R}\times\partial\Omega)}, \tag{1.31}$$

where the constant C_λ *depends on* K, λ *and* C_ϕ *but is independent of the concrete choice of* ϕ. *Moreover, the following estimates hold:*

$$\begin{aligned} C_1\left(\|u_0\|^2_{W^{(1/4,1/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega)} + \lambda^{1/2}\|u_0\|^2_{L^2_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega)}\right) \\ \leqslant \left\langle P_\lambda u_0\,,\ \varphi_{\varepsilon,y_0}u_0\right\rangle_{L^2(\mathbb{R}\times\partial\Omega)} \\ \leqslant C_2\left(\|u_0\|^2_{W^{(1/4,1/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega)} + \lambda^{1/2}\|u_0\|^2_{L^2_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega)}\right), \end{aligned} \tag{1.32}$$

where $\varphi_{\varepsilon,y_0}$ *is defined in* (1.20), $\varepsilon < \mu_0$, $y_0 \in \mathbb{R}\times\Omega$, *and the constants* C_1, C_2 *are independent of* $\lambda > 1$ *and* $y_0 \in \mathbb{R}\times\Omega$.

Proof. Let Π_Ω be an extension operator from the boundary $\mathbb{R}\times\partial\Omega$ inside of the domain $\mathbb{R}\times\Omega$ such that the function $v = \Pi_\Omega v_0$ solves Eq. (1.15) with $h=0$ and $v\big|_{\mathbb{R}\times\Omega} = v_0$. Then, according to Corollary 1.13, for every fixed $y_0 = (t_0, x_0)$ and every $v_0 \in W^{(1/4,1/2),2}(\mathbb{R}\times\partial\Omega)$ such that

$$\operatorname{supp} v_0 \subset [t_0, t_0+1]\times(\partial\Omega\cap B^1_{x_0}) \tag{1.33}$$

and sufficiently small $0 < \varepsilon < \mu_0(K)$, we have

$$\begin{aligned} \|v\|_{W^{(1/2,1),2}_{\varphi_{-\varepsilon,(t_0,x_0)}}(\mathbb{R}\times\Omega)} &\leqslant C\|v_0\|_{W^{(1/4,1/2),2}_{\varphi_{-\varepsilon,(t_0,x_0)}}(\mathbb{R}\times\partial\Omega)} \\ &\leqslant C_1\|v_0\|_{W^{(1/4,1/2),2}([t_0,t_0+1]\times(\partial\Omega\cap B^1_{x_0}))}, \end{aligned} \tag{1.34}$$

where the constants C and C_1 are independent of y_0. Multiplying Eq. (1.23) by v and integrating by $(t,x)\in\mathbb{R}\times\Omega$ and integrating by parts, we have

$$\begin{aligned} \gamma\langle\partial_n u, v_0\rangle_{L^2([t_0,t_0+1]\times(\partial\Omega\cap B^1_{x_0}))} = \langle\partial_t u, v\rangle_{L^2(\mathbb{R}\times\Omega)} + \gamma\langle\nabla_x u, \nabla_x v\rangle_{L^2(\mathbb{R}\times\Omega)} \\ +\lambda\langle u, v\rangle_{L^2(\mathbb{R}\times\Omega)}. \end{aligned} \tag{1.35}$$

Using now the obvious facts that

$$\|\partial_t u\|_{W^{(-1/2,0),2}_{\varphi_{\varepsilon,(t_0,x_0)}}(\mathbb{R}\times\Omega)} \leqslant C\|u\|_{W^{(1/2,0),2}_{\varphi_{\varepsilon,(t_0,x_0)}}(\mathbb{R}\times\Omega)}$$

and $[W^{(1/2,0),2}_{\varphi_{-\varepsilon,(t_0,x_0)}}(\mathbb{R}\times\Omega)]^* = W^{(-1/2,0),2}_{\varphi_{\varepsilon,(t_0,x_0)}}(\mathbb{R}\times\Omega)$, we derive from (1.35) that

$$\begin{aligned} |\langle\partial_n u, v_0\rangle_{L^2([t_0,t_0+1]\times(\partial\Omega\cap B^1_{x_0}))}| \leqslant C_\lambda\|u\|_{W^{(1/2,1),2}_{\varphi_{\varepsilon,(t_0,x_0)}}(\mathbb{R}\times\Omega)}\|v\|_{W^{(1/2,1),2}_{\varphi_{\varepsilon,(t_0,x_0)}}(\mathbb{R}\times\Omega)} \\ \leqslant C'_\lambda\|u_0\|_{W^{(1/4,1/2),2}_{\varphi_{\varepsilon,(t_0,x_0)}}(\mathbb{R}\times\partial\Omega)}\|v_0\|_{W^{(1/4,1/2),2}([t_0,t_0+1]\times(\partial\Omega\cap B^1_{x_0}))}, \end{aligned} \tag{1.36}$$

where the constants C_λ and C'_λ depend on λ, but are independent of y_0 (here we have used Corollary 1.13 in order to estimate u in terms of u_0

and estimate (1.34) for estimating v in terms of v_0). Thus we have established that

$$\|P_\lambda u_0\|^2_{W^{(-1/4,-1/2),2}([t_0,t_0+1]\times(\partial\Omega\cap B^1_{x_0}))} \leqslant C'_\lambda \int_{(t,s)\in\mathbb{R}\times\partial\Omega} \varphi_{\varepsilon,(t_0,x_0)}(t,s)\|u_0\|^2_{W^{(1/4,1/2),2}([t,t+1]\times(\partial\Omega\cap B^1_s))}\mathrm{d}t\,\mathrm{d}s.$$

Multiplying this estimate by $\phi(t_0,x_0)$, integrating over $(t_0,x_0)\in\mathbb{R}\times\partial\Omega$ and using Proposition 1.5 we obtain the desired estimate (1.31).

Thus, it remains to derive estimate (1.32). To this end we multiply Eq. (1.23) by $\varphi_{\varepsilon,y_0}u$ and integrate over $(t,x)\in\mathbb{R}\times\Omega$. After integration by parts we will have

$$\gamma\langle P_\lambda u_0,\varphi_{\varepsilon,y_0}u_0\rangle_{L^2(\mathbb{R}\times\partial\Omega)} = \gamma\langle|\nabla_x u|^2,\varphi_{\varepsilon,y_0}\rangle_{L^2(\mathbb{R}\times\Omega)} + \lambda\langle|u|^2,\varphi_{\varepsilon,y_0}\rangle_{L^2(\mathbb{R}\times\Omega)} + \gamma\langle\nabla_x u, u\nabla_x\varphi_{\varepsilon,y_0}\rangle_{L^2(\mathbb{R}\times\Omega)} - \langle|u|^2,\partial_t\varphi_{\varepsilon,y_0}\rangle_{L^2(\mathbb{R}\times\Omega)}.$$

Applying estimate (1.24) to the right-side and using the estimate (1.21) for $\varphi_{\varepsilon,y_0}$ we arrive at (1.32) (recall that $\varepsilon<\mu_0(K)$ is small). Thus, Corollary 1.16 is proved. □

2. HYPERBOLIC TRAJECTORIES AND SETS FOR RDSs

In this section we introduce a model RDS in a *bounded* domain Ω_0 which possesses a hyperbolic set Γ_0. Using this model RDS we formally construct a new RDS in an *unbounded* domain which possesses a hyperbolic set $\Gamma:=(\Gamma_0)^{\mathbb{Z}^n}$. This formal construction will be justified in the next sections.

Let Ω_0 be a regular bounded domain in $\mathbb{R}^n$ (without loss of generality we may assume that $0\in\Omega_0$ and $\operatorname{diam}\Omega_0<1$). Consider the following RDS in Ω_0:

$$\begin{cases} \partial_t u = \gamma\Delta_x u - u - f(t,u), \\ u\big|_{t=0} = u^0, \quad u\big|_{\partial\Omega_0} = 0. \end{cases} \tag{2.1}$$

Here $u=(u^1,\dots,u^k)$ is the unknown vector–valued function, Δ_x is the componentwise Laplacian with respect to the variable $x:=(x^1,\dots,x^n)$. The nonlinearity $f:\mathbb{R}\times\mathbb{R}^k\to\mathbb{R}^k$ is assumed to be 1-periodic with respect to t, i.e.,

$$f(t+1,u)\equiv f(t,u) \quad \text{for all } (t,u)\in\mathbb{R}\times\mathbb{R}^k \tag{2.2}$$

to vanish for $u=0$, i.e., $f(t,0)=0$, and to have bounded derivatives with respect to u and t such that

$$\|\partial_t f\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{R}^k)} + \|\mathrm{D}_u^k f\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{R}^k)} \leqslant C \quad \text{for} \quad k=0,1, \text{and } 2. \tag{2.3}$$

It is well known (see, e.g., [7]) that under these assumptions Eq. (2.1) possesses a unique solution $u:[0,\infty)\to \mathrm{L}^2(\Omega_0)$ for every $u^0\in \mathrm{L}^2(\Omega_0)$. It satisfies the dissipative estimate

$$\|u(t)\|_{\mathrm{L}^2(\Omega_0)} \leqslant C\|u^0\|_{\mathrm{L}^2(\Omega_0)}\, e^{-\alpha t} + C_f \quad \text{for } t\geqslant 0, \tag{2.4}$$

where $\alpha>0$ depends on Ω_0 only. Hence, the nonlinear solution operator

$$S_t^0 : \mathrm{L}^2(\Omega_0)\to \mathrm{L}^2(\Omega_0); \quad u^0\mapsto u(t) =: S_t^0(u^0), \tag{2.5}$$

is well defined. It is also known that under the above assumptions the Poincaré map $S_1^0 : \mathrm{L}^2(\Omega_0)\to \mathrm{L}^2(\Omega_0)$ associated with problem (2.1) possesses a compact global attractor $\mathcal{A}^0\in \mathrm{L}^2(\Omega_0)$, which is generated by all complete bounded solutions of (2.1), i.e.,

$$\mathcal{A}^0 = \Pi_0\mathcal{K}^0, \quad \Pi_0 u := u(0),$$

where the essential set $\mathcal{K}^0 := \{u\in \mathrm{C_b}(\mathbb{R}, \mathrm{L}^2(\Omega)) : u \text{ solves (2.1)}\}$ is not empty since $u\equiv 0$ lies in $\mathcal{K}^0$. For more information see, e.g., [7,40].

Finally, the standard parabolic regularity theory (see [24]) shows that there exists a constant $C_{\mathcal{K}}$ such that

$$\|u\|_{\mathrm{C}_\mathrm{b}^1(\mathbb{R}\times\Omega_0)} \leqslant C_{\mathcal{K}} \quad \text{for all } u\in\mathcal{K}^0. \tag{2.6}$$

In order to introduce the notion of hyperbolicity for invariant sets, we need the non-homogeneous analog of the first variation associated with the trajectory $u\in\mathcal{K}^0$:

$$\partial_t v = \gamma\Delta_x v - v - \mathrm{D}_u f(t,u(t,x))v + h(t,x) \text{ in } \mathbb{R}\times\Omega_0, \quad v\big|_{\mathbb{R}\times\partial\Omega_0} = 0. \tag{2.7}$$

Definition 2.1. A trajectory $u\in\mathcal{K}^0$ is called a *hyperbolic trajectory of system* (2.1), if there exists a constant $C_u>0$ and if for every $h\in \mathrm{L}^2(\mathbb{R}, \mathrm{W}^{-1,2}(\Omega_0))$ problem (2.7) has a unique solution $v\in \mathrm{L}^2(\mathbb{R}, \mathrm{W}_0^{1,2}(\Omega_0)) \cap \mathrm{W}^{1,2}(\mathbb{R}, \mathrm{W}^{-1,2}(\Omega_0))$ and this solution satisfies the estimate

$$\|v\|_{\mathrm{W}^{(0,1),2}(\mathbb{R}\times\Omega_0)} + \|\partial_t v\|_{\mathrm{W}^{(0,-1),2}(\mathbb{R}\times\Omega_0)} \leqslant C_u\|h\|_{\mathrm{W}^{(0,-1),2}(\mathbb{R}\times\Omega_0)}. \tag{2.8}$$

A subset $\Gamma_0^{\mathrm{tr}}\subset\mathcal{K}^0$ is called a (uniformly) *hyperbolic trajectory set for system* (2.1), if every trajectory $u\in\Gamma_0^{\mathrm{tr}}$ is hyperbolic and estimates (2.8) hold uniformly with respect to $u\in\Gamma_0^{\mathrm{tr}}$, i.e., with $C_u\leqslant C_{\Gamma_0}<\infty$. The set $\Gamma_0 := \Pi_0\Gamma_0^{\mathrm{tr}}\subset \mathrm{L}^2(\Omega_0)$ is called then a *hyperbolic set of problem* (2.1).

As above we need to reformulate estimate (2.8) in terms of weighted Sobolev spaces.

Lemma 2.2. *Let $u\in\mathcal{K}^0$ be a hyperbolic trajectory of system* (2.1) *with C_u as given in* (2.8). *Then there exists $\mu_0=\mu_0(C_u)>0$ such that for every weight function ϕ of exponential growth rate $\mu<\mu_0$ the following estimate is valid for the solution of problem* (2.7)

$$\|v\|_{\mathrm{W}^{(0,1),2}_{\phi}(\mathbb{R}\times\Omega_0)}+\|\partial_t v\|_{\mathrm{W}^{(0,-1),2}_{\phi}(\mathbb{R}\times\Omega_0)}\leqslant C'_u\|h\|_{\mathrm{W}^{(0,-1),2}_{\phi}(\mathbb{R}\times\Omega_0)}, \tag{2.9}$$

where the constant C'_u depends only on C_u and C_ϕ and is independent of the concrete choice of hyperbolic trajectory. In particular (2.9) *implies that*

$$\|v\|_{\mathrm{W}^{(0,1),2}_{\mathrm{b}}(\mathbb{R}\times\Omega_0)}+\|\partial_t v\|_{\mathrm{W}^{(0,-1),2}_{\mathrm{b}}(\mathbb{R}\times\Omega_0)}\leqslant C''_u\|h\|_{\mathrm{W}^{(0,-1),2}_{\mathrm{b}}(\mathbb{R}\times\Omega_0)}, \tag{2.10}$$

where C''_u also depends only on C_u.

The proof of this lemma is also based on a standard trick with variable changing $\widetilde{v}=\varphi_{\varepsilon,y_0}v$ in Eq. (2.7) analogously to Corollaries 1.13 and 1.14, so we leave the rigorous derivation of (2.9) and (2.10) to the reader.

Remark 2.3. It is not difficult to verify (using (2.9) with exponential weights $\phi(t):=e^{-\varepsilon|t|}$, ε is small positive) that the above definition of the hyperbolic trajectory $u(t)$ is equivalent to the existence of an exponential dichotomy for homogeneous equation of variations

$$\partial_t v=\gamma\Delta_x v-v-\mathrm{D}_u f(t,u(t,x))v \text{ in } \mathbb{R}\times\Omega_0,\quad v\big|_{\mathbb{R}\times\partial\Omega_0}=0,\ \ v(0)=v_0. \tag{2.11}$$

Namely, there exists a splitting the phase space $\mathrm{L}^2(\Omega_0)$ into a direct sum of two closed linear subspaces $V_+=V_+(u)$ and $V_-=V_-(u)$ such that for every $v_0\in V_+$ there exists a unique backward solution $v(t):=v_+(t)$ (defined for all negative t) and for every $v_-\in V_-$ there exists a forward solution $v(t):=v_+(t)$ (defined for all positive t) such that

$$\|v_\pm(\mp t)\|_{\mathrm{L}^2(\Omega_0)}\leqslant Ce^{-\varepsilon t}\|v_\pm\|_{\mathrm{L}^2(\Omega_0)},\quad t\geqslant 0. \tag{2.12}$$

Moreover, the positive constants C and ε depend only on the hyperbolicity constant C_u introduced on (2.8). Thus, the exponential dichotomy (2.12) is uniform with respect to all trajectories u belonging to a hyperbolic set Γ_0^{tr} and, consequently, our definition of a hyperbolic set is equivalent to the standard definition via stable and unstable foliations (see, e.g., [20]). We recall that our definition is adopted for the *nonautonomous* (e.g., time-periodic) equations, so the neutral foliation is absent and we factually have a *discrete* dynamical system generated by the Poincaré map S_1^0.

We also note that our definition is associated with the phase space $L^2(\Omega_0)$, although the concrete choice of that space is not essential due to the smoothing property for parabolic equations. In particular, the equivalent hyperbolicity formulation associated with the phase space $W^{1,2}(\Omega_0)$ reads: for every $h \in L^2(\mathbb{R} \times \Omega_0)$ there exists a unique solution $v \in W^{(1,2),2}(\mathbb{R} \times \Omega_0)$ of Eq. (2.7) and the following analog of (2.8) holds:

$$\|v\|_{W^{(1,2),2}(\mathbb{R}\times\Omega_0)} \leqslant C_u \|h\|_{L^2(\mathbb{R}\times\Omega_0)}. \tag{2.13}$$

Although (2.13) looks simpler than (2.8) (in particular, it does not contain Sobolev norms with fractional or negative exponents), we prefer to use the L^2-form formulated in Definition 2.1 since we will consider below the dynamical systems which are close only in rather weak sense and the $W^{1,2}$-norms will be too strong for our purposes.

In order to formulate our additional assumptions to system (2.1) we introduce the model dynamical system of Bernoulli shifts (see, e.g., [20] for details).

Definition 2.4. For $n \in \mathbb{N}$ define $\mathcal{M}^n := \{0,1\}^{\mathbb{Z}^n} = \{b: \mathbb{Z}^n \to \{0,1\}\}$ and equip it with the standard Tychonov topology. Using the n-dimensional "time" $l \in \mathbb{Z}^n$ we define the model dynamical system $\{\mathcal{T}_l : l \in \mathbb{Z}^n\}$ on $\mathcal{M}^n$ via

$$(\mathcal{T}_l b)(m) := b(l+m), \quad m \in \mathbb{Z}^n, \text{ where } b = b(\cdot) \in \mathcal{M}^n. \tag{2.14}$$

We will denote by $\{\mathcal{T}^i_k : k \in \mathbb{Z}\}$, $i = 1, \dots, n$, the one-parameter subgroups of (2.14) defined via $\mathcal{T}^i_k := \mathcal{T}_{ke_i}$, where e_i is a standard ith coordinate vector in $\mathbb{R}^n$.

Since we consider spatial-temporal systems we are lead to the set $(\mathcal{M}^1)^{\mathbb{Z}^n}$, which we identify with $\mathcal{M}^{n+1}$ via

$$(b(\cdot, l'))_{l' \in \mathbb{Z}^n} \in (\mathcal{M}^1)^{\mathbb{Z}^n} \quad \text{for } b \in \mathcal{M}^{n+1}. \tag{2.15}$$

We will then use $l = (l_0, l') \in \mathbb{Z}^{n+1}$ to indicate the time component l_0 and the spatial component $l' \in \mathbb{Z}^n$.

Our basic assumption for the construction of spatio-temporal chaos is now, that the dynamics of system (2.1) contains chaotic dynamics in the following sense.

Assumption 2.4A. There is a hyperbolic set Γ_0 for problem (2.1) in the sense of Definition 2.1. Moreover, there exists a homeomorphism τ_0:

$\mathcal{M}^1 \to \Gamma_0$ such that the dynamical systems $(\mathcal{M}^1, \mathcal{T}_1^0)$ and (Γ_0, S_1^0) are conjugate, i.e.,

$$\tau_0 \circ \mathcal{T}_1^0 = S_1^0 \circ \tau_0, \quad \text{where } (\mathcal{T}_1^0 b)(l_0) = b(l_0+1) \quad \text{for } l_0 \in \mathbb{Z}. \tag{2.16}$$

(Recall that S_1^0 is the Poincaré map of (2.1) with $S_1^0 u = u(\cdot + 1)$.)

Since every initial condition $u_0 \in \Gamma_0 \subset \mathrm{L}^2(\Omega_0)$ is associated with a unique trajectory $u \in \mathcal{K}^0$ we may also define the hyperbolic trajectory set Γ_0^{tr} and the homeomorphism $\tau_0^{\mathrm{tr}} : \mathcal{M}^1 \to \Gamma_0^{\mathrm{tr}}; a \mapsto U_a^0 \in \mathrm{C_b}(\mathbb{R}, \mathrm{L}^2(\Omega_0))$, which satisfies

$$\tau_0(a) = U_a^0(0) \quad \text{and } U_a^0(t+1) = U^0_{\mathcal{T}_1^0 a}(t) \quad \text{for } t \in \mathbb{R}. \tag{2.17}$$

To make τ_0^{tr} continuous, it is important to introduce a suitable topology on $\Gamma_0^{\mathrm{tr}} \subset \mathcal{K}^0$. This is the topology of uniform convergence in $\mathrm{L}^2(\Omega_0)$ on compact subset of $\mathbb{R}$. Since $\mathcal{K}^0$ and hence Γ_0^{tr} are contained in a bounded set of $\mathrm{C_b}(\mathbb{R}, \mathrm{L}^2(\Omega_0))$ this topology is easily obtained the weighted norms, e.g., $\mathrm{L}^\infty_{e^{-\varepsilon|t|}}(\mathbb{R}, \mathrm{L}^2(\Omega_0))$. We refer to [34] for the exact arguments.

Remark 2.5. We recall that the existence of a hyperbolic sets Γ_0 described in Assumption 2.4A is closely related with the existence of *transversal* homoclinic orbits for Eq. (2.1). Indeed, let $a_0 \in \mathcal{M}^1$ be a zero element (i.e. $a_0(n) \equiv 0$ for all $n \in \mathbb{Z}$). Then, according to (2.16), the corresponding hyperbolic trajectory $u_0 := U^0_{a_0}$ is a time-periodic trajectory of the RDS (2.1). Without loss of generality, we may assume that

$$u_0 := U^0_{a_0} \equiv 0 \tag{2.18}$$

(if (2.18) is not satisfied, it is sufficient to change a dependent variable $u \to u - u_0(t, x)$). Then, according to hyperbolicity assumption (2.8), $u_0 \equiv 0$ is a hyperbolic equilibrium of system (2.1) (in particular, (2.18) implies that $f(t, 0) \equiv 0$).

Let us now consider a basic homoclinic orbit $\bar{a}$ to a_0 in $\mathcal{M}^1$, determined by $\bar{a}(0) = 1$ and $\bar{a}(l) = 0$, $l \neq 0$. Then, the commutation relations (2.16) and (2.22) together with the fact that τ_0 is a homeomorphism guarantee that the associated trajectory $\bar{u} := U^0_{\bar{a}}$ will be a homoclinic orbit to $u_0 \equiv 0$. Moreover, the hyperbolicity assumption (2.8) implies that $\bar{u}$ is a *transversal homoclinic orbit* to zero solution and, consequently, decays exponentially as $t \to \pm\infty$:

$$\bar{u} \in \mathrm{C}^1_{e^{\mu|t|}}(\mathbb{R} \times \Omega_0) \tag{2.19}$$

for some positive μ. We now recall that every element $a \in \mathcal{M}^1$ can be presented as a sum of shifts of the basic homoclinic orbit $\bar{a}$ with coefficients from 0 and 1, namely

$$a = \sum_{l \in \mathbb{Z}} a(l) \mathcal{T}^0_{-l} \bar{a}. \tag{2.20}$$

Of course, the homeomorphism $\tau_0 : \mathcal{M}^1 \to \Gamma_0$ is not linear and we cannot write the analog of equality (2.20) for the solution $U_a \in \Gamma_0^{tr}$. Nevertheless, this solution is usually occurs *close* to the sum $\sum_{l \in \mathbb{Z}} a(l) \bar{u}(\cdot - l)$, i.e.

$$\| U_a - \sum_{l \in \mathbb{Z}} a(l) \bar{u}(\cdot - l) \|_{C^1(\mathbb{R} \times \Omega_0)} \leqslant \varepsilon_0, \tag{2.21}$$

where the small positive ε_0 is independent of $a \in \mathcal{M}^1$ and the trajectory U_a^0 is determined in a unique way by this condition. Thus, a hyperbolic set Γ_0^{tr} is generated by shifts of the basic transversal homoclinic orbit $\bar{u}$ summed with coefficients 0 and 1. Moreover, it is also well-known that the existence of a *single* transversal homoclinic orbit $\bar{u}$ to some periodic solution u_0 of problem (2.1) is sufficient for the existence of a hyperbolic set Γ_0^{tr} satisfying Assumption 2.4A, see [20, 39] for the details.

Remark 2.6. Assumption 2.4A which guarantees that the Dirichlet problem (2.1) in a bounded domain Ω_0 possesses a hyperbolic set homeomorphic to Bernoulli scheme $\mathcal{M}^1$ is the basis of our construction. As we explained in previous remark, existence of the hyperbolic sets of that type is strongly related with homoclinic orbits and they usually appear in concrete examples of ODEs or PDEs under the bifurcation of the appropriate homoclinic orbit, see e.g. [20, 39] and references therein. Moreover, a number of special constructions which allow to realize a given finite-dimensional vector field as a restriction of a RDS to its appropriate central manifold are also known, see [18, 37].

Unfortunately, although the existence of RDSs satisfying Assumption 2.4A seems well-known, it is not easy to give a *sharp* reference for this result. That is why, for the convenience of the reader, we briefly explain in Appendix A how to construct a RDS satisfying Assumption 2.4A if a system of ODEs is known which satisfies this assumption.

It is now easy to construct a RDS which has have the hyperbolic set $\Gamma := (\Gamma_0)^{\mathbb{Z}^n}$. We do this by considering a period array of uncoupled systems as follows. Later we will show that coupling does not destroy the hyperbolic set and such we are able to embed the problem into a RDS which is spatially homogeneous.

We use the spatial translation operator $T_h\colon x\mapsto x-h$ and define for $l\in\mathbb{Z}^n$ the sets $\Omega_l:=T_l\Omega_0$. Due to our assumptions $\operatorname{diam}\Omega_l<1$, the domains Ω_l do not intersect for different values of $l\in\mathbb{Z}^n$. Define the domains Ω_+ and Ω_- via

$$\Omega_+:=\cup_{l\in\mathbb{Z}^n}\Omega_l,\quad \Omega_-:=\operatorname{int}(\mathbb{R}^n\setminus\Omega_+).$$

Evidently Ω_+ and Ω_- are uniformly regular in the sense of Definition 1.1 and Ω_+ is disconnected.

Consider now the RDS in Ω_+:

$$\begin{cases}\partial_t u=\gamma\Delta_x u-u-f(t,u) & \text{in } \mathbb{R}\times\Omega_+,\\ u=0 & \text{on } \mathbb{R}\times\partial\Omega_+,\end{cases}\tag{2.22}$$

which in fact decouples into a countable number of copies of the initial RDS (2.1), since Ω_+ is the disjoint union $\cup_{l\in\mathbb{Z}^n}\Omega_0$. Indeed, any solution of (2.22) can be represented as

$$u(t)=\textstyle\sum_{l\in\mathbb{Z}^n}T_l u_l(t)\chi_{\Omega_l}\quad\text{where } (T_h v)(x)=v(x+h)\tag{2.23}$$

and χ_{Ω_l} is a characteristic function of the domain Ω_l. Moreover, for each $l\in\mathbb{Z}^n$ the function $u_l:[0,\infty)\to \mathrm{L}^2(\Omega_0)$ is a solution of (2.1).

Using the decoupling (2.23) and the estimate (2.4) one easily verifies that for every $u^0\in\mathrm{L}^2_{\mathrm{b}}(\Omega_+)$ problem (2.22) has a unique solution $u(t)$ which satisfies the dissipative estimate

$$\|u(t)\|_{\mathrm{L}^2_{\mathrm{b}}(\Omega_+)}\leqslant C\|u(0)\|_{\mathrm{L}^2_{\mathrm{b}}(\Omega_+)}\,e^{-\alpha t}+C_f.$$

Thus, the solution operator of (2.22) is well defined via

$$S_t:\mathrm{L}^2_{\mathrm{b}}(\Omega_+)\to\mathrm{L}^2_{\mathrm{b}}(\Omega_+),\quad u(0)\mapsto S_t u(0):=u(t).$$

The Poincaré map $S_1\colon\mathrm{L}^2_{\mathrm{b}}(\Omega_+)\to\mathrm{L}^2_{\mathrm{b}}(\Omega_+)$ associated with (2.22) possesses a global, locally compact attractor $\mathcal{A}\subset\mathrm{L}^2_{\mathrm{b}}(\Omega_+)$. The latter means that the set $\mathcal{A}$ is bounded in $\mathrm{L}^2_{\mathrm{b}}(\Omega_+)$ but is compact in the topology of $\mathrm{L}^2_{\mathrm{loc}}(\Omega_+)$ only (which is natural for the case of unbounded domains, see [31,33,46]). Moreover, the decoupling (2.23) defines a homeomorphism

$$\mathcal{A}\sim\left(\mathcal{A}^0\right)^{\mathbb{Z}^n}$$

and the local topology induced on $\mathcal{A}$ by the embedding $\mathcal{A}\subset\mathrm{L}^2_{\mathrm{loc}}(\Omega_+)$ coincides with the Tychonov topology induced on the product $\left(\mathcal{A}^0\right)^{\mathbb{Z}^n}$.

As in the case of $\mathcal{A}^0$, the attractor $\mathcal{A} \subset \mathrm{L}^2_{\mathrm{b}}(\Omega_+)$ is generated by all bounded complete solutions of problem (2.22), i.e.,

$$\mathcal{A} = \Pi_0 \mathcal{K}, \quad \mathcal{K} \sim \left(\mathcal{K}^0\right)^{\mathbb{Z}^n}.$$

Let us study now the relations between the hyperbolic trajectories of systems (2.1) and (2.22). To this end we introduce the nonhomogeneous equation of variations for (2.22) for $u \in \mathcal{K}$:

$$\partial_t v = \gamma \Delta_x v - v - \mathrm{D}_u f(t, u(t,x)) v + h(t,x) \text{ in } \mathbb{R} \times \Omega_+, \quad v = 0 \text{ on } \mathbb{R} \times \partial\Omega_+. \tag{2.24}$$

Lemma 2.7. *Let* $\Gamma_0^{\mathrm{tr}} \subset \mathcal{K}^0$ *be a nonempty hyperbolic trajectory set for* (2.1)*. Then, for any sequence* $(u_l)_{l \in \mathbb{Z}^n}$ *with* $u_l \in \Gamma_0^{\mathrm{tr}}$ *the function* $u \in \mathcal{K}$ *defined via* (2.23) *is a hyperbolic trajectory for Eq.* (2.22)*. Moreover, there is an exponent* $\mu_0 = \mu_0(C_{\Gamma_0}) > 0$ *such that for every weight function* ϕ *with the exponential growth rate* $\mu < \mu_0$ *the following estimate is valid for the solution* v *of equation* (2.24)*:*

$$\|v\|_{\mathrm{W}^{(0,1),2}_{\phi}(\mathbb{R} \times \Omega_+)} + \|\partial_t v\|_{\mathrm{W}^{(0,-1),2}_{\phi}(\mathbb{R} \times \Omega_+)} \leqslant C_\Gamma \|h\|_{\mathrm{W}^{(0,-1),2}_{\phi}(\mathbb{R} \times \Omega_+)}, \tag{2.25}$$

where the constant C_Γ *depends only on* C_{Γ_0} *and* C_ϕ *and is independent of the specific choice of hyperbolic trajectories* $(u_l)_{l \in \mathbb{Z}^n}$*. In particular,* (2.25) *implies that*

$$\|v\|_{\mathrm{W}^{(0,1),2}_{\mathrm{b}}(\mathbb{R} \times \Omega_+)} + \|\partial_t v\|_{\mathrm{W}^{(0,-1),2}_{\mathrm{b}}(\mathbb{R} \times \Omega_+)} \leqslant C^*_\Gamma \|h\|_{\mathrm{W}^{(0,-1),2}_{\mathrm{b}}(\mathbb{R} \times \Omega_+)}, \tag{2.26}$$

where C^*_Γ *depends only on* C_{Γ_0}.

Indeed, estimates (2.25) and (2.26) are immediate corollaries of (2.9), (2.10), and the fact that the weight $T_l \phi$ satisfies (1.3) with the same constant as ϕ.

The assertion of Lemma 2.7 admits to find a "large" hyperbolic set Γ for the RDS (2.22).

Corollary 2.8. *Under the Assumption of Lemma 2.4A the representation* (2.23) *defines a hyperbolic trajectory set*

$$\Gamma^{\mathrm{tr}} \sim \left(\Gamma_0^{\mathrm{tr}}\right)^{\mathbb{Z}^n}$$

for problem (2.22) *and consequently* $\Gamma := \Pi_0 \Gamma^{\mathrm{tr}} \subset \mathrm{L}^2_{\mathrm{b}}(\Omega_+)$ *is a hyperbolic set for* (2.22) *which is homeomorphic to* $\Gamma_0^{\mathbb{Z}^n}$.

Again it is important to equip $\Gamma^{\rm tr}\subset C_b(\mathbb{R},L^2_b(\Omega_+))$ and $\Gamma\subset L^2_b(\Omega_+)$ with the correct topologies which describe convergence on bounded subsets of $\mathbb{R}\times\Omega_+$ and Ω_+, respectively. As above the weighted topologies are the desired ones.

By construction, system (2.22) is invariant under the group $\{T_{l'}: l'\in\mathbb{Z}^n\}$ of discrete spatial translations as well as under the temporal shifts $\{S_{l_0}: l_0\in\mathbb{Z}\}$. Note that these two actions commute, such that we have a spatio-temporal action of $\mathbb{Z}^{n+1}$, which we denote by $\{\mathbb{S}_l=S_{l_0}\circ T_{l'}: l=(l_0,l')\in\mathbb{Z}^{n+1}\}$ and $\{\mathbb{T}_l: l=(l_0,l')\in\mathbb{Z}^{n+1}\}$ for the action on initial data (i.e., on $L^2_b(\Omega_+)$) and on trajectories (i.e., on $C_b(\mathbb{R},L^2_b(\Omega_+))$), respectively. In particular, the attractor $\mathcal{A}$ and the corresponding set of essential trajectories of (2.22) are also invariant with respect to these actions:

$$\mathbb{S}_l\mathcal{A}=\mathcal{A}\quad\text{and}\quad\mathbb{T}_l\mathcal{K}=\mathcal{K}\quad\text{for } l=(l_0,l')\in\mathbb{Z}^{n+1}.$$

Moreover the sets $\Gamma^{\rm tr}$ and Γ are also invariant with respect to these translations:

$$\mathbb{T}_l\Gamma^{\rm tr}=\Gamma^{\rm tr}\quad\text{and}\quad\mathbb{S}_l\Gamma=\Gamma\quad\text{for } l\in\mathbb{Z}^{n+1}. \tag{2.27}$$

More than that, these actions on $\Gamma=(\Gamma_0)^{\mathbb{Z}^n}$ and $\Gamma^{\rm tr}=(\Gamma_0^{\rm tr})^{\mathbb{Z}^n}$ are conjugated to the standard Bernoulli shift $\{\mathcal{T}_l: l\in\mathbb{Z}^{n+1}\}$ on $\mathcal{M}^{n+1}=(\mathcal{M}^1)^{\mathbb{Z}^n}$ via the obvious homeomorphisms $\tau:\mathcal{M}^{n+1}\to\Gamma$ and $\tau^{\rm tr}:\mathcal{M}^{n+1}\to\Gamma^{\rm tr}$ defined via $\tau(b)=\tau^{\rm tr}(b)|_{t=0}$ and

$$\tau(b)=U_b^{\rm tr}:(t,x)\mapsto\sum_{l'\in\mathbb{Z}^n}U^0_{b(\cdot,l')}(t,x+l')\chi_{\Omega_0}(x+l')$$

with $U^0_{b(\cdot,l')}$ as defined right before (2.17). We summarize these result as follows.

Corollary 2.9. *Let the Assumption 2.4A hold. Then, the multi-dimensional Bernoulli system* $(\mathcal{M}^{n+1},\{\mathcal{T}_k: k\in\mathbb{Z}^{n+1}\})$ *is conjugated to* $(\Gamma,\{\mathbb{S}_k=S_{k_0}\circ T_k: k=(k_0,k)\in\mathbb{Z}^{n+1}\})$ *associated with the RDS* (2.22) *via the homeomorphism* τ*, i.e.,*

$$S_{l_0}\circ\tau=\tau\circ\mathcal{T}_{(l_0,0)}\quad\textit{for } l_0\in\mathbb{Z}\quad\textit{and}\quad T_{l'}\circ\tau=\tau\circ\mathcal{T}_{(0,l')}\quad\textit{for } l'\in\mathbb{Z}^n. \tag{2.28}$$

Thus, we have constructed a RDS which possesses a hyperbolic set $\Gamma=\Gamma_0^{\mathbb{Z}^n}$ which shows spatio-temporal chaos. But unfortunately the unbounded domain Ω_+ is disconnected and consequently is not a "domain" in a usual sense which makes the obtained result artificial and uninteresting in itself. In the subsequent sections we will use the structural stability of hyperbolic sets to show that a weakly coupled system still has the same chaotic behavior.

3. THE LINEARIZATION OF THE WEAKLY COUPLED SYSTEM

Our next step is to construct an appropriate RDS in $\mathbb{R}^n$ which is in a sense close to problem (2.22) in Ω_+ and then to construct the hyperbolic set $\widetilde{\Gamma}$ for this new system using the structural stability of hyperbolic sets. We will search such a system in the following form:

$$\begin{cases} \partial_t u = \gamma\Delta_x u - u - f(t,u) & \text{for } x\in\Omega_+, \\ \partial_t u = \gamma\Delta_x u - \lambda u & \text{for } x\in\Omega_-, \\ u\big|_{\mathbb{R}\times\partial\Omega_+} = u\big|_{\mathbb{R}\times\partial\Omega_-}, \quad \partial_n u\big|_{\mathbb{R}\times\partial\Omega_+} + \partial_n u\big|_{\mathbb{R}\times\partial\Omega_-} = 0, \end{cases} \tag{3.1}$$

where $\lambda \gg 1$ is a large fixed parameter. Evidently, this problem can be rewritten as a RDS in $\mathbb{R}^n$:

$$\partial_t u = \gamma\Delta_x u - u - f_\lambda(t,u), \quad (t,x)\in\mathbb{R}\times\mathbb{R}^n, \tag{3.2}$$

where by definition

$$f_\lambda(t,x,u) := \begin{cases} f(t,u) & \text{for } x\in\Omega_+, \\ (\lambda-1)u & \text{for } x\in\Omega_-. \end{cases} \tag{3.3}$$

The continuity of u and $\nabla_x u$ on $\partial\Omega_+ = \partial\Omega_-$ is now included in the smoothness assumptions for the solutions. The function f_λ is 1-periodic with respect to t and x:

$$f_\lambda(t+1,x,u) = f_\lambda(t,x,u) \quad \text{and} \quad f_\lambda(t,T_l x,u) = f_\lambda(t,x,u) \tag{3.4}$$

for every $l\in\mathbb{Z}^n$ and every $(t,x,u)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^k$.

We now want to show that (3.2) is a small perturbation of the uncoupled system (2.22) on Ω_+. The heuristic argument is clear, since the large parameter $\lambda\gg 1$ corresponds to strong absorption and hence makes the solution very small in the domain Ω_-. Thus, we expect to approximate zero Dirichlet data on $\partial\Omega_+$. To make these arguments precise we recall, from Corollary 1.16, the linear Dirichlet–Neumann operator

$$P_\lambda^- : W_b^{(1/4,1/2),2}(\mathbb{R}\times\partial\Omega_-) \to W_b^{(-1/4,-1/2),2}(\mathbb{R}\times\partial\Omega_-),$$

where Ω is replaced by Ω_- and where we have used that the equation is linear in Ω_-.

By solving the second equation of (3.1) we rewrite this problem in the following equivalent form which is more convenient for our purposes:

$$\begin{cases} \partial_t u = \gamma\Delta_x u - f(t,u) \quad \text{for } (t,x)\in\mathbb{R}\times\Omega_+, \\ \partial_n u\big|_{\mathbb{R}\times\partial\Omega_+} + P_\lambda^-\left(u\big|_{\mathbb{R}\times\partial\Omega_+}\right) = 0. \end{cases} \tag{3.5}$$

Indeed, system (3.5) is a coupled version of system (2.22) in which the coupling is concentrated in the nonlocal linear boundary operator P_λ^-. Note that the nonlocality is in space and time.

We will now use that this coupling is small for $\lambda \gg 1$ such that we are able to show that the linearization of (3.5) at the functions $u\in\Gamma^{\mathrm{tr}}$ is still invertible. We will exploit this in the following section to construct a hyperbolic set $\widehat{\Gamma}_\lambda$ for Eq. (3.2) by structural-stability arguments.

Note that the functions $u\in\Gamma^{\mathrm{tr}}$ are not solutions of (3.5), however we may still study the inhomogeneous, variational equation around these functions:

$$\begin{cases} \partial_t\widetilde{v} = \gamma\Delta_x\widetilde{v} - \widetilde{v} - \mathrm{D}_u f(t,u(t,x))\widetilde{v} + h(t,x) & \text{on } \mathbb{R}\times\Omega_+, \\ \partial_n\widetilde{v} + P_\lambda^-\left(\widetilde{v}|_{\mathbb{R}\times\partial\Omega_+}\right) = g_0 & \text{on } \mathbb{R}\times\partial\Omega_+. \end{cases} \tag{3.6}$$

We will show below that the solution operator M_u^λ giving $\widetilde{v} = M_u^\lambda(g_0,h)$ is well-defined by comparing it with the solution operator M_u^∞ associated with the nonhomogeneous, variational Eq. (2.24), now including also inhomogeneous Dirichlet boundary conditions:

$$\begin{cases} \partial_t v = \gamma\Delta_x v - v - \mathrm{D}_u f(t,u(t,x))v + h(t,x) & \text{in } \mathbb{R}\times\Omega_+, \\ v = v_0 & \text{on } \mathbb{R}\times\partial\Omega_+. \end{cases} \tag{3.7}$$

Hence, M_u^∞ is defined via $v = M_u^\infty(v_0,h)$.

Our main result will be the comparison between $\widetilde{v} = M_u^\lambda(0,h)$ and $v = M_u^\infty(0,h)$ in the form

$$\|\widetilde{v} - v\| = \|M_u^\infty(0,h) - M_u^\lambda(0,h)\| \leqslant C\lambda^{-\beta_p}\|h\|$$

in suitable function spaces. This result is obtained by reducing the problem to the boundary $\mathbb{R}\times\partial\Omega_+$ via the operator $P_u^f\colon \theta_0 \mapsto \partial_n M_u^\infty(\theta_0,0)$. Thus, $w_0 = (v-\widetilde{v})|_{\mathbb{R}\times\partial\Omega_+}$ will satisfy

$$\partial_n v + P_u^f w_0 + P_\lambda^- w_0 = 0 \quad \text{on } \mathbb{R}\times\partial\Omega_+.$$

Our *a priori* estimates for P_λ^- from Proposition 1.15 and Corollary 1.16 imply that the inner product $\langle P_-^\lambda w_0, w_0\rangle$ is of order $\lambda^{1/2}\|w_0\|^2$ as $\lambda\to\infty$. This fact, together with the positivity of the principal part of P_u^f (see Lemma 3.3) allow us to conclude that the operator $P_u^f + P_\lambda^-$ is invertible (and, consequently, w_0 is uniquely determined by $\partial_n v|_{\mathbb{R}\times\partial\Omega_+}$) if λ is large enough. Hence, going backwards, we are able to find $\widetilde{v} = M_u^\lambda(0, h)$.

We now make the above statements rigorous. The main result of this section is the following theorem.

Theorem 3.1. *Let $\Gamma^{\rm tr}$ be the hyperbolic set constructed above. Then there exist constants $\lambda_0 \gg 1$ and $\varepsilon_0, C_\Gamma > 0$ such that for all $\lambda > \lambda_0$, all $\varepsilon \in [0, \varepsilon_0]$, all $y_0 \in \mathbb{R}^{n+1}$, all solutions $u \in \Gamma^{\rm tr}$, and all $h \in \mathrm{W}_{\rm b}^{(0,-1),2}(\mathbb{R} \times \Omega_+)$ Eq.* (3.6) *with $g_0 \equiv 0$ has a unique solution $\widetilde{v}$ and it satisfies*

$$\|\partial_t \widetilde{v}\|_{\mathrm{W}^{(0,-1),2}_{\varphi_\varepsilon, y_0}(\mathbb{R}\times\Omega_+)} + \|\widetilde{v}\|_{\mathrm{W}^{(0,1),2}_{\varphi_\varepsilon, y_0}(\mathbb{R}\times\Omega_+)} \leqslant C_\Gamma \|h\|_{\mathrm{W}^{(0,-1),2}_{\varphi_\varepsilon, y_0}(\mathbb{R}\times\Omega_+)}. \tag{3.8}$$

Moreover, for each $p \in (p_{\min}, p_{\max})$ with $p_{\min} = \frac{2n+4}{n+1}$ and $p_{\max} = \frac{2n+4}{n}$ there exists a constant c_p^* such that

$$\|v - \widetilde{v}\|_{\mathrm{L}^p_{\varphi_{p\varepsilon/2}, y_0}(\mathbb{R}\times\Omega_+)} \leqslant c_p^* C_\Gamma \lambda^{-\beta_p} \|h\|_{\mathrm{W}^{(0,-1),2}_{\varphi_\varepsilon, y_0}(\mathbb{R}\times\Omega_+)} \qquad \text{where } \beta_p = \frac{n(p_{\max} - p)}{4p}, \tag{3.9}$$

and $v = M_u^\infty(0, h)$ is the solution of the linearized problem (2.24) (*which exists due to Lemma 2.7*).

Before we start the proof of this theorem, we provide two lemmas. All the constants will be independent of $u \in \Gamma^{\rm tr}$ and $y_0 \in \mathbb{R}^{n+1}$.

Lemma 3.2. *There exist $\varepsilon_0, C_\Gamma > 0$ such that for all boundary data $\theta_0 \in \mathrm{W}_{\rm b}^{(1/4,1/2),2}(\mathbb{R} \times \partial\Omega_+)$ Eq.* (3.7) *with $h \equiv 0$ has a unique solution θ and the following estimate is valid:*

$$\|\partial_t \theta\|_{\mathrm{W}^{(0,-1),2}_{\varphi_\varepsilon, y_0}(\mathbb{R}\times\Omega_+)} + \|\theta\|_{\mathrm{W}^{(0,1),2}_{\varphi_\varepsilon, y_0}(\mathbb{R}\times\Omega_+)} \leqslant C_\Gamma \|\theta_0\|_{\mathrm{W}^{(1/4,1/2),2}_{\varphi_\varepsilon, y_0}(\mathbb{R}\times\partial\Omega_+)}, \tag{3.10}$$

Thus, the linear operator

$$P_u^f : \mathrm{W}_{\rm b}^{(1/4,1/2),2}(\mathbb{R} \times \partial\Omega_+) \to \mathrm{W}_{\rm b}^{(-1/4,-1/2),2}(\mathbb{R} \times \partial\Omega_+) \tag{3.11}$$

is well defined via $P_u^f \theta_0 := \partial_n \theta\big|_{\mathbb{R}\times\partial\Omega_+}$ and bounded.

Proof. Indeed, estimate (3.10) is a standard corollary of estimates (1.19) and (2.25). The boundedness of P_u^f follows from the trace theorem, analogously to Corollary 1.16. □

Lemma 3.3. *We decompose the operator* P_u^f *as* $P_u^f = P_1^+ + P_u'$*, where the operator* P_1^+ *is the Dirichlet–Neumann operator* P_λ *of* (1.30) *with* $\lambda = 1$ *and* $\Omega = \Omega_+$ *(i.e., with* $f \equiv 0$*). Then there exist constants* $\varepsilon_0, C, C_1, C_2 > 0$ *such that for all* $\varepsilon \in [0, \varepsilon_0]$ *and all* $\theta_0 \in \mathrm{W}_\mathrm{b}^{(1/4,1/2),2}(\mathbb{R} \times \partial\Omega_+)$ *the following estimates hold:*

$$\|P_u'\theta_0\|_{\mathrm{W}^{(1/4,1/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)} \leqslant C\|\theta_0\|_{\mathrm{W}^{(1/4,1/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)}, \tag{3.12}$$

$$C_1\|\theta_0\|^2_{\mathrm{W}^{(1/4,1/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)} \leqslant \langle P_1^+\theta_0, \varphi_{\varepsilon,y_0}\theta_0\rangle_{\mathrm{L}^2(\mathbb{R}\times\partial\Omega_+)} \leqslant C_2\|\theta_0\|^2_{\mathrm{W}^{(1/4,1/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)}. \tag{3.13}$$

Proof. To estimate P_1^+ let θ_1 be the solution of the problem $\partial_t\theta_1 = \gamma\Delta\theta_1 - \theta_1$ in $\mathbb{R}\times\Omega_+$ and $\theta_1 = \theta_0$ in $\mathbb{R}\times\partial\Omega_+$ such that $P_1^+\theta_0 := \partial_n\theta_1\big|_{\Omega_+}$. Multiplying the equation by $\varphi_{\varepsilon,y_0}\theta_1$, integrating over $\mathbb{R}\times\Omega_+$ and arguing as in the proof of (1.32), we derive estimate (3.13).

To estimate P_u' we introduce the function $\theta_2 := \theta - \theta_1$ such that by definition $P_u'\theta_0 = \partial_n\theta_2|_{\mathbb{R}\times\partial\Omega_+}$ and which satisfies the equation

$$\partial_t\theta_2 = \gamma\Delta_x\theta_2 - \theta_2 - \mathrm{D}_u f(t, u(t,x))\theta \text{ in } \mathbb{R}\times\Omega_+, \quad \theta_2 = 0 \text{ on } \mathbb{R}\times\partial\Omega_+, \tag{3.14}$$

where $\theta = M_u^\infty(\theta_0, 0)$.

Because of Lemma 3.2 and the fact that $\mathrm{D}_u f \in \mathrm{L}^\infty$ we have the estimate

$$\|\mathrm{D}_u f(\cdot, u(\cdot,\cdot))\theta\|_{\mathrm{L}^2_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} \leqslant C_1\|\theta_0\|_{\mathrm{W}^{(1/4,1/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)}. \tag{3.15}$$

Applying estimate (1.19) with $s = 2$ to Eq. (3.14) and using (3.15) we obtain

$$\|\theta_2\|_{\mathrm{W}^{(1,2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} \leqslant C_2\|\theta_0\|_{\mathrm{W}^{(1/4,1/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)}.$$

With the standard trace theorem we conclude

$$\|\partial_n\theta_2\|_{\mathrm{W}^{(1/4,1/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)} \leqslant C_3\|\theta_0\|_{\mathrm{W}^{(1/4,1/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)}, \tag{3.16}$$

which is the desired estimate (3.12). □

Proof of Theorem 3.1. Let $v = M_u^\infty(0,h)$ and $w = v - \widetilde{v}$ satisfies the equation

$$\begin{cases} \partial_t w = \gamma\Delta_x w - w - \mathrm{D}_u f(t,u(t,x))w & \text{in } \mathbb{R}\times\Omega_+, \\ \partial_n w + P_\lambda^- \left(w\big|_{\mathbb{R}\times\partial\Omega_+}\right) + \partial_n v = 0 & \text{on } \mathbb{R}\times\partial\Omega_+. \end{cases} \tag{3.17}$$

Because of Lemma 2.7 and the trace theorem (analogously to Corollary 1.16) we have

$$\|\partial_n v\|_{\mathrm{W}^{(-1/4,-1/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)} \leqslant C\|h\|_{\mathrm{W}^{(0,-1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)}. \tag{3.18}$$

Using the operator P_u^f we reduce problem (3.17) to the equivalent pseudo-differential equation on the boundary $\mathbb{R}\times\partial\Omega_+$, namely $P_u^f w_0 + P_\lambda^- w_0 + \partial_n v = 0$ with $w_0 := w\big|_{\mathbb{R}\times\partial\Omega_+}$. Using the decomposition $P_u^f = P_1^+ + P_u'$ of Lemma 3.3 we arrive at

$$P_\lambda^- w_0 + P_1^+ w_0 + P_u' w_0 + \partial_n v = 0, \quad \text{on } \mathbb{R}\times\partial\Omega_+. \tag{3.19}$$

Taking a scalar product in $\mathrm{L}^2(\mathbb{R}\times\partial\Omega_+)$ of Eq. (3.19) with the function $\varphi_{\varepsilon,y_0} w_0$ and using the Cauchy–Schwarz inequality and the estimates (1.32), (3.12), and (3.13) we derive

$$\begin{aligned} &\|w_0\|^2_{\mathrm{W}^{(1/4,1/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)} + \lambda^{1/2}\|w_0\|^2_{\mathrm{L}^2_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)} \\ &\quad \leqslant C_3\|w_0\|^2_{\mathrm{L}^2_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)} + C_3\|\partial_n v\|^2_{\mathrm{W}^{(-1/4,-1/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)}. \end{aligned} \tag{3.20}$$

Inserting estimate (3.18) into (3.20) we derive that for $\lambda > (C_3+1)^2 =: \lambda_0$ the following estimate is valid:

$$\|w_0\|^2_{\mathrm{W}^{(1/4,1/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)} + \lambda^{1/2}\|w_0\|^2_{\mathrm{L}^2_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)} \leqslant C_4\|h\|^2_{\mathrm{W}^{(0,-1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)}$$

for an appropriate constant C_4 which is independent of $\lambda > \lambda_0$. Interpolating with $\alpha \in [0,1]$ we obtain

$$\|w_0\|_{\mathrm{W}^{(\alpha/4,\alpha/2),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\partial\Omega_+)} \leqslant C\lambda^{-(1-\alpha)/4}\|h\|_{\mathrm{W}^{(0,-1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)}, \tag{3.21}$$

$u \in \Gamma^{\mathrm{tr}}$. Using $w_0 = w\big|_{\mathbb{R}\times\partial\Omega_+}$, choosing $\alpha = 1$ in (3.21) and applying estimate (3.10), we find

$$\|\partial_t w\|_{\mathrm{W}^{(0,-1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} + \|w\|_{\mathrm{W}^{(1/2,1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} \leqslant C\|h\|_{\mathrm{W}^{(0,-1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} \tag{3.22}$$

which together with $\widetilde{v} = v - w$ and (2.25) implies (3.8).

To prove estimate (3.9) we use (3.21) with $0<\alpha<1$ and (1.19) with $s=(\alpha+1)/2$ to derive

$$\|w\|_{W^{((\alpha+1)/4,(\alpha+1)/2),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} \leqslant C'\lambda^{-(1-\alpha)/4}\|h\|_{W^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}. \tag{3.23}$$

The presence of the subordinated term $D_u f(t,u(t,x))w$ in (3.10) in comparison with (1.15) is not essential thanks to estimate (2.25). Applying now an embedding theorem for anisotropic Sobolev spaces (cf. [24]) we obtain

$$\|w\|_{L^{p_\alpha}_{\varphi_{p_\alpha\varepsilon/2},y_0}(\mathbb{R}\times\Omega)} \leqslant C_p\|w\|_{W^{((\alpha+1)/4,(\alpha+1)/2),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}, \tag{3.24}$$

where $p_\alpha:=\frac{2(n+2)}{n+1-\alpha}\in(p_{\min},p_{\max})$ and the constant C_p is independent of $y_0\in\mathbb{R}^{n+1}$. This estimate, together with inequality (3.23) proves estimate (3.9). Having the *a priori* estimate (3.8) for the solutions of (3.6) one can verify the existence of a solution $\widetilde{v}$ in a standard way. Theorem 3.1 is proved. □

Corollary 3.4. *Under the assumptions of Theorem 3.1 the following estimates hold:*

$$\|\partial_t\widetilde{v}\|_{W^{(0,-1),2}_b(\mathbb{R}\times\Omega_+)}+\|\widetilde{v}\|_{W^{(0,1),2}_b(\mathbb{R}\times\Omega_+)} \leqslant C_\Gamma\|h\|_{W^{(0,-1),2}_b(\mathbb{R}\times\Omega_+)} \tag{3.25}$$

and

$$\|v-\widetilde{v}\|_{L^p_b(\mathbb{R}\times\Omega_+)} \leqslant C_\Gamma\lambda^{-\beta_p}\|h\|_{W^{(0,-1),2}_b(\mathbb{R}\times\Omega_+)}, \tag{3.26}$$

where $p\in(p_{\min},p_{\max}):=(\frac{2n+4}{n+1},\frac{2n+4}{n})$, $\beta_p=\frac{n(p_{\max}-p)}{4p}>0$ *and all constants are independent of the concrete choice of* $u\in\Gamma^{tr}$.

Indeed, estimates (3.25) and (3.26) are immediate corollaries of Theorem 3.1 and Proposition 1.9.

In conclusion of this section we consider problem (3.6) with nonhomogeneous boundary conditions, i.e., we estimate $w=M^\lambda_u(w_0,0)$.

Corollary 3.5. *Let the assumptions of Theorem 3.1 hold. Then for every* $\lambda>\lambda_0$ *and for every* $w_0\in W^{(-1/4,-1/2),2}_b(\mathbb{R}\times\partial\Omega_+)$ *the solution* $w=M^\lambda_u(w_0,0)$ *of* (3.6) *exists uniquely and satisfies the estimate:*

$$\begin{aligned}\|\partial_t w\|_{W^{(0,-1),2}_b(\mathbb{R}\times\Omega_+)}+\|w\|_{W^{(0,1),2}_b(\mathbb{R}\times\Omega_+)}+\lambda^{\beta_p}\|w\|_{L^p_b(\mathbb{R}\times\Omega_+)}\\ \leqslant C_\Gamma\|w_0\|_{W^{(-1/4,-1/2),2}_b(\mathbb{R}\times\partial\Omega_+)}.\end{aligned} \tag{3.27}$$

Moreover, for $\varepsilon\in[0,\varepsilon_0]$ *the following weighted analog of* (3.27) *holds:*

$$\begin{aligned}&\|\partial_t w\|_{W^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}+\|w\|_{W^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}+\lambda^{\beta_p}\|w\|_{L^p_{\varphi_{p\varepsilon/2,y_0}}(\mathbb{R}\times\Omega_+)}\\&\quad\leqslant C_\Gamma\|w_0\|_{W^{(-1/4,-1/2),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\partial\Omega_+)},\end{aligned}\tag{3.28}$$

where $p\in(p_{\min},p_{\max})$ *and* $\beta_p=\frac{n(p_{\max}-p)}{4p}>0$ *as above.*

Indeed, estimates (3.27) and (3.28) have been obtained in the proof of Theorem 3.1 (compare equations (3.6) and (3.17)).

4. STRUCTURAL STABILITY

In this section we construct a hyperbolic trajectory set $\Gamma^{\mathrm{tr}}_\lambda$ for the coupled problem (3.5) which will be homeomorphic to the hyperbolic trajectory set Γ^{tr} of the uncoupled problem (2.22).

The main result of the section is the following theorem.

Theorem 4.1. *Let* Γ_0 *be a nonempty hyperbolic set for problem* (2.1). *Then there is a constant* $\lambda_1\gg 1$ *such that for every* $\lambda>\lambda_1$ *there is a hyperbolic trajectory set* $\Gamma^{\mathrm{tr}}_\lambda$ *for Eq.* (3.5) *which is homeomorphic to* $\Gamma^{\mathrm{tr}}=(\Gamma_0^{tr})^{\mathbb{Z}^n}$*:*

$$\kappa^{\mathrm{tr}}_\lambda:\Gamma^{\mathrm{tr}}\leftrightarrow\Gamma^{\mathrm{tr}}_\lambda.\tag{4.1}$$

This homeomorphism commutes with the discrete space–time translations $\{\mathbb{T}_l\colon l=(l_0,l')\in\mathbb{Z}^{n+1}\}$ *(defined via* $(\mathbb{T}_{(l_0,l')}u)(t,x):=u(t+l_0,x+l')$*), i.e.,*

$$\kappa^{\mathrm{tr}}_\lambda\circ\mathbb{T}_l=\mathbb{T}_l\circ\kappa^{\mathrm{tr}}_\lambda\quad\textit{for } l\in\mathbb{Z}^{n+1}.\tag{4.2}$$

Moreover, $\kappa^{\mathrm{tr}}_\lambda$ *is bi-Lipschitz continuous in the weighted topology, i.e., there exist constants* $C_1,C_2>0$, $\lambda_1>0$, *and* $\varepsilon_0>0$ *such that for all* $\lambda>\lambda_1$, $\varepsilon\in(0,\varepsilon_0)$ *and all* $u_1,u_2\in\Gamma^{\mathrm{tr}}$ *the following estimate is valid:*

$$\begin{aligned}&C_1\left(\|\partial_t u_1-\partial_t u_2\|_{W^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}+\|u_1-u_2\|_{W^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}\right)\\&\quad\leqslant\|\partial_t\widetilde{u}_1-\partial_t\widetilde{u}_2\|_{W^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}+\|\widetilde{u}_1-\widetilde{u}_2\|_{W^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}\\&\quad\leqslant C_2\left(\|\partial_t u_1-\partial_t u_2\|_{W^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}+\|u_1-u_2\|_{W^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}\right),\end{aligned}\tag{4.3}$$

where $\widetilde{u}_i=\kappa^{\mathrm{tr}}_\lambda(u)$ *and* $i=1,2.$

Proof. Indeed, let $u \in \Gamma^{\mathrm{tr}}$ be a hyperbolic trajectory of the uncoupled system (2.22). We will search for the corresponding hyperbolic trajectory $\widetilde{u} := \kappa_\lambda^{\mathrm{tr}}(u)$ in the form $\widetilde{u} = u + w$. Then, the correction w has to satisfy the equation

$$\begin{cases} \partial_t w = \gamma \Delta_x w - w - [f(t, u(t,x) + w(t,x)) - f(t, u(t,x))] & \text{in } \mathbb{R} \times \Omega_+, \\ P_\lambda^- \left(w\big|_{\mathbb{R} \times \partial\Omega_+} \right) + \partial_n w + \partial_n u = 0 & \text{on } \mathbb{R} \times \partial\Omega_+. \end{cases} \tag{4.4}$$

Define the function $\widetilde{w}_\lambda$ as $M_u^\lambda(-\partial_n u, 0)$, i.e., as solution of (3.6) with $h \equiv 0$. Corollary 3.5 and estimate (2.6) give

$$\|\widetilde{w}_\lambda\|_{L_b^p(\mathbb{R} \times \Omega_+)} \leqslant C_p \lambda^{-\beta_p} \|\partial_n u\|_{W_b^{(-1/4, -1/2), 2}(\mathbb{R} \times \Omega_+)} \leqslant C_p^* \lambda^{-\beta_p}, \tag{4.5}$$

where $p \in (p_{\min}, p_{\max})$ is fixed and C_p^* is independent of λ and $u \in \Gamma^{\mathrm{tr}}$.

Now the function $\theta = w - \widetilde{w}_\lambda$ has to satisfy the equation

$$\begin{cases} \partial_t \theta = \gamma \Delta_x \theta - \theta - \mathrm{D}_u f(t, u(t,x))\theta - h(t, x, \theta) & \text{in } \mathbb{R} \times \Omega_+, \\ P_\lambda^- \left(\theta\big|_{\mathbb{R} \times \partial\Omega_+} \right) + \partial_n \theta = 0 & \text{on } \mathbb{R} \times \partial\Omega_+, \end{cases} \tag{4.6}$$

where $h(t, x, \theta) := f(t, u(t,x) + \widetilde{w}_\lambda(t,x) + \theta) - f(t, u(t,x)) - \mathrm{D}_u f(t, u(t,x))\theta$. Recalling the solution operator M_u^λ for system (3.6), Eq. (4.6) can be rewritten in the form

$$\theta + M_u^\lambda(0, [f(\cdot, u + \widetilde{w}_\lambda + \theta) - f(\cdot, u) - \mathrm{D}_u f(\cdot, u)\theta]) = 0.$$

We are going to solve this equation with the help of the implicit function theorem. For fixed $p \in (p_{\min}, p_{\max})$ we define the mapping

$$\Phi : \begin{cases} [1, \infty] \times L_b^p(\mathbb{R} \times \Omega_+) \to L_b^p(\mathbb{R} \times \Omega_+), \\ (\lambda, \theta) \mapsto \theta + M_u^\lambda(0, [f(\cdot, u + \widetilde{w}_\lambda + \theta) - f(\cdot, u) - \mathrm{D}_u f(\cdot, u)\theta]). \end{cases}$$

Recall that M_u^∞ is the solution operator of problem (2.24) and that $\widetilde{w}_\infty \equiv 0$. It follows now from (2.3) (3.8), (3.9) and (4.5) that $\Phi \in C^0([1, \infty] \times L_b^p, L_b^p)$, $\mathrm{D}_\theta \Phi \in C^0([1, \infty] \times L_b^p, \mathrm{Lin}\,(L_b^p, L_b^p))$, $\Phi(\infty, 0) = 0$ and $\mathrm{D}_\theta \Phi(\infty, 0) = \mathrm{Id}$. Consequently, due to the implicit function theorem there is $\lambda_0 \gg 1$ such that for every $\lambda > \lambda_1$ there is a unique $\theta_\lambda \in L_b^p(\mathbb{R} \times \Omega_+)$ such that

$$\Phi(\lambda, \theta_\lambda) = 0 \quad \text{and} \quad \|\theta_\lambda\|_{L_b^p(\mathbb{R} \times \Omega_+)} \leqslant C \lambda^{-\beta_p}, \tag{4.7}$$

where the constant C is independent of $u \in \Gamma^{\mathrm{tr}}$. Consequently, $\theta = \theta_\lambda$ solves (4.6). Moreover, $\theta_\lambda = -M_u^\lambda(0, h)$ implies via (3.8) and (3.9) that

$$\|\partial_t \theta_\lambda\|_{W_b^{(0,-1),2}(\mathbb{R} \times \Omega_+)} + \|\theta_\lambda\|_{W_b^{(0,1),2}(\mathbb{R} \times \Omega)} \leqslant C_1 \lambda^{-\beta_p}, \tag{4.8}$$

since h can be estimated as follows. From (2.3) and (4.5) we obtain

$$\|h\|_{L^2_b(\mathbb{R}\times\Omega_+)} = \|f(\cdot,u+\widetilde{w}_\lambda+\theta_\lambda) - f(\cdot,u) - D_u f(\cdot,u)\theta_\lambda]\|_{L^2_b(\mathbb{R}\times\Omega_+)} \leqslant C\left(\|\theta_\lambda\|_{L^2_b(\mathbb{R}\times\Omega)} + \|\widetilde{w}_\lambda\|_{L^2_b(\mathbb{R}\times\Omega)}\right).$$

Together with (4.7) and (4.5) this implies estimate (4.8) for sufficiently large λ.

Now define $\kappa^{\rm tr}_\lambda(u) := \widetilde{u} := u + \widetilde{w}_\lambda + \theta_\lambda$, which, by construction, is a solution of equation (3.5) and hence of the RDS (3.2). Moreover, due to (4.5) and (4.8) we have the closeness condition

$$\|u-\widetilde{u}\|_{L^p_b(\mathbb{R}\times\Omega_+)} \leqslant C\lambda^{-\beta_p}. \tag{4.9}$$

Let us verify that $\widetilde{u}$ is a hyperbolic trajectory for system (3.5). According to Definition 2.1 we have to consider the linearized problem

$$\begin{cases} \partial_t\widetilde{v} = \gamma\Delta_x\widetilde{v} - \widetilde{v} - D_u f(t,\widetilde{u})\widetilde{v} + h(t,x) & \text{on } \mathbb{R}\times\Omega_+, \\ P^-_\lambda\left(\widetilde{v}\big|_{\mathbb{R}\times\partial\Omega_+}\right) + \partial_n\widetilde{v} = 0 & \text{on } \mathbb{R}\times\partial\Omega_+. \end{cases} \tag{4.10}$$

Indeed, it follows from (2.3) and (4.9) and from the embedding theorem for the anysotropic Sobolev spaces that

$$\begin{aligned} \|[D_u f(\cdot,u) - D_u f(\cdot,\widetilde{u})]\widetilde{v}\|_{L^2(\mathbb{R}\times\Omega_+)} &\leqslant C\lambda^{-\beta}\|\widetilde{v}\|_{L^{p_{\max}}(\mathbb{R}\times\Omega_+)} \\ &\leqslant C_1\Big(\|\partial_t\widetilde{v}\|_{W^{(0,-1),2}(\mathbb{R}\times\Omega_+)} \\ &\qquad + \|\widetilde{v}\|_{W^{(0,1),2}(\mathbb{R}\times\Omega_+)}\Big) \end{aligned} \tag{4.11}$$

for some positive $\beta > 0$. Applying this to the right-hand side of (4.10) and employing the hyperbolicity of u as given in estimate (3.8) with $\phi = 1$ (i.e., $\varepsilon = 0$), we easily obtain

$$\|\partial_t\widetilde{v}\|_{W^{(0,-1),2}(\mathbb{R}\times\Omega_+)} + \|\widetilde{v}\|_{W^{(0,1),2}(\mathbb{R}\times\Omega_+)} \leqslant C'_\Gamma\|h\|_{W^{(0,-1),2}(\mathbb{R}\times\Omega_+)},$$

where the constant C'_Γ is independent of $\lambda > \lambda_0$ and of the concrete choice of $\widetilde{u} = \kappa^{\rm tr}_\lambda(u) \in \Gamma^{\rm tr}_\lambda$. Thus, $\widetilde{u}$ is a hyperbolic trajectory of (3.5).

Thus, we have shown that $\Gamma^{\rm tr}_\lambda := \kappa^{\rm tr}_\lambda(\Gamma^{\rm tr})$ is a hyperbolic set for the coupled problem (3.5). The commutation properties (4.2) are immediate corollaries of our construction of $\kappa^{\rm tr}_\lambda$ and of the uniqueness part of the implicit function theorem. Thus, it remains only to verify the local Lipschitz continuity (4.3). For this, let $u_1, u_2 \in \Gamma^{\rm tr}$ be two hyperbolic solutions of (2.22) and let $\widetilde{u}_i := \kappa^{\rm tr}_\lambda(u_i)$, $i = 1, 2$, be the corresponding hyperbolic solutions of (3.5). Define also the functions $\widetilde{w}^i_\lambda$ as the solutions of problem (4.4) associated with u_i, $i=1,2$, respectively and the functions $\theta_i := \widetilde{u}_i - u_i - \widetilde{w}^i_\lambda$. Then, the function $\widetilde{w}_\lambda := \widetilde{w}^1_\lambda - \widetilde{w}^2_\lambda$ satisfies the equation

$$\begin{cases} \partial_t \widetilde{w}_\lambda = \gamma \Delta_x \widetilde{w}_\lambda - \widetilde{w}_\lambda - \mathrm{D}_u f(t,u_1)\widetilde{w}_\lambda + h, \\ P^-_\lambda \left(\widetilde{w}_\lambda \big|_{\mathbb{R}\times\partial\Omega_+} \right) + \partial_n \widetilde{w}_\lambda \big|_{\mathbb{R}\times\partial\Omega_+} + \left(\partial_n u_1 \big|_{\mathbb{R}\times\partial\Omega_+} - \partial_n u_2 \big|_{\mathbb{R}\times\partial\Omega_+} \right) = 0, \end{cases} \tag{4.12}$$

where $h(t,x) := [\mathrm{D}_u f(t,u_1(t,x)) - \mathrm{D}_u f(t,u_2(t,x))]\widetilde{w}^2_\lambda(t,x)$. We claim that there are $C, \varepsilon_0 > 0$ such that for every $\varepsilon \in [0,\varepsilon_0]$ the following estimate holds:

$$\begin{aligned} &\|\partial_t \widetilde{w}_\lambda\|_{\mathrm{W}^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} + \|\widetilde{w}_\lambda\|_{\mathrm{W}^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} + \lambda^{\beta_p} \|\widetilde{w}_\lambda\|_{\mathrm{L}^p_{\varphi_{p\varepsilon/2},y_0}(\mathbb{R}\times\Omega_+)} \\ &\quad \leqslant C \left(\|\partial_t u_1 - \partial_t u_2\|_{\mathrm{W}^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} + \|u_1 - u_2\|_{\mathrm{W}^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} \right), \end{aligned} \tag{4.13}$$

where $p \in (p_{\min}, p_{\max})$ is fixed and C and ε_0 are independent of $u_i \in \Gamma^{\mathrm{tr}}$, $\lambda > \lambda_0$ and of $y_0 \in \mathbb{R}^{n+1}$.

In order to derive (4.13) we need the following standard smoothing property for the solutions of (2.22). □

Lemma 4.2. *Let $u_i \in \mathcal{K}$, $i=1,2$, be two essential solutions of problem* (2.22). *Then there is $\varepsilon_0 > 0$ such that for every $\varepsilon \in [0,\varepsilon_0]$ the following estimates hold:*

$$\begin{aligned} C_1 \|u_1 - u_2\|_{\mathrm{C}^1_{\varphi_{\varepsilon/2},y_0}(\mathbb{R}\times\Omega_+)} &\leqslant \|\partial_t u_1 - \partial_t u_2\|_{\mathrm{W}^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} \\ &\quad + \|u_1 - u_2\|_{\mathrm{W}^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} \\ &\leqslant C_2 \|u_1 - u_2\|_{\mathrm{C}^1_{\varphi_{\varepsilon/2},y_0}(\mathbb{R}\times\Omega_+)}, \end{aligned} \tag{4.14}$$

where the constants ε_0 and C_i are independent of $u_i \in \mathcal{K}$ and of $y_0 \in \mathbb{R}^{n+1}$.

Proof. Indeed, the function $v := u_1 - u_2$ satisfies the linear equation

$$\partial_t v = \gamma \Delta_x v - v - A(t,x)v \text{ in } \mathbb{R}\times\Omega_+, \quad v = 0 \text{ on } \mathbb{R}\times\partial\Omega_+ \tag{4.15}$$

with $A(t,x) := \int_0^1 \mathrm{D}_u f(t, su_1(t,x) + (1-s)u_2(t,x))\mathrm{d}s$ and hence, by (2.3) and estimates (2.6) for the solutions u_i, we have $A \in \mathrm{C}^1_{\mathrm{b}}(\mathbb{R}\times\Omega_+, \mathbb{R}^{m\times m})$.

Applying the standard parabolic regularity theorems (see, e.g., [24]) to Eq. (4.15), we derive estimates (4.14) and Lemma 4.2 is proved. □

Estimate (4.14) together with estimate (4.5) for $\widetilde{w}_\lambda^2$ imply that

$$\|h\|_{L^2_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} \leqslant C\lambda^{-\beta_p}\left(\|\partial_t u_1-\partial_t u_2\|_{W^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}+\|u_1-u_2\|_{W^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}\right), \tag{4.16}$$

where h is defined after (4.12), $\beta_p>0$ and C is independent of λ, y_0 and u_i. Applying estimates (3.8), (3.9) and (3.28) to Eq. (4.12) and using (4.16) we obtain the desired estimate (4.13).

It remains to estimate $\theta:=\theta_1-\theta_2$ which satisfies the equation

$$\begin{cases} \partial_t\theta=\gamma\Delta_x\theta-\theta-D_u f(t,u_1)\theta+\widetilde{h} & \text{in } \mathbb{R}\times\Omega_+,\\ P^-_\lambda\left(\theta\big|_{\mathbb{R}\times\partial\Omega_+}\right)+\partial_n\theta=0 & \text{on } \mathbb{R}\times\partial\Omega_+; \end{cases} \tag{4.17}$$

where $\widetilde{h}=h_1+h_2+h_3$ with

$$\begin{aligned} &h_1:=[f(\cdot,u_1+\widetilde{w}_\lambda^2+\theta_2)-f(\cdot,u_1+\widetilde{w}_\lambda^1+\theta_2)],\\ &h_2:=-[f(\cdot,u_1+\widetilde{w}_\lambda^1+\theta_1)-f(\cdot,u_1+\widetilde{w}_\lambda^1+\theta_2)-D_u f(\cdot,u_1)\theta],\\ &h_3:=f(\cdot,u_1+\widetilde{w}_\lambda^2+\theta_2)-f(\cdot,u_2+\widetilde{w}_\lambda^2+\theta_2)+f(\cdot,u_1)-f(\cdot,u_2). \end{aligned} \tag{4.18}$$

We estimate every term h_i separately. Due to the global Lipschitz continuity of f and estimate (4.13) for $\widetilde{w}_\lambda$ we have

$$\|h_1\|_{L^2_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} \leqslant C_1\lambda^{-\beta_p}\left(\|\partial_t u_1-\partial_t u_2\|_{W^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}+\|u_1-u_2\|_{W^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}\right). \tag{4.19}$$

Moreover, due to the uniform Hölder continuity of $D_u f$ one has the estimate

$$\begin{aligned} &|f(t,u_1+\widetilde{w}_\lambda^1+\theta_1)-f(t,u_1+\widetilde{w}_\lambda^1+\theta_2)-D_u f(t,u_1)\theta|\\ &\quad\leqslant\int_0^1|D_u f(t,u_1+\widetilde{w}_\lambda^1+s\theta_1+(1-s)\theta_2)-D_u f(t,u_1)|\,ds\,|\theta|\\ &\quad\leqslant C\left(|\widetilde{w}_\lambda^1|^\delta+|\theta_1|^\delta+|\theta_2|^\delta\right)|\theta|, \end{aligned}$$

where $0<\delta\leqslant 1$ is arbitrary and the constant C depends only on f. Fixing δ small enough and using estimates (4.5) and (4.7) and the Hölder inequality we arrive at

$$\begin{aligned} \|h_2\|_{L^2_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} &\leqslant C\left(\|\widetilde{w}_\lambda^1\|^\delta_{L^p_b(\mathbb{R}\times\Omega_+)}+\|\theta_1\|^\delta_{L^p_b(\mathbb{R}\times\Omega_+)}+\|\theta_2\|^\delta_{L^p_b(\mathbb{R}\times\Omega_+)}\right)\|\theta\|_{L^p_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}\\ &\leqslant C_2\lambda^{-\delta\beta_p}\|\theta\|_{L^p_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}. \end{aligned} \tag{4.20}$$

Finally using the Lipschitz continuity of $D_u f$ we obtain the estimate

$$|h_3(t,x)|\leqslant C\left(|\widetilde{w}_\lambda^2(t,x)|+|\theta_2(t,x)|\right)|u_1(t,x)-u_2(t,x)|$$

and consequently, due to (4.5), (4.13) we find

$$\|h_3\|_{L^2_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} \leqslant C_3\lambda^{-\beta_p}\Big(\|\partial_t u_1-\partial_t u_2\|_{W^{(0,-1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} + \|u_1-u_2\|_{W^{(0,1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)}\Big). \tag{4.21}$$

In the three estimates (4.19), (4.20) and (4.21) the exponent $p\in(p_{\min}, p_{\max})$ is fixed and the constants C_1, C_2, and C_3 are independent of $y_0\in\mathbb{R}^{n+1}$, $\lambda>\lambda_1$ and $u_i\in\Gamma^{\mathrm{tr}}$.

Applying estimate (3.8) to Eq. (4.17) and using inequalities (4.19), (4.20) and (4.21) for the right-hand side $h=h_1+h_2+h_3$ we obtain

$$\|\partial_t\theta_1-\partial_t\theta_2\|_{W^{(0,-1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} + \|\theta_1-\theta_2\|_{W^{(0,1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} \leqslant C\lambda^{-\beta_p}\left(\|\partial_t u_1-\partial_t u_2\|_{W^{(0,-1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} + \|u_1-u_2\|_{W^{(0,1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)}\right). \tag{4.22}$$

Using $\widetilde{u}_1-\widetilde{u}_2=[u_1-u_2]+[\widetilde{w}^1_\lambda-\widetilde{w}^2_\lambda]+[\theta_1-\theta_2]$, estimates (4.13) and (4.22) imply

$$\begin{aligned} C_1\|u_1-u_2\|_{L^p_{\varphi_{p\varepsilon/2,y_0}}(\mathbb{R}\times\Omega_+)} &\leqslant C_1'\|\widetilde{u}_1-\widetilde{u}_2\|_{L^p_{\varphi_{p\varepsilon/2,y_0}}(\mathbb{R}\times\Omega_+)} \\ &\leqslant \|\partial_t\widetilde{u}_1-\partial_t\widetilde{u}_2\|_{W^{(0,-1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} + \|\widetilde{u}_1-\widetilde{u}_2\|_{W^{(0,1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} \\ &\leqslant C_2\Big(\|\partial_t u_1-\partial_t u_2\|_{W^{(0,-1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} + \|u_1-u_2\|_{W^{(0,1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)}\Big). \end{aligned} \tag{4.23}$$

To finish the proof of Theorem 4.1 it remains to note that estimate (1.22) with $s=1$ applied to Eq. (4.15) gives

$$\|\partial_t u_1-\partial_t u_2\|_{W^{(0,-1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} + \|u_1-u_2\|_{W^{(0,1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}\times\Omega_+)} \leqslant C_3\|u_1-u_2\|_{L^p_{\varphi_{p\varepsilon/2,y_0}}(\mathbb{R}\times\Omega_+)}.$$

This estimate together with (4.23) provide (4.3); and thus Theorem 4.1 is proved. □

Until now we have discussed the problem (3.5) on Ω_+ with the non-local boundary conditions depending on λ. Now we return to the RDS (3.2) defined on all of $\mathbb{R}^n$, but with the nonlinearity f_λ defined in (3.3). The associated initial value problem reads (3.2), i.e.,

$$\begin{cases} \partial_t u=\gamma\Delta_x u-u-f_\lambda(t,x,u) & \text{for } (t,x)\in(0,\infty)\times\mathbb{R}^n, \\ u(0,x)=u^0(x) & \text{for } x\in\mathbb{R}^n. \end{cases} \tag{4.24}$$

First of all we note that (2.3) and (3.3) imply that for every $\lambda > 0$ and every $u^0 \in \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n)$ problem (4.24) has a unique solution $u\colon [0,\infty) \to \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n)$ and the following estimate holds:

$$\|u(t)\|_{\mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n)} \leqslant C\|u(0)\|_{\mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n)} e^{-\alpha t} + C_\lambda, \quad \alpha > 0 \tag{4.25}$$

and consequently the solution operator

$$\widehat{S}^\lambda_t : \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n) \to \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n), \quad u^0 \mapsto u(t) \tag{4.26}$$

is well defined (see, e.g., [46]). Moreover, it is known (see, e.g., [32,33, 46]) that the Poincaré map $\widehat{S}_1$ associated with problem (4.24) possesses a global, locally compact attractor $\widehat{\mathcal{A}}_\lambda \subset \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n)$ which is generated by the set $\widehat{\mathcal{K}}_\lambda \subset \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}\times\mathbb{R}^n)$ of all essential solutions of problem (3.2), i.e.,

$$\widehat{\mathcal{A}}_\lambda = \Pi_0 \widehat{\mathcal{K}}_\lambda.$$

Moreover, the following result is a standard corollary of the parabolic regularity theorems (see, e.g., [24]).

Proposition 4.3. *Every trajectory $\widehat{u} \in \widehat{\mathcal{K}}_\lambda$ lies in $\mathrm{C}^1_{\mathrm{b}}(\mathbb{R}\times\mathbb{R}^n)$ and satisfies the estimate $\|\widehat{u}\|_{\mathrm{C}^1_{\mathrm{b}}(\mathbb{R}\times\mathbb{R}^n)} \leqslant C_\lambda$, where C_λ depends on λ but is independent of the concrete choice of $\widehat{u} \in \widehat{\mathcal{K}}_\lambda$. Moreover, the set $\widehat{\mathcal{K}}_\lambda$ is compact in the local topology of $\mathrm{C}^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^n)$, namely $\widehat{\mathcal{K}}_\lambda \Subset \mathrm{C}^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^n)$.*

Note that is is sufficient to have f_λ smooth in u and t. The jumps over the boundary $\partial\Omega_+$ do prevent higher regularity, but not the C^1-regularity.

Let us study now the hyperbolic trajectory set $\widehat{\Gamma}^{\mathrm{tr}}_\lambda \subset \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}\times\mathbb{R}^n)$ for Eq. (4.24) associated with the corresponding set $\Gamma^{\mathrm{tr}}_\lambda \subset \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}\times\Omega_+)$.

Corollary 4.4. *Let $\lambda_1 \gg 1$ be the same as in Theorem 4.1. Then for every $\lambda > \lambda_1$ Eq.* (4.24) *possesses a hyperbolic set $\widehat{\Gamma}^{\mathrm{tr}}_\lambda \subset \widehat{\mathcal{K}}_\lambda$ which is homeomorphic to Γ^{tr} via*

$$\widehat{\kappa}^{\mathrm{tr}}_\lambda \colon \Gamma^{\mathrm{tr}} \leftrightarrow \widehat{\Gamma}^{\mathrm{tr}}_\lambda.$$

Moreover, the homeomorphism $\widehat{\kappa}^{\mathrm{tr}}_\lambda$ satisfies the commutation properties (4.2) *and is Lipschitz continuous in the local topology, i.e., for each $\lambda > \lambda_1$ there exist constants C_1 and C_2 such that for all $u_1, u_2 \in \Gamma^{\mathrm{tr}}$ we have*

$$\begin{aligned}
&C_1\Big(\|\partial_t u_1 - \partial_t u_2\|_{\mathrm{W}^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} + \|u_1-u_2\|_{\mathrm{W}^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}\Big)\\
&\quad\leqslant \|\widehat{\kappa}^{\mathrm{tr}}_\lambda(u_1) - \widehat{\kappa}^{\mathrm{tr}}_\lambda(u_2)\|_{\mathrm{C}^1_{\varphi_{\varepsilon/2},y_0}(\mathbb{R}\times\mathbb{R}^n)}\\
&\quad\leqslant C_2\Big(\|\partial_t u_1 - \partial_t u_2\|_{\mathrm{W}^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} + \|u_1-u_2\|_{\mathrm{W}^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)}\Big).
\end{aligned} \tag{4.27}$$

Note that in contrast to (4.3) here C_1, C_2 depend on λ. This is because we switch to the noncompatible C^1 topology, cf. also Proposition 4.3.

Proof. Indeed, let $u \in \Gamma^{\mathrm{tr}}$ and let $\widetilde{u} = \kappa_\lambda^{\mathrm{tr}}(u) \in \Gamma_\lambda^{\mathrm{tr}}$. Define the function $\widehat{u} := \widehat{\kappa}_\lambda^{\mathrm{tr}}(u) \in \widehat{\Gamma}_\lambda^{\mathrm{tr}}$ via the trivial linear extension $E_\lambda : \mathrm{L}^2_{\mathrm{b}}(\mathbb{R} \times \Omega_+) \to \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}\times\mathbb{R}^n)$ and let $\widetilde{u} \mapsto \widehat{u}$ be defined as follows:

$$\widehat{u}(t,x) := \begin{cases} \widetilde{u}(t,x), & \text{if } x \in \Omega_+, \\ v_u(t,x), & \text{if } x \in \Omega_-, \end{cases}$$

where v_u is the unique solution v of the problem

$$\partial_t v = \gamma \Delta_x v - \lambda v \text{ in } \mathbb{R} \times \Omega_- \quad \text{and} \quad v\big|_{\mathbb{R}\times\partial\Omega_-} = \widetilde{u}\big|_{\mathbb{R}\times\partial\Omega_+}.$$

This works well since the RDS (4.24) is linear in Ω_-. Because of (1.24), we have

$$\|\partial_t \widehat{u}\|_{\mathrm{W}^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}^{n+1})} + \|\widehat{u}\|_{\mathrm{W}^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}^{n+1})} \leqslant C_\lambda \Big(\|\widetilde{u}\|_{\mathrm{W}^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} + \|\widetilde{u}\|_{\mathrm{W}^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} \Big). \tag{4.28}$$

According to the construction of the operator P_λ^- the function $\widehat{u}$ is a solution of problem (3.2) and consequently $\widehat{u} \in \widehat{\mathcal{K}}_\lambda$. Moreover, it is not difficult to verify, using Theorem 4.1 and estimate (1.24), that $\widehat{u}$ is a hyperbolic trajectory of the system (4.24), i.e., for every $h \in \mathrm{W}^{(0,-1),2}(\mathbb{R}\times\mathbb{R}^n)$ the problem

$$\partial_t v = \gamma \Delta_x v - v - \mathrm{D}_u f_\lambda(t,x,\widehat{u}(t,x))v + h(t,x) \quad \text{on } \mathbb{R}^{n+1} \tag{4.29}$$

possesses a unique solution v which satisfies the estimate

$$\|\partial_t v\|_{\mathrm{W}^{(0,-1),2}(\mathbb{R}^{n+1})} + \|v\|_{\mathrm{W}^{(0,1),2}(\mathbb{R}^{n+1})} \leqslant C_\Gamma \|h\|_{\mathrm{W}^{(0,-1),2}(\mathbb{R}^{n+1})}, \tag{4.30}$$

where the constant C_Γ depends on λ but are independent of the concrete choice of $u \in \Gamma^{\mathrm{tr}}$. Thus, $\widehat{\Gamma}_\lambda^{\mathrm{tr}} := \widehat{\kappa}_\lambda^{\mathrm{tr}}(\Gamma^{\mathrm{tr}})$ is a hyperbolic set for problem (4.24).

Let now $\widehat{u}_i = \widehat{\kappa}_\lambda^{\mathrm{tr}}(u_i)$. Then combining (4.3) and (4.30) we obtain

$$\begin{aligned} &\widetilde{C}_1 \Big(\|\partial_t u_1 - \partial_t u_2\|_{\mathrm{W}^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} + \|u_1 - u_2\|_{\mathrm{W}^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} \Big) \\ &\quad \leqslant \|\partial_t \widehat{u}_1 - \partial_t \widehat{u}_2\|_{\mathrm{W}^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}^{n+1})} + \|\widehat{u}_1 - \widehat{u}_2\|_{\mathrm{W}^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}^{n+1})} \\ &\quad \leqslant \widetilde{C}_2 \Big(\|\partial_t u_1 - \partial_t u_2\|_{\mathrm{W}^{(0,-1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} + \|u_1 - u_2\|_{\mathrm{W}^{(0,1),2}_{\varphi_\varepsilon,y_0}(\mathbb{R}\times\Omega_+)} \Big), \end{aligned} \tag{4.31}$$

where the constants $\widetilde{C}_i$ depend on λ but is independent of the concrete choice of $u_i \in \Gamma^{\rm tr}$. To finish the proof of estimate (4.27) we use the parabolic regularity theory which gives

$$\|\widehat{u}_1-\widehat{u}_2\|_{C^1_{\varphi_{\varepsilon/2,y_0}}(\mathbb{R}^{n+1})} \leqslant C\Big(\|\partial_t\widehat{u}_1-\partial_t\widehat{u}_2\|_{W^{(0,-1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}^{n+1})}+\|\widehat{u}_1-\widehat{u}_2\|_{W^{(0,1),2}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}^{n+1})}\Big), \tag{4.32}$$

where C is independent of $\widehat{u}_i \in \widehat{\mathcal{K}}$. Indeed, (4.31) and (4.32) provide (4.28); and thus, Corollary 4.4 is proved. □

5. EXAMPLE OF SPACE–TIME CHAOS IN RDSs

So far we have constructed homeomorphisms between the set $\Gamma^{\rm tr} = (\Gamma_0^{\rm tr})^{\mathbb{Z}^n}$ from Corollary 2.8 and $\widehat{\Gamma}^{\rm tr}_\lambda$. Thus, we worked in the space of trajectories. We now return to the phase space of the RDS (4.24) and construct an hyperbolic set there. Moreover, we will use the homeomorphism between $\Gamma^{\rm tr}$ and the standard Bernoulli scheme to deduce space–time chaos.

Recall that $\widehat{\Pi}_0\colon C_b(\mathbb{R}, L^2_b(\mathbb{R}^n)) \to L^2_b(\mathbb{R}^n)$ maps trajectories u onto their initial data $u^0 = u(0)$.

Corollary 5.1. *Let the assumptions of Corollary 4.4 hold and let $\widehat{\Gamma}^{\rm tr}_\lambda$ be the hyperbolic set of the problem* (4.24) *constructed in Corollary* 4.4. *Define $\widehat{\Gamma}_\lambda := \widehat{\Pi}_0\widehat{\Gamma}^{\rm tr}_\lambda \subset \widehat{\mathcal{A}} \subset L^2_b(\mathbb{R}^n)$, then there is a homeomorphism* (*with respect to the associated local topologies*)

$$\widehat{\kappa}_\lambda : \widehat{\Gamma}_\lambda \leftrightarrow \Gamma = (\Gamma_0)^{\mathbb{Z}^n}$$

which commutes with the discrete group of spatial translations $\{T_{l'}\colon l' \in \mathbb{Z}^n\}$ and with the Poincaré maps generated by the Eq. (2.22) *and* (4.24), *respectively:*

$$\widehat{\kappa}_\lambda \circ S_{l_0} = \widehat{S}^\lambda_{l_0} \circ \widehat{\kappa}_\lambda \quad \text{and} \quad \widehat{\kappa}_\lambda \circ T_{l'} = T_{l'} \circ \widehat{\kappa}_\lambda \quad \text{for } l_0 \in \mathbb{Z} \text{ and } l' \in \mathbb{Z}^n.$$

Proof. Indeed, it is known (see, e.g., [4,47]) that the maps $\widehat{\Pi}_0\colon \widehat{\mathcal{K}}_\lambda \to \widehat{\mathcal{A}}_\lambda$ and $\Pi_0 : \mathcal{K} \to \mathcal{A}$ are one-to-one. Since the sets $\widehat{\mathcal{K}}_\lambda$ and $\mathcal{K}$ are compact in the local topologies, these maps are in fact homeomorphisms. The desired homeomorphism $\widehat{\kappa}_\lambda$ can be defined now via

$$\widehat{\kappa}_\lambda := \Pi_0 \circ \widehat{\kappa}^{\rm tr}_\lambda \circ (\widehat{\Pi}_0)^{-1}, \tag{5.1}$$

where $\widehat{\kappa}^{\rm tr}_\lambda$ is constructed in Corollary 4.4. The commutation properties are immediate corollaries of (4.2). □

Combining the Corollaries 2.9 and 5.1 we arrive at the final result on this section which states the existence of spatial–temporal chaos for the RDS (4.24). Note that the system is space and time periodic with periodicity 1 in each direction.

Theorem 5.2. *Let Assumption 2.4A be valid such that the RDS* (2.1) *in the bounded domain* Ω_0 *has a hyperbolic set. Moreover, let the assumptions of Corollary 4.4 hold* (*i.e., the coupling is weak since* $\lambda > \lambda_1 \gg 1$) *such that* $\widehat{\Gamma}_\lambda$ *and* $\widehat{\kappa}_\lambda$ *can be constructed as above. Then, the RDS* (4.24) *admits space–time chaos in the following precise sense: The multidimensional Bernoulli system* ($\mathcal{M}^{n+1}, \{\mathcal{T}_l \colon l \in \mathbb{Z}^{n+1}\}$) (*as defined in Definition* 2.4) *is topological conjugate to* ($\widehat{\Gamma}_\lambda, \{\widehat{\mathbb{S}}^\lambda_l \colon l = (l_0, l') \in \mathbb{Z}^{n+1}\}$) *via the homeomorphism* $\widehat{\tau}\colon \mathcal{M}^{n+1} \to \widehat{\Gamma}_\lambda$ (*in the local topologies*). *Here,*

$$\widehat{\mathbb{S}}_{(l_0,0)} = \widehat{S}^\lambda_{l_0} \quad \textit{and} \quad \widehat{\mathbb{S}}_{(0,l')} = T_{l'} \qquad \textit{for } (l_0, l') \in \mathbb{Z}^{n+1},$$

and conjugacy means the commutations

$$\widehat{\tau} \circ \mathcal{T}_{(l_0,0)} = \widehat{S}^\lambda_{l_0} \circ \widehat{\tau}, \quad \widehat{\tau} \circ \mathcal{T}_{(0,l')} = T_{l'} \circ \widehat{\tau} \quad \textit{for } l_0 \in \mathbb{Z} \textit{ and } l' \in \mathbb{Z}^n. \tag{5.2}$$

Remark 5.3. As in the case of purely temporal dynamics (see Remark 2.5) the spatio-temporal hyperbolic sets $\widehat{\Gamma}_\lambda$ and $\widehat{\Gamma}^{tr}_\lambda$ are closely related with the multidimensional analog of homoclinic orbits the so-called bump solutions. Indeed, let the assumption (2.18) be satisfied. Then, the image $\widehat{\tau}(b_0)$ of a zero element $b_0 \equiv 0$ of $\mathcal{M}^{n+1}$ under the homeomorphism $\widehat{\tau}$ constructed in Theorem 5.2, obviously, also equals zero: $\widehat{\tau}(b_0) = 0$. Analogously, the image $\widehat{\tau}(\bar{b})$ of the basic bump element $\bar{b} \in \mathcal{M}^{n+1}$ ($\bar{b}(0) = 1$, $\bar{b}(l) = 0$, $l \neq 0$) gives a spatio-temporal bump orbit $\bar{u} := \widehat{\tau}^{tr}(\bar{b})$ for problem (4.24). Moreover, it follows from the hyperbolicity of condition on $\bar{u}$ and u_0 (exactly as in purely temporal case) that the solution $\bar{u}$ is an *exponential* bump, i.e.

$$\bar{u} \in C^1_{e^{\mu|t|+\mu|x|}}(\mathbb{R}^{n+1}) \tag{5.3}$$

for some positive μ. Thus, our construction gives, in particular the existence of an exponentially decaying space–time bump solution for Eq. (4.24). Furthermore, analogously to (2.20), every $b \in \mathcal{M}^{n+1}$ can be presented as follows:

$$b = \sum_{l \in \mathbb{Z}^{n+1}} b(l) \mathcal{T}_{-l} \bar{b}. \tag{5.4}$$

And, analogously to (2.21), it is not difficult to verify that every solution $U_b := \widehat{\tau}^{tr}(b),\ b \in \mathcal{M}^{n+1}$ satisfies

$$\|U_b - \sum_{(l_0,l')\in\mathbb{Z}^{n+1}} b(l_0,l')\bar{u}(\cdot - l_0, \cdot - l')\|_{L^2_b(\mathbb{R}^{n+1})} < \varepsilon'_0,$$

where $\varepsilon'_0 > 0$ is a small positive number depending on ε_0 introduced in (2.21). Thus, analogously to the pure temporal case, the hyperbolic set $\widehat{\Gamma}^{tr}_\lambda$ is also generated by all spatio-temporal shifts of the basic bump solution $\bar{u}$ summed with coefficients 0 or 1.

Finally we want to show that it is also possible to construct spatio-temporal chaos in RDS with a nonlinearity f which is smooth in t, x and u, since by now our function f_λ is not continuous with respect to $x \in \mathbb{R}^n$. Moreover, the periodicity in space and time might appear to be artificial and it is desirable to find a RDS which is autonomous and spatially homogeneous. However, this can only be done by giving up the hyperbolicity, because of the arising continuous translation groups in space and time.

To construct a smooth RDS we approximate the nonlinearity f_λ by a new nonlinearity which is polynomial in the state variable u while the periodicity in t and x is obtained by polynomials in the trigonometric functions $\cos(2\pi x_j)$ and $\sin(2\pi x_j)$ where $j = 0, \dots, n$ and $x_0 = t$. To be more precise, consider $R_N: \mathbb{R}^{n+1} \times \mathbb{R}^k \to \mathbb{R}^k$ in the form

$$\begin{aligned} R_N(t,x,u) := \widetilde{R}_N(&\sin(2\pi t), \cos(2\pi t),\\ &\sin(2\pi x_1), \cos(2\pi x_1), \dots, \sin(2\pi x_n), \cos(2\pi x_n), u)\\ &+ \varepsilon |u|^{2N} u, \end{aligned} \tag{5.5}$$

where $\widetilde{R}_N$ is an algebraic polynomial of degree $2N$ with $N \in \mathbb{N}$ and $\varepsilon = \varepsilon_N > 0$. Thus, the function R_N is 1-periodic with respect to (t,x). By (2.3) it is possible to find polynomials $\widetilde{R}_N$ and $\varepsilon_N > 0$ in such a way that for every $R > 0$ and every $q > 1$ we have

$$\begin{aligned} &\|f_\lambda - R_N\|_{L^q_b(\mathbb{R}^{n+1}\times V_R)} \to 0,\ \|D_u f_\lambda - D_u R_N\|_{L^q_b(\mathbb{R}^{n+1}\times V_R)} \to 0,\\ &\|D^2_u f_\lambda - D^2_u R_N\|_{L^q_b(\mathbb{R}^{n+1}\times V_R)} \to 0 \quad \text{for } N \to \infty, \end{aligned} \tag{5.6}$$

where $V_R := \{u \in \mathbb{R}^k\colon |u| \leqslant R\}$.

Consider now the following family of RDS in $\mathbb{R}^n$:

$$\partial_t \overline{u} = \gamma \Delta_x \overline{u} - \overline{u} - R_N(t,x,\overline{u})), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,\ \overline{u}\big|_{t=0} = u^0. \tag{5.7}$$

By the construction of R_N we have the dissipativity condition

$$R_N(t,x,v).v \geqslant -C_N + \frac{\varepsilon_N}{2}|u|^{2N+2} \quad \text{for all } (t,x,v)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^k \tag{5.8}$$

for an appropriate constant C_N. As a consequence (see, e.g., [46]) the RDS (5.7) possesses, for every $u^0\in \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n)$, a unique global solution $\overline{u}:[0,\infty)\to \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n)$, $t\in\mathbb{R}_+$, which satisfies the dissipativity estimate:

$$\|\overline{u}(t)\|_{\mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n)} \leqslant C\|\overline{u}(0)\|_{\mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n)}e^{-\overline{\alpha}t} + \overline{C}_N \quad \text{for some } \overline{\alpha},\overline{C}_N>0,$$

and the solution operator $\overline{S}_t:\mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n)\to\mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n);\, u^0\mapsto\overline{u}(t)$ of problem (5.7) is well defined. Moreover, the Poincaré map $\overline{S}_1$ admits a global, locally compact attractor $\overline{\mathcal{A}}_N\subset\mathrm{L}^2_{\mathrm{b}}(\Omega)$ which is generated by the set $\overline{\mathcal{K}}_N$ of all essential solutions trajectories of (5.7):

$$\overline{\mathcal{A}}_N = \Pi_0\overline{\mathcal{K}}_N. \tag{5.9}$$

Moreover, in analogy to Proposition 4.3 we have

$$\overline{\mathcal{K}}_N\subset \mathrm{C}^1_{\mathrm{b}}(\mathbb{R}^{n+1}), \quad \|\overline{u}\|_{\mathrm{C}^1_{\mathrm{b}}(\mathbb{R}^{n+1})}\leqslant \overline{C}'_N \text{ for all } \overline{u}\in\overline{\mathcal{K}}_N, \quad \mathcal{K}_N\Subset \mathrm{C}^1_{\mathrm{loc}}(\mathbb{R}^{n+1}). \tag{5.10}$$

The following result is a complete analog of the structural-stability Theorem 4.1 for the case of systems (4.24) and (5.7). It states the existence of a hyperbolic trajectory set. The subsequent theorem then concludes that the smooth, polynomial RDS (5.7) has spatio-temporal chaos.

Proposition 5.4. *Let the above assumptions hold. Then, there is* $N_0>0$ *such that for every* $N>N_0$ *Eq.* (5.7) *possesses a hyperbolic trajectory set* $\overline{\Gamma}^{\mathrm{tr}}_N\subset\overline{\mathcal{K}}_N$ *which is homeomorphic to the hyperbolic trajectory set* $\widehat{\Gamma}^{\mathrm{tr}}:=\widehat{\Gamma}^{\mathrm{tr}}_\lambda$ *constructed in Corollary 4.4* (*recall that* $\lambda>\lambda_1$ *is fixed now*)*:*

$$\overline{\kappa}^N_{\mathrm{tr}}:\widehat{\Gamma}^{\mathrm{tr}}\leftrightarrow\overline{\Gamma}^{\mathrm{tr}}_N.$$

This homeomorphism satisfies the commutation properties (4.2) *and is Lipschitz continuous in the local topology, i.e., if* $\widehat{u}_i\in\widehat{\Gamma}^{\mathrm{tr}}$ *and* $\overline{u}_i:=\overline{\kappa}^N_{\mathrm{tr}}(\widehat{u}_i)$ *then the following estimate is valid for a sufficiently small* $\varepsilon<\varepsilon_0$*:*

$$C_1\|\widehat{u}_1-\widehat{u}_2\|_{\mathrm{C}^1_{\varphi_{\varepsilon,y_0}}(\mathbb{R}^{n+1})}\leqslant\|\overline{u}_1-\overline{u}_2\|_{\mathrm{C}^1_{\varphi_{\varepsilon,y_0}}(\mathbb{R}^{n+1})}\leqslant C_2\|\widehat{u}_1-\widehat{u}_2\|_{\mathrm{C}^1_{\varphi_{\varepsilon,y_0}}(\mathbb{R}^{n+1})}, \tag{5.11}$$

where the constants C_i, $i=1,2$ *are independent of* N, $y_0\in\mathbb{R}^{n+1}$ *and of the concrete choice of hyperbolic trajectories* $\widehat{u}_i\in\widehat{\Gamma}^{\mathrm{tr}}$.

Proof. The assertion of this theorem can be derived from Corollary 4.4 using the structural-stability arguments in complete analogy to the proof of Theorem 4.1. In fact, the proof is much simpler since (5.6) guarantees the closeness of systems (4.24) and (5.7) in a more regular topology that we had in the case of Eq. (3.5) and (2.22). Indeed, as in Section 4, in order to construct the required homeomorphism, we need to find, for every hyperbolic trajectory $\widehat{u} \in \widehat{\Gamma}^{tr}$ of (4.24), the associated trajectory $\bar{u}$ of Eq. (5.7) such that $\widehat{u}(t,x)$ is "close" to $\bar{u}(t,x)$ for all $(t,x) \in \mathbb{R}^{n+1}$. Moreover, due to the smoothing property for parabolic equations, the hyperbolicity estimate (4.30) for the variation Eq. (4.29) can be reformulated as follows, for every $h \in \mathrm{L}^p_\mathrm{b}(\mathbb{R}^{n+1})$ there exists a unique solution $v \in \mathrm{W}^{(1,2),p}_\mathrm{b}(\mathbb{R}^{n+1})$ of

$$\partial_t v = \gamma \Delta_x v - v - \mathrm{D}_u f_\lambda(t,x,\widehat{u}(t,x))v + h(t,x)$$

and the following estimate holds:

$$\|v\|_{\mathrm{W}^{(1,2),p}_\mathrm{b}(\mathbb{R}^{n+1})} \leqslant C\|h\|_{\mathrm{L}^p_\mathrm{b}(\mathbb{R}^{n+1})},$$

where the constant C depends only on the hyperbolicity constant C_Γ from (4.30) and p. Thus, due to (5.6), we may require $\bar{u}$ to be close to $\hat{u}$ in the metric of $\mathrm{W}^{(1,2),p}_\mathrm{b}(\mathbb{R}^{n+1})$ (where $p = p(n)$ is fixed large enough that the embedding $\mathrm{W}^{(1,2),p}(\mathbb{R}^{n+1}) \subset C_b(\mathbb{R}, C^1_b(\mathbb{R}^n))$ holds) and find it by the implicit function theorem exactly as we did in Section 4. So we leave the details to the reader and only note that the above scheme gives the closeness of $\widehat{u}$ and $\bar{u}$ only in $\mathrm{C}_\mathrm{b}(\mathbb{R}, \mathrm{C}^1_\mathrm{b}(\mathbb{R}^n))$ and estimate (5.11) in the $\mathrm{W}^{(1,2),p}_{\varphi_{\varepsilon,y_0}}(\mathbb{R}^{n+1})$ which is weaker than $\mathrm{C}^1_{\varphi_{\varepsilon,y_0}}(\mathbb{R}^{n+1})$ since it does not contain the estimate of the C-norms of the t-derivative. In order to obtain the required C-estimate for the t-derivatives postulated in Proposition 5.4, it only remains to differentiate Eqs. (4.24) and (5.7) with respect to t (based on the observation that f_λ and R_N are smooth with respect to t and u) and apply the $\mathrm{L}^p_{\varphi_{\varepsilon,y_0}}$ regularity estimate once more. □

Returning to the hyperbolic sets in the phase space we derive the following corollaries.

Theorem 5.5. *Let the assumptions of Proposition 5.4 hold such that the hyperbolic trajectory set* $\overline{\Gamma}^{\mathrm{tr}}_N$ *can be constructed. Let* $\overline{\Gamma}_N := \Pi_0\overline{\Gamma}^{\mathrm{tr}}_N \subset \mathcal{A}_N$ *and* $\overline{\mathbb{S}}_{(l_0,l')} = \overline{S}_{l_0} \circ T_{l'}$ *for* $(l_0,l') \in \mathbb{Z}^{n+1}$. *Then, there is a homeomorphism* $\overline{\tau}: \mathcal{M}^{n+1} \to \overline{\Gamma}_N$ *such that multidimensional Bernoulli system* $(\mathcal{M}^{n+1}, \{\mathcal{T}_l : l \in \mathbb{Z}^{n+1}\}$ *is conjugated to the hyperbolic dynamics* $(\overline{\Gamma}_N, \{\overline{\mathbb{S}}_l : l \in \mathbb{Z}^{n+1}\})$ *in* (5.7), *i.e.,* $\overline{\tau} \circ \mathcal{T}_l = \overline{\mathbb{S}}_l \circ \overline{\tau}$ *for all* $l \in \mathbb{Z}^{n+1}$.

Indeed, this assertion follows immediately from Proposition 5.4 by applying Π_0 and by the previously established conjugacy of $\widehat{\Gamma}^{\mathrm{tr}}$ to $\mathcal{M}^{n+1}$.

Finally we construct the desired autonomous and homogeneous RDS. We start with the simple construction of an autonomous, homogeneous RDS which has the vector

$$(\sin(2\pi t), \cos(2\pi t), \sin(2\pi x_1), \cos(2\pi x_1), \dots, \cos(2\pi x_n))$$

as a particular solution.

Lemma 5.6. *For every $N \in \mathbb{N}$ there exists a $(2n+2)$-dimensional RDS (i.e., $v(t,x) \in \mathbb{R}^{2n+2}$)*

$$\partial_t v = \gamma \Delta_x v - v - Q_N(v), \; (t,x) \in \mathbb{R}^{n+1} \tag{5.12}$$

with $Q_N(v) = \widetilde{Q}_N(v) + \varepsilon_N |v|^{2N} v$ where $\varepsilon_N > 0$ and $\widetilde{Q}_N$ is a polynomial of degree $2N$, which possesses the particular solution v^{per} given via

$$v^{\mathrm{per}}(t,x) := (\sin(2\pi t), \cos(2\pi t), \sin(2\pi x_1), \cos(2\pi x_1), \dots, \sin(2\pi x_n), \cos(2\pi x_n)).$$

Proof. The construction is totally explicit, since for each (t,x) we have $|v^{\mathrm{per}}(t,x)|^2 = n+1$. We let $Q_N(v) = -Av + \varepsilon_N v |v|^{2N-2}(|v|^2 - (n+1))$ and choose $A \in \mathbb{R}^{(2n+2)\times(2n+2)}$ such that $(\partial_t - \gamma \Delta_x + 1) v^{\mathrm{per}} = A v^{\mathrm{per}}$, namely $A = \mathrm{diag}\left(\begin{pmatrix} 0 & 2\pi \\ -2\pi & 0 \end{pmatrix}, 4\gamma\pi^2, \dots, 4\gamma\pi^2\right)$. □

Fix now the integer $N > N_0$ and define the vector $U := (v, \overline{u}) \in \mathbb{R}^{k+2n+2}$ and the polynomial $F: \mathbb{R}^{k+2n+2} \to \mathbb{R}^{k+2n+2}$ of the degree $2N+1$ via

$$F(U) = F(v, \overline{u}) := \big(Q_N(v), \widetilde{R}_N(v, \overline{u})\big), \tag{5.13}$$

where $\widetilde{R}_N$ is defined in (5.5). Our final system is the autonomous $(k+2n+2)$-dimensional RDS (i.e., $U \in \mathbb{R}^{k+2n+2}$)

$$\begin{cases} \partial_t U = \gamma \Delta_x U - U - F(U) & \text{in } (0,\infty) \times \mathbb{R}^n \\ U(0,x) = U^0(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{5.14}$$

Then, by construction we have, for $0 < \delta < \varepsilon_N$,

$$F(U) \cdot U \geqslant -C + \delta |U|^{2N+2} \quad \text{for all } U \in \mathbb{R}^{k+2n+2} \tag{5.15}$$

and consequently for every $U_0 \in \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n)$ problem (5.14) possesses a unique global solution $U: [0,\infty) \to \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^n)$. Moreover, the semigroup $(\mathcal{S}_t)_{t \geqslant 0}$ with

$$\mathcal{S}_t: \mathrm{L}^2_b(\mathbb{R}^n) \to \mathrm{L}^2_b(\mathbb{R}^n); \quad U^0 \mapsto \mathcal{S}_t U^0 := U(t), \tag{5.16}$$

associated with problem (5.14) is well defined and possesses a global, locally compact attractor $\mathcal{A}^U \subset L^2_b(\mathbb{R}^n)$ and the analogs of (5.9) and (5.10) are valid (see, e.g., [46]).

However, due to the skew-product structure of system (5.14), where the v-component is independent of $\bar{u}$, and due to Lemma 5.6, the attractor $\mathcal{A}_N$ is embedded in $\mathcal{A}^U$ via

$$\mathcal{A}_N \sim v^{\text{per}}(0, \cdot) \times \mathcal{A}_N \subset \mathcal{A}^U.$$

Combining this embedding with Theorem 5.5 we finally obtain the following result.

Theorem 5.7. *Let the above assumptions hold. Then, the multidimensional Bernoulli shift* $(\mathcal{M}^{n+1}, \{\mathcal{T}_l : l \in \mathbb{Z}^{n+1}\})$ *is topologically embedded via a homeomorphic embedding* $\tau : \mathcal{M}^{n+1} \to \mathcal{A}^U$

$$\tau \circ \mathcal{T}_{(l_0, l')} = (\mathcal{S}_{l_0} \circ T_{l'}) \circ \tau \quad \textit{for all } l = (l_0, l') \in \mathbb{Z}^{n+1}, \tag{5.17}$$

where $\mathcal{S}_t$ *denotes the semigroup for equation* (5.14).

Corollary 5.8. *Let* $\mathbb{S}_{(t,h)} := \mathcal{S}_t \circ T_h$ *be the* $(n+1)$*-parametrical extended spatio-temporal semigroup associated with equation* (5.14). *Then this semigroup acts on the attractor* $\mathcal{A}^U$ *and the topological entropy of this action is finite and strictly positive:*

$$0 < h_{\text{top}}\left(\mathbb{S}_{(t,h)}, \mathcal{A}^U\right) < \infty. \tag{5.18}$$

Indeed, the finiteness of the value (5.18) is proved in [45]. In order to verify that this value is strictly positive, it remains to recall that the topological entropy is preserved under homeomorphisms (cf. e.g., [20]). So (5.17) implies that

$$\begin{aligned} &h_{\text{top}}(\{\mathbb{S}_{(l_0,l')}\}_{(l_0,l') \in \mathbb{Z}^{n+1}}, \mathcal{A}^U) \geqslant h_{\text{top}}(\{\mathbb{S}_{(l_0,l')}\}_{(l_0,l') \in \mathbb{Z}^{n+1}}, \tau(\mathcal{M}^{n+1})) \\ &\quad = h_{\text{top}}(\mathcal{T}_{(l_0,l')}, \mathcal{M}^{n+1}) = 1. \end{aligned} \tag{5.19}$$

Since the topological entropy of the continuous semigroup $\{\mathbb{S}_{(t,h)} : (t, h) \in \mathbb{R}^{n+1}\}$ obviously coincides with that of its discrete subgroup $\{\mathbb{S}_{(l_0,l')} : (l_0, l') \in \mathbb{Z}^{n+1}\}$, (5.19) implies the left inequality in (5.18).

APPENDIX A. HYPERBOLIC SETS FOR RDS IN BOUNDED DOMAINS

In this section, we give an explicit scheme which allows us to construct RDSs in bounded domains with hyperbolic sets in the sense of Definition 2.1. To be more precise, for a given system of ODEs with polynomial nonlinearity which possesses a hyperbolic set $\Gamma_{\rm ODE}$ and every bounded domain Ω_0, we construct a RDS in Ω_0 with Dirichlet boundary conditions which possesses a hyperbolic set $\Gamma_{\rm PDE}$ homeomorphic to $\Gamma_{\rm ODE}$. Since the existence of hyperbolic sets for ODEs with polynomial nonlinearities is well known, this scheme provides a rich class of hyperbolic sets for RDSs in bounded domains with Dirichlet boundary conditions.

Our strategy is based on the detailed analysis of the following special form of a RDS:

$$\partial_t u = \gamma(\Delta_x u - \lambda_1 u) - f(t,u) \quad \text{in } \mathbb{R}\times\Omega_0, \;\; u=0 \text{ on } \mathbb{R}\times\partial\Omega_0, \tag{A.1}$$

where $u=(u^1,\ldots,u^k)$ is the unknown vector-valued function, $f(t,u)$ is a given nonlinearity (which is assumed to be periodic with respect to t: $f(t+T,u)\equiv f(t,u)$), $\lambda_1<0$ is the first eigenvalue of the Laplacian in Ω_0, and $\gamma\gg 1$ is a large parameter. Let P_+ and P_- be the L^2-orthoprojectors to the first and the rest eigenvectors of the Laplacian respectively and let $u(t)=u_+(t)+u_-(t)$ with $u_\pm(t):=P_\pm u(t)$. Then, these functions satisfy

$$\begin{cases} \partial_t u_+ = -P_+ f(t, u_+ + u_-), \\ \partial_t u_- = \gamma(\Delta_x u_- + \lambda_1 u_-) - P_- f(t, u_+ + u_-). \end{cases} \tag{A.2}$$

It is intuitively clear that, for large γ the u_--component of a solution u of Eq. (A.2) will be small and, consequently, this solution will be close to the appropriate solution v of the system of ODEs

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = -f_+(t,v) \quad \text{with } f_+(t,v) := \langle f(t, e_1\cdot v), e_1\rangle_{L^2(\Omega_0)} = \int_{\Omega_0} f(t, e_1(x)v)e_1(x)\mathrm{d}x, \tag{A.3}$$

where e_1 is the normed, first eigenvector of the Laplacian and $v(t)=(v^1(t),\ldots,v^k(t))$. Thus, if system (A.3) possesses a hyperbolic set, it should preserve also for system (A.2) for large γ by structural-stability arguments. The following theorem gives a rigorous justification of these arguments.

Theorem A.1. *Let the nonlinearity f in Eq.* (A.1) *be T-periodic with respect to time, satisfy regularity assumption* (2.3) *and, in addition* $f(t,0)\equiv$

0. *Moreover, let* $\Omega_0 \subset \mathbb{R}^n$ *be an open bounded domain. Assume also that the associated system of ODE* (A.3) *possesses a hyperbolic set* Γ_{ODE}. *Then, for all sufficiently large* γ, *system* (A.2) *possesses a hyperbolic set* $\Gamma_{\mathrm{PDE}} = \Gamma_{\mathrm{PDE}}(\gamma)$ *which is homeomorphic to* Γ_{ODE}.

Proof. In order to deduce the assertion of the theorem, we first need to study the second equation of (A.2).

Lemma A.2. *Let the assumptions of Theorem A.1 hold. Then, there exists* $\gamma_0 > 0$ *such that, for all* $\gamma \geqslant \gamma_0$ *and every function* $v \in \mathrm{L}^\infty(\mathbb{R}, \mathbb{R}^k)$, *the equation*

$$\partial_t u_- = \gamma(\Delta_x u_- - \lambda_1 u_-) - P_- f(t, u_- + v \cdot e_1) \tag{A.4}$$

possesses a unique solution $u_- \in \mathrm{L}^\infty(\mathbb{R}, P_- \mathrm{L}^2(\Omega_0))$ *and the following estimate holds:*

$$\|u_-(t)\|_{\mathrm{L}^2(\Omega_0)} \leqslant C\gamma^{-1/2} \sup\{ e^{-\alpha(t-s)} \|v(s)\| \colon s \in (-\infty, t]\}, \tag{A.5}$$

where positive constants C *and* α *are independent of* γ *and* t. *Thus, the nonlinear solution operator* $\mathbb{T}_\gamma \colon \mathrm{L}^\infty(\mathbb{R}, \mathbb{R}^k) \to \mathrm{L}^\infty(\mathbb{R}, P_- \mathrm{L}^2(\Omega_0))$ *is well-defined via* $\mathbb{T}_\gamma v := u_-$. *Moreover, this operator is of class* C^1 *and satisfies the following estimate:*

$$\|\mathbb{T}_\gamma\|_{\mathrm{C}^1(\mathrm{L}^\infty(\mathbb{R},\mathbb{R}^k), \mathrm{L}^\infty(\mathbb{R}, \mathrm{L}^2(\Omega_0)))} \leqslant C_1 \gamma^{-1/2}, \tag{A.6}$$

where the constant C_1 *is also independent of* γ.

Proof. Indeed, multiplying Eq. (A.5) scalarly in $\mathrm{L}^2(\Omega_0)$ by $u_-(t)$ and using that $\|\nabla_x u_-\|^2_{\mathrm{L}^2(\Omega_0)} \geqslant \lambda_2 \|u_-(t)\|^2_{\mathrm{L}^2(\Omega_0)}$ with $\lambda_2 > -\lambda_1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u_-(t)\|^2_{\mathrm{L}^2} + \gamma(\lambda_2 + \lambda_1)\|u_-(t)\|^2_{\mathrm{L}^2} \leqslant C\|f(v(t) \cdot e_1 + u_-(t))\|^2_{\mathrm{L}^2}. \tag{A.7}$$

Since, f and $\mathrm{D}_u f$ are globally bounded and $f(t, 0) \equiv 0$ the right-hand side of (A.7) can be estimated as follows

$$C\|f(v(t) \cdot e_1 + u_-(t)\|^2_{\mathrm{L}^2} \leqslant C_f \|v(t) e_1 + u_-(t)\|^2_{\mathrm{L}^2} \leqslant C_2(\|v(t)\|^2 + \|u_-(t)\|^2_{\mathrm{L}^2}).$$

Inserting this estimate to (A.7), we deduce that, for sufficiently large γ,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u_-(t)\|^2_{\mathrm{L}^2} + \kappa\gamma \|u_-(t)\|^2_{\mathrm{L}^2} \leqslant C' \|v(t)\|^2, \tag{A.8}$$

where the positive constants κ and C are independent of γ. Applying the Gronwall inequality to (A.8), we deduce estimate (A.5). Thus, operator $\mathbb{T}_\gamma$ is indeed well-defined. The estimate for its Fréchet derivative can be obtained analogously. This proves Lemma A.2. □

Thus, Lemma A.2 provides a Lyapunov–Schmidt reduction which allows to rewrite problem (A.2) in the following nonlocal form:

$$\frac{\mathrm{d}}{\mathrm{d}t}v_+(t) = -\langle f(t, v_+(t)\cdot e_1 + (\mathbb{T}_\gamma v_+)(t)), e_1\rangle_{\mathrm{L}^2(\Omega_0)}, \tag{A.9}$$

where $u_+(t,x) = v_+(t)\cdot e_1(x)$, $v_+(t)\subset\mathbb{R}^k$. Estimate (A.6) shows that this equation is really a small perturbation of Eq. (A.2). Let now the trajectory $v: t\mapsto v(t)$ belong to the (trajectory) hyperbolic set $\Gamma^{\mathrm{tr}}_{\mathrm{ODE}}$ of the ODE (A.2). Then, we seek for the associated hyperbolic trajectory v_+ of (A.9) in the form $v_+ := v+\theta$ where θ is a small perturbation which has to satisfy the following equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\theta &= -\mathrm{D}_v f_+(t,v)\theta - \langle f(t, v(t)\cdot e_1 + (\mathbb{T}_\gamma(v+\theta))(t))\\ &\quad - f(t, v(t)\cdot e_1) - f'_u(t, v(t)\cdot e_1)\theta, e_1\rangle_{\mathrm{L}^2(\Omega_0)},\end{aligned} \tag{A.10}$$

We recall that the linear part of Eq. (A.10) is invertible (in $\mathrm{L}^\infty(\mathbb{R},\mathbb{R}^k)$) due to the hyperbolicity assumption on v and the nonlinear part is small (for large γ)) due to Lemma A.2. Thus, applying the implicit function theorem analogously to Section 4, we establish that for all large γ (i.e., $\gamma > \gamma_0 \gg 1$) Eq. (A.10) is uniquely solvable in a small neighborhood of v and the solution θ satisfies

$$\|\theta\|_{\mathrm{L}^\infty(\mathbb{R},\mathbb{R}^k)} \leqslant C'\gamma^{-1/2}, \tag{A.11}$$

where the constant C' is independent of γ. Recalling that the function θ determine in a unique way the associated solution u (namely $u = u_+ + u_- = (v+\theta)e_1 + \mathbb{T}_\gamma(v+\theta)$) of the whole system (A.2) and using estimates (A.11) and (A.5), we obtain a unique trajectory $u(t)$ of Eq. (A.1) which satisfies

$$\|u(t) - v(t)\cdot e_1\|_{\mathrm{L}^2(\Omega_0)} \leqslant C''\gamma^{-1/2} \quad \text{for } t\in\mathbb{R}. \tag{A.12}$$

Using this estimate, it is very easy to verify that this trajectory is hyperbolic in the sense of Definition 2.1. Thus, for every $v\in\Gamma^{\mathrm{tr}}_{\mathrm{ODE}}$, we have constructed a hyperbolic trajectory $u = \kappa_\gamma(v)$ of the RDS (A.1) and, consequently, the required hyperbolic set $\Gamma^{\mathrm{tr}}_{\mathrm{PDE}} := \kappa_\gamma(\Gamma^{\mathrm{tr}}_{\mathrm{ODE}})$ is also constructed and it only remains to verify that κ_γ is a homeomorphism. To this end, arguing as in Section 4, we can verify that, for every $v_1, v_2 \in \Gamma^{\mathrm{tr}}_{\mathrm{ODE}}$, the associated trajectories $u_i := \kappa_\gamma(v_i)$ satisfy the following analog of (4.23):

$$c\,\|v_1 - v_2\|_{\mathrm{L}^\infty_{\phi^\varepsilon}(\mathbb{R},\mathbb{R}^k)} \leqslant \|u_1 - u_2\|_{\mathrm{L}^\infty_{\phi^\varepsilon}(\mathbb{R},\mathrm{L}^2(\Omega_0))} \leqslant C\|v_1 - v_2\|_{\mathrm{L}^\infty_{\phi^\varepsilon}(\mathbb{R},\mathbb{R}^k)}. \tag{A.13}$$

Here $\phi^\varepsilon(t) = e^{-\varepsilon|t|}$ and the estimate holds for sufficiently small positive ε where the constants c and C are independent of v_1 and v_2. Estimate

(A.13) shows that κ_γ is indeed a homeomorphism, which finishes the proof of the Theorem A.1. □

The following corollary shows that the L^2-norm in (A.12) can be replaced by stronger ones like the C-norm or the C^1-norm.

Corollary A.3. *Let the assumptions of Theorem A.1 hold and let the domain Ω_0 be sufficiently smooth. Then, for every $v \in \Gamma^{tr}_{ODE}$ and $u := \kappa_\gamma(v)$, we have*

$$\|u(t) - v(t) \cdot e_1\|_{C^1(\Omega_0)} \leqslant C\gamma^{-1/2}, \tag{A.14}$$

where the constant C is independent of γ and v.

Proof. Indeed, according to our assumptions on the nonlinearity f, we have

$$\|P_- f(t, u(t))\|_{L^\infty(\Omega_0)} \leqslant C_1, \tag{A.15}$$

where C_1 is independent of $u \in \Gamma_{PDE}$ and t. Scaling the time as $t = \gamma^{-1} t'$ in the second equation of (A.2), we have

$$\partial_{t'} u_- = \Delta_x u_- + \lambda_1 u_- - \gamma^{-1} P_- f(\gamma^{-1} t', u(t')). \tag{A.16}$$

Applying now the appropriate parabolic regularity theorem to Eq. (A.16) and using (A.15) and the fact that the L^∞-norm is invariant under scaling, we infer

$$\|u_-(t)\|_{C^1(\Omega_0)} \leqslant C_1 \gamma^{-1} \quad \text{for all } u \in \Gamma_{PDE} \text{ and all } t \in \mathbb{R}. \tag{A.17}$$

Together with (A.12) this gives (A.14) and finishes the proof of the corollary. □

Thus, we see that the hyperbolic set Γ_{PDE} of Eq. (A.1) is localized in a small C^1-neighborhood of the initial hyperbolic set Γ_{ODE} (more precisely, of the set $\Gamma_{ODE} \cdot e_1$). Consequently, the behavior of the function $f(t, u)$ outside of this neighborhood is not essential and we can drop out the growth restrictions of the function f in conditions of Theorem A.1. In particular, we can use this theorem in the class of nonlinearities which are *polynomials* with respect to u. The following simple lemma shows that every polynomial nonlinearity can be realized in the form of a right-hand side of Eq. (A.3).

Lemma A.4. *For every nonlinearity $Q(t, v) = (Q_1(t, v), \dots, Q_N(t, v))$ which is polynomial with respect to $v = (v_1, \dots, v_N)$ there exists a polynomial function $f_Q(t, v)$ such that*

$$Q(t, v) = P_+ f_Q(t, v \cdot e_1) \quad \textit{for all } t \in \mathbb{R} \textit{ and } v \in \mathbb{R}^k. \tag{A.18}$$

Proof. Indeed, it is sufficient to verify (A.18) for scalar monomials $Q(t,v):=a(t)v_1^{j_1}\cdots v_N^{j_n}$. We will seek for the required polynomial f_Q in the form $f_Q(t,u)=b(t)u_1^{j_1}\cdots u_N^{j_n}$. Then, obviously

$$P_+f_Q(t,v\cdot e_1)=b(t)v_1^{j_1}\cdots v_N^{j_n}\cdot(e_1^{j_1+\dots+j_N},e_1)$$

It only remains to recall that the first eigenvector of the Laplacian is strictly positive (i.e. $e_1(x)>0$, $x\in\Omega$). Consequently, the scalar product in the right-hand side is also strictly positive and (A.18) hold with $b(t):=a(t)[(e_1^{j_1+\dots+j_N},e_1)]^{-1}$. □

Since the existence of hyperbolic sets for the systems of ODEs with polynomial nonlinearities is obvious (see e.g. [20]), we have proven the following corollary.

Corollary A.5. *For every sufficiently smooth bounded domain* Ω_0*, there exists a RDS of the form* (2.1) (*or, which is the same,* (A.1)) *which possesses a hyperbolic set homeomorphic to the Bernoulli scheme* $\mathcal{M}^1$ *and thus satisfies Assumption 2.4A of Section* 2.

APPENDIX B. KOLMOGOROV'S ϵ-ENTROPY AND THE TOPOLOGICAL ENTROPY OF ATTRACTORS OF RDS IN $\mathbb{R}^n$

We recall the definitions and main results for Kolmogorov's ϵ-entropy and topological entropy of attractors of reaction-diffusion equations. Further details can be found e.g. in [23,42,47,48]. We start with the definition of Kolmogorov's ϵ-entropy.

Definition B.1. Let K be a compact set in a metric space M. Then, for every positive ϵ, it can be covered by the finite number of ϵ-balls of M. Let $N_\epsilon(K,M)$ be the minimal number of that balls. Then, by definition, Kolmogorov's ϵ-entropy is the logarithm of that number:

$$\mathbb{H}_\epsilon(K,M):=\log_2 N_\epsilon(K,M). \tag{B.1}$$

The detailed exposition of this entropy including its asymptotics for typical sets in functional spaces can be found in [23]. We only recall that the fractal dimension $d_f(K)$ of the set K in M can be expressed in terms of that entropy as follows:

$$d_f(K):=\limsup_{\epsilon\to 0}\frac{\mathbb{H}_\epsilon(K,M)}{\log_2\frac{1}{\epsilon}}. \tag{B.2}$$

Roughly speaking, a set K has a fractal dimension $\kappa \in \mathbb{R}_+$ if $\mathbb{H}_\epsilon(K, M) \sim (\frac{1}{\epsilon})^\kappa$.

It is well-known that the attractors of dissipative dynamical systems generated by evolution equations of mathematical physics usually have finite fractal dimension:

$$d_f(\mathcal{A}, \mathrm{L}^\infty(\Omega)) \leqslant C \quad \text{and} \quad \mathbb{H}_\epsilon(\mathcal{A}, \mathrm{L}^\infty(\Omega)) \leqslant C \log_2 \frac{1}{\epsilon}. \tag{B.3}$$

In contrast to this, the attractors in unbounded domains (e.g. $\Omega = \mathbb{R}^n$) are usually infinite-dimensional and, instead of (B.3), we usually have the following estimate of the ϵ-entropy.

Proposition B.2. *Let the nonlinearity f in equation* (0.1) *in $\mathbb{R}^n$ satisfies some natural dissipativity* (*and growth*) *assumptions. Then, ϵ-entropy of restrictions of the attractor $\mathcal{A}$ to R-balls $B_0^R \subset \mathbb{R}^n$ centered at* 0 *satisfies the estimate*

$$\mathbb{H}_\epsilon(\mathcal{A}\big|_{B_0^R}, \mathrm{L}^\infty(B_0^R)) \leqslant C\Big(R + \log_2 \frac{1}{\epsilon}\Big)^n \log_2 \frac{1}{\epsilon}, \tag{B.4}$$

where the constant C is independent of ϵ and R.

We recall that, in unbounded domains, the attractor $\mathcal{A}$ is compact in the local topology only, so the restrictions to bounded domains (e.g., R-balls) are necessary in order to have finite Kolmogorov's ϵ-entropy. Moreover, under very weak assumption Eq. (0.1) possesses at least one spatially homogeneous exponentially unstable equilibrium; then the attractor $\mathcal{A}$ satisfies the following *lower* bound for the ϵ-entropy:

$$\mathbb{H}_\epsilon\big(\mathcal{A}\big|_{B_0^R}, \mathrm{L}^\infty\big(B_0^R\big)\big) \geqslant C' R^n \log_2 \frac{1}{\epsilon} \tag{B.5}$$

for some positive C' depending on the concrete form of the equation. This estimate shows that (B.4) is sharp for the case $R \gtrsim \log_2 \frac{1}{\epsilon}$. For the case $R \ll \log_2 \frac{1}{\epsilon}$ (e.g. $R = 1$) estimate (B.5) is far from being optimal and can be replaced by the following one: for every positive μ there exists positive C_μ such that

$$\mathbb{H}_\epsilon\big(\mathcal{A}\big|_{B_0^1}, \mathrm{L}^\infty\big(B_0^1\big)\big) \geqslant C_\mu\Big(\log_2 \frac{1}{\epsilon}\Big)^{n+1-\mu}. \tag{B.6}$$

Thus, estimate (B.4) is, in a sense sharp for all values of R and ϵ. It is also worth to note that the type of the asymptotics given by (B.5)–(B.7) seems to have a universal nature and depends very weakly on the concrete type of equation considered. Indeed, up to the moment, the above asymptotics are verified for various classes of reaction-diffusion equation, damped hyperbolic equations and even elliptic systems in unbounded domains, see [17, 34, 44, 47].

We now turn to the *topological* entropy of attractors in unbounded domains. We first recall the general definition adapted to the case of multiparametrical semigroups.

Definition B.3. Let (M, d) be a compact metric space and let an $(n+1)$-parametrical semigroup $\{\mathbb{S}_{(t,h)}: t \geqslant 0,\ h \in \mathbb{R}^n\}$ acts (continuously) in M. For every $T \geqslant 0$, we define a new equivalent metric d_T on M via

$$d_T(m_1, m_2) := \sup_{(t,h)\in[0,T]^{n+1}} d(\mathbb{S}_{(t,h)}m_1, \mathbb{S}_{(t,h)}m_2). \tag{B.7}$$

Then, since (M, d_T) remains compact, its ϵ-entropy $\mathbb{H}_\epsilon(M, (M, d_T))$ is well defined and finite. The topological entropy of $\mathbb{S}_{(t,h)}$ on M is defined as

$$h_{top}(\mathbb{S}_{(t,h)}, M) := \lim_{\epsilon\to 0} \limsup_{T\to\infty} \frac{1}{T^{n+1}} \mathbb{H}_\epsilon(M, (M, d_T)). \tag{B.8}$$

We also recall that the topological entropy depends only on the topology on M and is independent of the concrete choice of the metric d.

The following result shows that for the case of the spatio-temporal dynamical system associated with the RDS (0.1) acting on the attractor $\mathcal{A}$, the topological entropy can be calculated in a much simpler way.

Proposition B.4. *Let $\mathcal{K} \subset \mathrm{L}^\infty(\mathbb{R}^{n+1})$ be the set of all essential solutions of* (0.1). *Then, the topological entropy of the extended spatio-temporal semigroup on the attractor satisfies*

$$h_{top}(\mathbb{S}_{(t,h)}, \mathcal{A}) = \lim_{\epsilon\to 0} \lim_{T\to\infty} \frac{1}{T^{n+1}} \mathbb{H}_\epsilon\Big(\mathcal{K}\big|_{[0,T]^{n+1}}, \mathrm{L}^\infty([0,T]^{n+1})\Big). \tag{B.9}$$

The proof of this formula can be found in [47] or [48]. The universal entropy estimate (B.4) together with expression (B.9) allows us to establish the finiteness of the topological entropy of the attractor.

Corollary B.5. *The expression* (B.9) *is finite for the spatio-temporal semigroup associated with* (0.1)*, i.e.,*

$$h_{top}(\mathbb{S}_{(t,h)}, \mathcal{A}) \leqslant C' < \infty. \tag{B.10}$$

Indeed, estimate (B.4) together with the standard Lipschitz continuity of solutions of (0.1) with respect to the initial data implies that

$$\mathbb{H}_\epsilon\Big(\mathcal{K}\big|_{[0,T]^{n+1}}, \mathrm{L}^\infty\big([0,T]^{n+1}\big)\Big) \leqslant C'\Big(T + \log_2 \frac{1}{\epsilon}\Big)^{n+1} \tag{B.11}$$

which immediately implies (B.10), see [48] for details.

To conclude we note that the lower bounds (B.5) and (B.6) are strong enough in order to verify that the topological entropy of the n-parametrical group $\{T_h\colon h \in \mathbb{R}^n\}$ of *spatial* translations on the attractor is infinite, i.e.,

$$h_{top}(T_h, \mathcal{A}) = \lim_{\epsilon\to 0}\lim_{R\to\infty} \frac{1}{R^n}\mathbb{H}_\epsilon\left(\mathcal{A}\big|_{[0,R]^n}, \mathrm{L}^\infty\left([0,R]^n\right)\right) = \infty. \tag{B.12}$$

Yet, these bounds are not sufficient to show that the spatio-temporal entropy (B.9) is strictly positive. Indeed, in order to obtain this positivity, we need $\mathbb{H}_\epsilon(\mathcal{K}, \mathrm{L}^\infty([0,T]^{n+1}))$ to be proportional to T^{n+1} as $T\to\infty$ and (B.5) gives only

$$\mathbb{H}_\epsilon\left(\mathcal{K}, \mathrm{L}^\infty\left([0,T]^{n+1}\right)\right) \geqslant \mathbb{H}_\epsilon(\mathcal{A}, \mathrm{L}^\infty([0,T]^{n+1})) \geqslant C'\left(T+\log_2\frac{1}{\epsilon}\right)^n \log_2\frac{1}{\epsilon}. \tag{B.13}$$

However, this leads to the much too weak lower bound $C(\epsilon)T^n$ for $T\to\infty$.

In contrast to that, it is not difficult to verify that, for the example of a RDS constructed above the estimate

$$\mathbb{H}_\epsilon\left(\mathcal{K}\big|_{[0,T]^{n+1}}, \mathrm{L}^\infty([0,T]^{n+1})\right) \geqslant C''\left(T-\log_2\frac{1}{\epsilon}\right)^{n+1} \tag{B.14}$$

holds. This gives an alternative proof of the fact that the topological entropy (B.9) is strictly positive for our example.

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