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Existence results for a class of rate-independent material models with nonconvex elastic energies

By *Gilles Francfort* at Villetaneuse and *Alexander Mielke* at Berlin

1. Introduction

In mechanics, rate-independent evolutionary problems have always played an important role, e.g., in Coulomb friction or in perfect plasticity. The intrinsic nonsmoothness made these models difficult to handle mathematically. Only the development of variational inequalities, see e.g. [13] paved a way for their treatment. In particular, the theory of linear elastoplasticity led to major advances, see [21], [38], [40]. However, despite a consistent mathematical formulation of general material models within the framework of *standard generalized materials* (cf. [19]), the theory remained restricted to the case of linear evolutionary variational inequalities which are usually written in the form

$$(1.1) \qquad \forall v \in X\colon \quad \langle Ay(t) - \ell(t) \,|\, v - \dot{y}(t) \rangle + \mathscr{R}(v) - \mathscr{R}\big(\dot{y}(t)\big) \geqq 0,$$

where X is a Hilbert space with scalar product $\langle \cdot | \cdot \rangle$, A is a bounded, symmetric and positive definite operator and $\ell \in \mathrm{C}^1([0,T],X)$ is the exterior forcing term. The dissipation functional $\mathscr{R} : X \to [0,\infty]$ is assumed to be homogeneous of degree 1, lower semi-continuous and convex. Rate independence means that if y is a solution for the loading ℓ, then for each strictly monotone time reparametrization α the function $y \circ \alpha$ solves (1.1) for the loading $\ell \circ \alpha$. To see this, note that the left-hand side in (1.1) is homogeneous of degree 1 in $(\dot{y}, v)$.

Another way to write (1.1) is in the form of a differential inclusion, where $\partial\mathscr{R}$ denotes the subdifferential:

$$(1.2) \qquad -\mathrm{D}\mathscr{E}\big(t, y(t)\big) \in \partial\mathscr{R}\big(\dot{y}(t)\big) \quad \text{with } \mathscr{E}(t,y) = \frac{1}{2}\langle Ay \,|\, y \rangle - \langle \ell(t) \,|\, y \rangle.$$

Many applications can be put into this form, but $\mathscr{E}$ should be more general, for instance nonquadratic and nonconvex. Moreover, for the dissipation functional $\mathscr{R}$ a general metric

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in the form $\mathcal{R}(y, \dot{y})$ is desirable. In fact, for models in damage and fracture, see [9], [12], [15], [16], [22], it is desirable to avoid all differentiable structures, i.e., the state space X should not be assumed to be a subset of a vector space.

In this work we develop a mathematical technique for approaching those more general problems. It is motivated by the abstract topological approach in [25] which builds on the previous Banach space version in [34], [36]. In [11] new technical tools were developed on a special problem in quasistatic crack growth, and it is the purpose of this work to show that the abstract versions of these tools apply to a quite general class of problems. The general framework is that of the *energetic formulation* of (1.2) which takes the following form. A function $y : [0, T] \to X$ is called an *energetic solution* to the rate-independent problem associated with $\mathcal{E}$ and $\mathcal{R}$, if the *stability condition* $(\mathrm{S})_t$ and the *energy balance* $(\mathrm{E})_t$ hold for all $t \in [0, T]$:

$$(\mathrm{S})_t \qquad \forall \hat{y} \in X: \quad \mathcal{E}\big(t, y(t)\big) \leqq \mathcal{E}(t, \hat{y}) + \mathcal{D}\big(y(t), \hat{y}\big),$$

$$(\mathrm{E})_t \qquad \mathcal{E}\big(t, y(t)\big) + \mathrm{Diss}_{\mathcal{D}}(y; [0, t]) = \mathcal{E}\big(0, y(0)\big) + \int_0^t \frac{\partial}{\partial s} \mathcal{E}\big(s, y(s)\big)\, \mathrm{d}s.$$

Here $\mathcal{D}$ is a general (nonsymmetric) distance function on $X \times X$ which generalizes $\mathcal{R}$ via $\mathcal{D}(y_0, y_1) = \mathcal{R}(y_1 - y_0)$. The $\mathcal{D}$-dissipation of a curve y is defined like a variation via

$$\mathrm{Diss}_{\mathcal{D}}(y; [s, t]) = \sup\left\{ \sum_{j=1}^{N} \mathcal{D}\big(y(t_{j-1}), y(t_j)\big) \,\middle|\, N \in \mathbb{N}, s \leqq t_0 < \cdots < t_n \leqq t \right\}.$$

It was observed in [35] that the energetic formulation (S) and (E) is equivalent to (1.1) and also to (1.2) if $\mathcal{E}$ is quadratic and convex and $\mathcal{D}$ is given via $\mathcal{R}$, as above. In general situations, (S) and (E) can be considered as a weak form of the differential inclusion (1.2). It has the advantage that it is derivative free for the potentials $\mathcal{E}$, $\mathcal{D}$ and the solution y. Only the derivative $\frac{\partial}{\partial t}\mathcal{E}$ of the real-valued energy with respect to the time t needs to be defined. Moreover, for a time discretization $0 = t_0 < t_1 < \cdots < t_N = T$, we can use the time-incremental minimization problems

$$(\mathrm{IP}) \qquad \text{Find } y_k \in X \text{ as minimizer of } y \mapsto \mathcal{E}(t, y) + \mathcal{D}(y_{k-1}, y),$$

to approximate the solutions. In fact, (IP) is the basis of our existence result.

To explain the main result of the paper we use the standard decomposition of the state variable y into the pair (φ, z), where $\varphi \in \mathcal{F}$ usually corresponds to an elastic deformation and $z \in \mathcal{Z}$ corresponds to an internal variable associated with the dissipation distance $\mathcal{D}$, i.e., $\mathcal{D}\big((\varphi_0, z_0), (\varphi_1, z_1)\big) = \mathcal{D}(z_0, z_1)$. Typical applications that occur in continuum mechanics (see [7], [22], [26], [27], [28], [30], [32]) have the following form:

$$\mathcal{E}(t, \varphi, z) = \int_\Omega W\big(x, \nabla\varphi(x), z(x)\big) + \frac{\sigma}{\alpha} |\nabla z(x)|^\alpha \, \mathrm{d}x - \langle \ell(t), \varphi \rangle$$

and

$$\mathscr{D}(z_0, z_1) = \int_\Omega \hat{d}\big(x, z_0(x), z_1(x)\big)\,\mathrm{d}x,$$

where $\Omega \subset \mathbb{R}^d$ denotes the body. The energy densities W and $\hat{d}$ satisfy suitable growth and convexity conditions; further, $\sigma \geqq 0$ and is strictly positive when some smoothing of the internal variable is desired.

From the stability condition (S) we immediately conclude that for a given $z(t)$ the function $\varphi(t)$ must be a global minimizer of $\varphi \mapsto \mathscr{E}\big(t, \cdot, z(t)\big)$. In fact, this is the only condition on $\varphi(t)$, which shows that it is intrinsically impossible to control temporal oscillation of the approximate sequences φ^N obtained via (IP). However, under suitable assumptions, it is easy to estimate $\mathrm{Diss}_{\mathscr{D}}(z^N, [0, T])$, which provides a bound on the total variation. Thus, a suitable version of Helly's selection principle allows us to find a convergent subsequence $\big(z^{N_k}(t)\big)_{k \in \mathbb{N}}$. The selection for ($t$-dependent) subsequences of $\big(\varphi_k^N(t)\big)_{k \in \mathbb{N}}$ is much more subtle, as we have to guarantee that the nonlinear term $\partial_t \mathscr{E}\big(t, \varphi^{N_k}(t), z^{N_k}(t)\big)$ converges to the correct limit as well.

To be more specific we assume that there exist Banach spaces X_1, X_2 and X_3 such that X_2 is compactly embedded in X_3 and that $\mathscr{F} = X_1$ and $\mathscr{Z} \subset X_2$. The energy functional $\mathscr{E} : [0, T] \times X_1 \times X_2 \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ has weakly compact sublevels, and hence is lower semi-continuous. Moreover, we assume that the power of the external forces given by $\partial_t \mathscr{E}(t, y)$ can be controlled by the energy via two constants $c_E^{(1)}$ and $c_E^{(0)} > 0$ as follows:

$$|\partial_t \mathscr{E}(t, y)| \leqq c_E^{(1)}\big(c_E^{(0)} + \mathscr{E}(t, y)\big). \tag{1.3}$$

The dissipation $\mathscr{D}$ is assumed to be coercive on the larger space X_3, i.e.,

$$\exists c_{\mathscr{D}} > 0\ \forall z_0, z_1 \in X_2\colon \quad \mathscr{D}(z_0, z_1) \geqq c_{\mathscr{D}} \|z_1 - z_0\|_{X_3}.$$

Additionally, $\mathscr{D} : X_2 \times X_2 \to [0, \infty]$ is assumed to satisfy the triangle inequality and to be continuous with respect to the weak topology of X_2.

These assumptions guarantee that the incremental problem (IP) has solutions and that these solutions satisfy appropriate a priori bounds, see Theorem 3.2. The compactness properties allow us to extract suitable subsequences and to find a limit function $(\varphi, z) : [0, T] \to X_1 \times X_2$. Using the weak continuity of $\mathscr{D}$ on X_2 it is not difficult to show the stability (S) for the limit function. To obtain the energy balance (E) we use the new observation in the spirit of [11], namely

$$\left.\begin{array}{l} (u_m, z_m) \rightharpoonup (u, z) \ \text{ in } \ X_1 \times X_3 \\ \text{and } \mathscr{E}(t, u_m, z_m) \to \mathscr{E}(t, u, z) < \infty \end{array}\right\} \quad \Rightarrow \quad \partial_t \mathscr{E}(t, u_m, z_m) \to \partial_t \mathscr{E}(t, u, z). \tag{1.4}$$

With this result the final step for proving (E) can be performed and existence of solutions can be established, see Theorem 4.1. In the abstract part, we derive (1.4) from a strengthened version of (1.3) (cf. (3.3)) which uses regularity with respect to t, see Proposition 3.3. In Section 4 the same result is obtained under low time regularity by using continuity arguments for $(F, z) \mapsto \mathrm{D}_F W(F, z)$.

The abstract result in (1.4) should be compared with the following, better known result in the calculus of variations (cf. [4], [8], [11]). If $\mathscr{I}(u) = \int_\Omega W(\nabla u)\,\mathrm{d}x$ defines a quasiconvex functional on $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^m)$ which is also Gateaux differentiable, then the following implication holds:

$$\left.\begin{array}{l} v_m \rightharpoonup v \text{ in } \mathrm{W}^{1,p}(\Omega;\mathbb{R}^m) \\ \text{and } \mathscr{I}(v_m) \to \mathscr{I}(v) < \infty \end{array}\right\} \quad \Rightarrow \quad \mathrm{D}\mathscr{I}(v_m) \rightharpoonup \mathrm{D}\mathscr{I}(v) \text{ in } \mathrm{W}^{1,p}(\Omega;\mathbb{R}^m)^*.$$

In the case of time-dependent Dirichlet data this can be used to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}\big(\varphi_{\mathrm{Dir}}(t)+v,z\big) = \big\langle \mathrm{D}_u\mathscr{E}\big(\varphi_{\mathrm{Dir}}(t)+v,z\big),\dot{\varphi}_{\mathrm{Dir}}\big\rangle.$$

This idea will be used in Section 4, where we assume that $W(x,\cdot,z):\mathbb{R}^{d\times d}\to\mathbb{R}$ is quasiconvex and satisfies the assumptions

$$(1.5) \qquad c|F|^p - C \leqq W(x,F,z) \leqq C(1+|F|^p),$$

which implies $|\mathrm{D}_F W(x,F,z)| \leqq \tilde{C}(1+|F|^{p-1})$ and hence Gateaux differentiability, see [10].

In Section 5 we will treat the case of finite strain, where the additive decomposition $u = \varphi_{\mathrm{Dir}}(t)+v$ is no longer appropriate. There, we use the nonlinear decomposition $\varphi = \varphi_{\mathrm{Dir}}(t,\cdot)\circ\psi$ and take full advantage of the abstract version (1.4). In particular, (1.5) needs to be generalized to allow for the local non-interpenetration condition, i.e., $W(x,F,z) = +\infty$ whenever $\det F \leqq 0$. Thus, we assume that $W(x,\cdot,z):\mathbb{R}^{d\times d}\to\mathbb{R}_\infty$ is polyconvex and satisfies the estimates

$$(1.6) \qquad c|F|^p - C \leqq W(x,F,z) \quad \text{and} \quad |\mathrm{D}_F W(x,F,z)F^{\mathsf{T}}| \leqq c_W^{(1)}\big(c_W^{(0)} + W(x,F,z)\big),$$

where differentiability must hold only where $W(x,F,z)<\infty$, i.e., $\det F > 0$. Combining the latter estimate with suitable regularity assumptions on the time-dependent Dirichlet data $t\mapsto\varphi_{\mathrm{Dir}}(t)$, the multiplicative decomposition $\varphi = \varphi_{\mathrm{Dir}}(t)\circ\psi$ provides exactly the desired estimates for the abstract framework, for instance (1.3). Thus, it is possible to prove existence of energetic solutions also for the case of finite-strain elasticity with time-dependent Dirichlet data.

To motivate the subsequent analysis, we provide in the next section some mechanical background, which explains the choice of the functionals and function spaces in the remainder of the paper. Thus, Section 2 can be viewed as an introduction aimed at the mechanically inclined reader.

2. The mechanical model

We consider a hyperelastic material with elastic energy W. It occupies a domain $\Omega \in \mathbb{R}^d$. Its elasticity is controlled by an internal variable, akin to a mesoscopic averaged phase variable z, which may vary from point to point, so that W is a function of the deformation gradient $F \in \mathbb{R}^{d\times d}$ and of z in Z, a closed convex subset of $\mathbb{R}^m$. In the spirit of [15], we could also view z as a damage parameter, and think of the material as one undergoing continuous damage. We assume, to begin with, that W is the free energy of the material.

Following a "classical" thermodynamic reasoning, we associate to the state variable z the following thermodynamic force:

$$A := -\frac{\partial}{\partial z} W(\varphi, z) - \partial I_Z(z),$$

where $I_Z : \mathbb{R}^m \to [0, \infty]$ is the convex indicator function of Z. We then introduce a convex dissipation potential R, a function of $\dot{z}$. Then, classically,

$$A(t) \in \partial R\big(\dot{z}(t)\big)$$

at each point of Ω. Since the model we have in mind is rate-independent, we also assume that R is positively homogeneous of degree one in the variable $\dot{z}$, i.e., $R(\gamma\dot{z}) = \gamma R(\dot{z})$ for $\gamma \geqq 0$. The associated dissipation functional is obtained via integration over the body, i.e.,

$$\mathscr{R}\big(\dot{z}(t)\big) = \int_\Omega R\big(\dot{z}(t,x)\big)\,\mathrm{d}x.$$

We further assume that, at each time, the domain is in elastic equilibrium with the data, which consist of a time-dependent body load $f_{\mathrm{vol}}(t)$, a time-dependent surface load f_{surf} on the part Γ_{Neu} of the boundary $\partial\Omega$, and a time-dependent boundary displacement $\varphi_{\mathrm{Dir}}(t)$ on the remaining part Γ_{Dir}. We view $\varphi_{\mathrm{Dir}}(t)$ as the trace on $\partial\Omega$ of a field, still denoted by $\varphi_{\mathrm{Dir}}(t)$, defined on all of $\bar{\Omega}$. We are not concerned here with the most general class of admissible loads, and will favor simplicity over completeness. Thus, the pair solution $\big(\varphi(t), z(t)\big)$, where φ stands for the deformation, satisfies the following system:

$$\begin{cases} -\operatorname{div}\left[\dfrac{\partial W}{\partial F}\big(\nabla\varphi(t), z(t)\big)\right] = f_{\mathrm{vol}}(t) & \text{in } \Omega, \\ \varphi(t) = \varphi_{\mathrm{Dir}}(t) & \text{on } \Gamma_{\mathrm{Dir}}, \\ \dfrac{\partial W}{\partial F}\big(\nabla\varphi(t), z(t)\big)n = f_{\mathrm{surf}}(t) & \text{on } \Gamma_{\mathrm{Neu}}, \end{cases} \tag{2.1}$$

and

$$-\frac{\partial W}{\partial z}\big(\nabla\varphi(t), z(t)\big) \in \partial R\big(\dot{z}(t)\big) + \partial I_Z\big(z(t)\big) \quad \text{and} \quad z(0) = z_0 \quad \text{in } \Omega. \tag{2.2}$$

In many situations the above described model is intractable, either because the suitable tools are not yet developed or because the model is badly behaved resulting in the formation of microstructure, see [6], [7], [28], [29], [33], [39]. Only in cases with quadratic energies (see [35], [38], [40]) or in very special models (see e.g., [36]) existence results can be obtained without regularizing terms. In the footstep of prior thermomechanical studies (see e.g. [17], [18], [20], [23]) a regularizing term is introduced in the form of a gradient of the internal variable. This amounts to adding to the left-hand side of the first relation in (2.2) a term of the form $\sigma\operatorname{div}(|\nabla z|^{\alpha-2}\nabla z)$, $\alpha \geqq 1$, so that (2.2) becomes

$$\sigma \operatorname{div}(|\nabla z|^{\alpha-2}\nabla z) - \frac{\partial W}{\partial z}\big(\nabla\varphi(t), z(t)\big) \in \partial R\big(\dot z(t)\big) + \partial I_Z\big(z(t)\big), \quad \text{and} \quad z(0) = z_0 \quad \text{in } \Omega, \tag{2.3}$$

together with suitable boundary conditions on $\partial\Omega$, for example homogeneous Neumann conditions. Assuming for now that the evolution (2.1), (2.3) makes sense and that $\big(\varphi(t), z(t)\big)$ do exist over the time of existence of the data, say $[0, T]$, and that they (and the loads) are smooth enough for all that follows to be meaningful, we define the potential energy at time t as

$$\mathscr{E}\big(t, \varphi(t), z(t)\big) := \int_\Omega W\big(\nabla\varphi(t), z(t)\big)\,\mathrm{d}x + \frac{\sigma}{\alpha}\int_\Omega |\nabla z(t)|^\alpha\,\mathrm{d}x - \langle l(t), \varphi(t)\rangle$$

where, from now onward,

$$\langle l(t), \varphi\rangle := \int_\Omega f_{\mathrm{vol}}(t).\varphi\,\mathrm{d}x + \int_{\Gamma_{\mathrm{Neu}}} f_{\mathrm{surf}}(t).\varphi\,\mathrm{d}x. \tag{2.4}$$

We also define the dissipation as

$$\mathrm{Diss}(z; [0, t]) := \int_0^t \mathscr{R}\big(\dot z(s)\big)\,\mathrm{d}s = \int_0^t \int_\Omega R\big(\dot z(s, x)\big)\,\mathrm{d}x\,\mathrm{d}s.$$

Note that, since $R(\cdot)$ is convex and positively homogeneous of degree 1, we have

$$R(v) = y.v \quad \text{for all } y \in \partial R(v),$$

and $\mathbb{E} = \partial R(0)$ is often called the elastic domain. Then, a straightforward computation using (2.1) and (2.3) yields, with obvious notation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathscr{E}\big(t, \varphi(t), z(t)\big) + \mathrm{Diss}(z; [0, t])\Big) = \int_\Omega \mathrm{D}W\big(\nabla\varphi(t), z(t)\big).\nabla\dot\varphi_{\mathrm{Dir}}(t)\,\mathrm{d}x - \langle \dot l(t), \varphi(t)\rangle - \langle l(t), \dot\varphi_{\mathrm{Dir}}(t)\rangle =: \mathscr{P}(t).$$

After integrating over $[0, t]$ this also reads

$$\mathscr{E}\big(t, \varphi(t), z(t)\big) + \mathrm{Diss}(z; [0, t]) = \mathscr{E}\big(0, \varphi(0), z(0)\big) + \int_0^t \mathscr{P}(s)\,\mathrm{d}s,$$

which, through an elementary integration by parts, becomes a statement of the first law of thermodynamics (energy balance).

As will be seen in the sequel, energy conservation will be recovered, albeit in a slightly weaker form; indeed, we will not recover the kind of smoothness that allows us to differentiate $\mathrm{Diss}(z; [0, t])$. Instead, we will obtain the following:

$$\mathscr{E}\big(t, \varphi(t), z(t)\big) + \mathrm{Diss}_{\mathscr{R}}(z; [0, t]) = \mathscr{E}\big(0, \varphi(0), z(0)\big) + \int_0^t \mathscr{P}(s)\,\mathrm{d}s, \tag{2.5}$$

where $\mathrm{Diss}_{\mathscr{R}}$ stands for the total variation of the dissipation, that is

$$\mathrm{Diss}_{\mathscr{R}}(z;[0,t]) := \sup\left\{ \sum_{i=1}^{N} \mathscr{R}\big(z(t_i)-z(t_{i-1})\big) \,\middle|\, \{t_i\}_{i=0,\dots,N} \text{ partition of } [0,t] \right\}.$$

A natural way to attack the evolution problem (2.1), (2.3) is via time discretization. Taking for example a partition of $[0,T]$ into $0 = t_0^n \leqq \cdots \leqq t_n^n = T$, and setting

$$\tau_i^n := t_{i+1}^n - t_i^n, \quad (f_{\mathrm{vol}})_i^n := f_{\mathrm{vol}}(t_i^n), \quad (f_{\mathrm{surf}})_i^n := f_{\mathrm{surf}}(t_i^n),$$
$$l_i^n := l(t_i^n), \quad (\varphi_{\mathrm{Dir}})_i^n := \varphi_{\mathrm{Dir}}(t_i^n),$$

we would be led to the solving of the following system. We start with $z_0^n := z_0$ in Ω and for $i \in \{1,\dots,N\}$ we have

$$(2.6)\qquad \begin{cases} -\operatorname{div}\left[\dfrac{\partial W}{\partial F}(\nabla\varphi_{i+1}^n, z_{i+1}^n)\right] = (f_{\mathrm{vol}})_{i+1}^n & \text{in } \Omega, \\ \varphi_{i+1}^n = (\varphi_{\mathrm{Dir}})_{i+1}^n & \text{on } \Gamma_{\mathrm{Dir}}, \\ \dfrac{\partial W}{\partial F}(\nabla\varphi_{i+1}^n, z_{i+1}^n)n = (f_{\mathrm{surf}})_{i+1}^n & \text{on } \Gamma_{\mathrm{Neu}}, \end{cases}$$

and

$$(2.7)\qquad \sigma \operatorname{div}[|\nabla z_{i+1}^n|^{\alpha-2}\nabla z_{i+1}^n] - \frac{\partial W}{\partial z}(\nabla\varphi_{i+1}^n, z_{i+1}^n) \in \partial R\left(\frac{z_{i+1}^n - z_i^n}{\tau_i^n}\right) + \partial I_Z(z_{i+1}^n).$$

A straightforward variation shows that the system above is in particular a necessary first order optimality condition for $(\varphi_{i+1}^n, z_{i+1}^n)$ to be a local minimizer for

$$\int_\Omega W(\nabla\varphi, z)\,\mathrm{d}x + \frac{\sigma}{\alpha}\int_\Omega |\nabla z|^\alpha\,\mathrm{d}x - \langle l_{i+1}^n, \varphi\rangle + \tau_i^n \int_\Omega R\left(\frac{z - z_i^n}{\tau_i^n}\right)\mathrm{d}x$$

among all admissible pairs (φ, z). But, since $\mathscr{R}$ is homogeneous of degree 1, the time step τ_i^n drops out in the last term of the above expression and the optimality condition reads as that of

$$\mathscr{E}(t_{i+1}^n, \varphi, z) + \mathscr{R}(z - z_i^n) = \int_\Omega W(\nabla\varphi, z)\,\mathrm{d}x + \frac{\sigma}{\alpha}\int_\Omega |\nabla z|^\alpha\,\mathrm{d}x - \langle l_{i+1}^n, \varphi\rangle + \int_\Omega R(z - z_i^n)\,\mathrm{d}x.$$

Our only deviation from this consists in assuming that, at each discrete time, $(\varphi_{i+1}^n, z_{i+1}^n)$ is a *global minimizer* for

$$\mathscr{E}(t_{i+1}^n, \varphi, z) + \mathscr{R}(z - z_i^n).$$

This is an admittedly unjustified assumption, but it does not contradict any known thermodynamical principle, and has proved useful in various contexts (see e.g. [15], [16], [36]). Moreover, it opens up the mathematical toolbox of the direct method in the calculus of variations which would not be available for local minimizers. Of course, suitable properties imply equivalence, see [35].

Our analysis is based on the discrete approximation proposed above and consists in showing that, as the time-step tends to zero, these approximations converge to a time-continuous evolution, which provides a "weak" solution for (2.1), (2.3), namely the energetic formulation (S) and (E) given in the introduction.

From a mechanical standpoint, the dissipation potential R could also depend on $z \in Z \subset \mathbb{R}^m$, i.e.,

$$\mathrm{Diss}_{\mathscr{R}}(z;[s,t]) = \int_s^t \int_\Omega R\big(x, z(\tau,x), \dot{z}(\tau,x)\big)\,\mathrm{d}x\,\mathrm{d}\tau.$$

We also included the dependence on $x \in \Omega$ for clarity. Mathematically

$$R(x,\cdot,\cdot) : \Omega \times \mathrm{T}Z \to [0,\infty)$$

corresponds to a local Finsler-Minkowski metric on Z. We associate with two different states z_0 and z_1 at one material point $x \in \Omega$ a specific dissipated energy $\hat{d}(x, z_0, z_1)$, called *dissipation distance*:

$$\hat{d}(x,z_0,z_1) = \inf\left\{ \int_{\tau=0}^{1} R\big(x, z(\tau), \dot{z}(\tau)\big)\,\mathrm{d}\tau \,\middle|\, z \in \mathrm{C}^0([0,1];Z) \cap \mathrm{W}^{1,\infty}\big((0,1);\mathbb{R}^m\big),\ z(0) = z_0,\ z(1) = z_1 \right\}.$$

The minimal dissipated energy by a change from the internal state $\tilde{z}_0 : \Omega \to Z$ to the internal state $\tilde{z}_1 : \Omega \to Z$ is then called the (global) dissipation distance

$$\mathscr{D}(\tilde{z}_0, \tilde{z}_1) = \int_\Omega \hat{d}\big(x, \tilde{z}_0(x), \tilde{z}_1(x)\big)\,\mathrm{d}x.$$

From the definition it is clear that $\hat{d}$ and $\mathscr{D}$ satisfy $\hat{d}(x,z,z) = 0$ and $\mathscr{D}(\tilde{z},\tilde{z}) = 0$ and the triangle inequality

$$\hat{d}(x,z_0,z_2) \leqq \hat{d}(x,z_0,z_1) + \hat{d}(x,z_1,z_2) \quad \text{and} \quad \mathscr{D}(\tilde{z}_0,\tilde{z}_2) \leqq \mathscr{D}(\tilde{z}_0,\tilde{z}_1) + \mathscr{D}(\tilde{z}_1,\tilde{z}_2).$$

Note that the symmetries $\hat{d}(x,z_0,z_1) = \hat{d}(x,z_1,z_0)$ and $\mathscr{D}(\tilde{z}_0,\tilde{z}_1) = \mathscr{D}(\tilde{z}_1,\tilde{z}_0)$, which would follow from $R(x,z,-\dot{z}) = R(x,z,\dot{z})$, are *not* assumed and not used in our work, since in many mechanical situations (like damage or hardening) they are not satisfied. See [27], [28], [32] for more details and applications in finite-strain elastoplasticity where Z is the group of matrices with determinant 1.

If $Z \subset \mathbb{R}^m$ is convex and R is independent of z, the convexity of R in the rate $\dot{z}$ immediately implies $\hat{d}(x,z_0,z_1) = R(x, z_1 - z_0)$ and hence

$$\mathscr{D}(\tilde{z}_0,\tilde{z}_1) = \int_\Omega R\big(x, \tilde{z}_1(x) - \tilde{z}_0(x)\big)\,\mathrm{d}x = \mathscr{R}(\tilde{z}_1 - \tilde{z}_0).$$

The case discussed above is thus recovered. From now on, we will work in this general setting, starting directly from $\hat{d}$ and $\mathscr{D}$.

3. Abstract energetic formulation

In this section, we abstract the problem described in Section 2 into a generic framework. The first step is to choose a basic time-independent function space $\mathscr{F}$. In the illustrative example, this would be $\mathscr{F} = \mathrm{W}_0^{1,p}(\Omega; \mathbb{R}^d)$, once the variable φ has been replaced by $u = \varphi - \varphi_{\mathrm{Dir}}(t)$. Also, the internal variable z would live in $\mathscr{Z} = \mathrm{L}^1(\Omega; Z)$.

The energy functional $\mathscr{E} : [0, T] \times \mathscr{F} \times \mathscr{Z} \to [-c_E^{(0)}, \infty]$, with $c_E^{(0)} \geqq 0$, and the dissipation distance $\mathscr{D} : \mathscr{Z} \times \mathscr{Z} \to [0, \infty]$ are assumed to be lower semicontinuous with respect to the chosen topology $\mathscr{T}_{\mathscr{F}} \times \mathscr{T}_{\mathscr{Z}}$ on $\mathscr{F} \times \mathscr{Z}$. For the example above, we will choose the weak topology on $\mathrm{W}^{1,p}$ and the strong topology of $\mathrm{L}^1(\Omega; \mathbb{R}^m)$ restricted to $\mathscr{Z}$.

Note that only continuity properties are used on u and z since no linear structure is imposed on $\mathscr{F} \times \mathscr{Z}$. We will, however, assume differentiability of the function $t \mapsto \mathscr{E}(t, u, z)$.

(3.1) There exist constants $c_E^{(0)}, c_E^{(1)} > 0$, such that

$$\mathscr{E}(t, u, z) < \infty \quad \Rightarrow \quad |\partial_t \mathscr{E}(t, u, z)| \leqq c_E^{(1)} \big(c_E^{(0)} + \mathscr{E}(t, u, z)\big).$$

Specifically, we mean that, if for some (t_*, u_*, z_*) we have $\mathscr{E}(t_*, u_*, z_*) < \infty$, then $t \to \mathscr{E}(t, u_*, z_*) \in \mathbb{R}_\infty$ is differentiable in t_* and that the derivative satisfies the given bound. In particular, this implies that $t \mapsto \mathscr{E}(t, u_*, z_*)$ is bounded and differentiable on $[0, T]$. The importance of (3.1) is that it provides uniform continuity of $t \mapsto \mathscr{E}(t, u, z)$ on sublevels of $\mathscr{E}$. Estimating $\mathscr{E}(t_1, u, z) = \mathscr{E}(t_2, u, z) + \int_{t_1}^{t_2} \partial_s \mathscr{E}(s, u, z)\, \mathrm{d}s$ with (3.1) and applying Gronwall's lemma we obtain

$$\mathscr{E}(t_2, u, z) \leqq \big(c_E^{(0)} + \mathscr{E}(t_1, u, z)\big) \mathrm{e}^{c_E^{(1)} |t_2 - t_1|} - c_E^{(0)}. \tag{3.2}$$

This estimate will be the basis for the a priori estimate of the stored energy and the dissipated energy.

At a later stage of the existence proof, we need the following strengthened version of (3.1), which asks for uniform continuity of $\partial_t \mathscr{E}$ on sublevels of $\mathscr{E}$:

(3.3) $\forall E > 0\ \exists$ a modulus of continuity $\omega_E : [0, T] \to [0, \infty)$:

$$\mathscr{E}(0, u, z) \leqq E \quad \Rightarrow \quad \forall t_1, t_2 \in [0, T] : |\partial_t \mathscr{E}(t_1, u, z) - \partial_t \mathscr{E}(t_2, u, z)| \leqq \omega_E(t_1 - t_2).$$

Here, a function $\omega : [0, T] \to [0, \infty)$ is called a modulus of continuity, if it is nondecreasing and $\omega(\tau) \to 0$ for $\tau \searrow 0$.

The dissipation distance $\mathscr{D}$ satisfies $\mathscr{D}(z, z) = 0$ and the triangle inequality

$$\forall z_0, z_1, z_2 \in X_3: \quad \mathscr{D}(z_0, z_2) \leqq \mathscr{D}(z_0, z_1) + \mathscr{D}(z_1, z_2). \tag{3.4}$$

We define the sets $\mathscr{S}(t)$ of stable states at time t via

$$\mathscr{S}(t) := \{(u,z) \in \mathscr{F} \times \mathscr{Z} \mid \mathscr{E}(t,u,z) < \infty \text{ and } \forall(\tilde{u},\tilde{z}) : \mathscr{E}(t,u,z) \leqq \mathscr{E}(t,\tilde{u},\tilde{z}) + \mathscr{D}(z,\tilde{z})\}$$

as well as the stable graph $\mathscr{S}_{[0,T]} := \bigcup_{[0,T]} \big(t, \mathscr{S}(t)\big) \subset [0,T] \times \mathscr{F} \times \mathscr{Z}$.

Definition 3.1. A function $(u,z) : [0,T] \to \mathscr{F} \times \mathscr{Z}$ is called an *energetic solution* of the rate-independent problem associated with $\mathscr{E}$ and $\mathscr{D}$, if $t \mapsto \partial_t \mathscr{E}\big(t, u(t), z(t)\big)$ lies in $\mathrm{L}^1\big((0,T), \mathbb{R}\big)$ and if for all $t \in [0,T]$ we have $(\mathrm{S})_t$ and $(\mathrm{E})_t$:

$(\mathrm{S})_t$ Stability: $\big(u(t), z(t)\big) \in \mathscr{S}(t)$.

$(\mathrm{E})_t$ Energy balance:

$$\mathscr{E}\big(t, u(t), z(t)\big) + \mathrm{Diss}_{\mathscr{D}}(z;[0,t]) = \mathscr{E}\big(0, u(0), z(0)\big) + \int_0^t \partial_t \mathscr{E}\big(s, u(s), z(s)\big)\,\mathrm{d}s.$$

Since the dissipation is always nonnegative, energy balance and assumption (3.1) imply that $e(t) \leqq e(0) + \int_0^t c_E^{(1)}\big(c_E^{(0)} + e(\tau)\big)\,\mathrm{d}\tau$ where $e(t) := \mathscr{E}\big(t, u(t), z(t)\big)$. A Gronwall type argument applied to $c_E^{(0)} + e(t)$ provides

$$\mathscr{E}\big(t, u(t), z(t)\big) \leqq \big(c_E^{(0)} + \mathscr{E}\big(0, u(0), z(0)\big)\big)\mathrm{e}^{c_E^{(1)} t} - c_E^{(0)}.$$

Inserting this back into $(\mathrm{E})_t$, we obtain

$$\mathrm{Diss}_{\mathscr{D}}(z;[0,t]) \leqq \big(c_E^{(0)} + \mathscr{E}\big(0, u(0), z(0)\big)\big)\mathrm{e}^{c_E^{(1)} t}.$$

Discrete counterparts of these estimates will be pivotal in deriving a priori estimates for the approximate incremental solutions.

Our existence proof relies on the following incremental minimization problem. As a short-hand notation we use "$\mathrm{Argmin}\{\Phi(v) \mid v \in \mathscr{V}\}$" to denote the set of all minimizers of a functional $\Phi : \mathscr{V} \to \mathbb{R}_\infty$, i.e., with $\alpha = \inf\{\Phi(v) \mid v \in \mathscr{V}\}$ we define

$$\underset{\mathscr{V}}{\mathrm{Argmin}}\, \Phi = \mathrm{Argmin}\{\Phi(v) \mid v \in \mathscr{V}\} := \{v \in \mathscr{V} \mid \Phi(v) = \alpha\}.$$

For a given partition $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ and an initial value $z_0 \in \mathscr{Z}$ we choose $u_0 \in \mathrm{Argmin}_{\mathscr{F}}\, \mathscr{E}(0, \cdot, z_0)$ and define the following incremental problem (which always has solutions, by lower semi-continuity properties and coercivity):

(IP) For $k = 1, \ldots, N$ find

$$(u_k, z_k) \in \mathrm{Argmin}\{\mathscr{E}(t_k, u, z) + \mathscr{D}(z_{k-1}, z) \mid (u,z) \in \mathscr{F} \times Z\}.$$

The proof of the existence results consists of six steps and the abstract existence result for (S) and (E) is given at the end of this section, see Theorem 3.4.

Step 1. A priori estimates. Since (u_k, z_k) are minimizers, it is easy to derive the following result, firstly established in [34], which shows that the fully implicit incremental problem is a very convenient discretization from the analytical standpoint.

Theorem 3.2. *Assume* $(u_0, z_0) \in \mathscr{S}(0)$, *then every solution* $(u_k, z_k)_{k=0,1,\dots,N}$ *of* (IP) *satisfies the discrete versions* $(\mathrm{S})_{\mathrm{discr}}$ *and* $(\mathrm{E})_{\mathrm{discr}}$ *of stability* (S) *and energy equality* (E), *namely for all* $k \in \{1, \dots, N\}$ *we have*

$$(\mathrm{S})_{\mathrm{discr}} \qquad (u_k, z_k) \in \mathscr{S}(t_k),$$

$$(\mathrm{E})_{\mathrm{discr}} \qquad \int_{t_{k-1}}^{t_k} \partial_t \mathscr{E}(s, u_k, z_k)\,\mathrm{d}s \leqq e_k + \delta_k - e_{k-1} \leqq \int_{t_{k-1}}^{t_k} \partial_t \mathscr{E}(s, u_{k-1}, z_{k-1})\,\mathrm{d}s$$

where $e_k := \mathscr{E}(t_k, u_k, z_k)$ *and* $\delta_k := \mathscr{D}(z_{k-1}, z_k)$. *Moreover, we have the a priori estimates*

$$\mathscr{E}(t_k, u_k, z_k) \leqq (e_0 + c_E^{(0)})\mathrm{e}^{c_E^{(1)} t_k} - c_E^{(0)} \quad \textit{and} \quad \sum_{j=1}^{N} \mathscr{D}(z_{j-1}, z_j) \leqq (e_0 + c_E^{(0)})\mathrm{e}^{c_E^{(1)} T}. \tag{3.5}$$

Proof. Since (u_k, z_k) is a minimizer, the estimate $e_k + \delta_k \leqq \mathscr{E}(t_k, \tilde{u}, \tilde{z}) + \mathscr{D}(z_{k-1}, \tilde{z})$ holds for all $(\tilde{u}, \tilde{z})$. Using the triangle estimate $\mathscr{D}(z_{k-1}, \tilde{z}) \leqq \delta_k + \mathscr{D}(z_k, \tilde{z})$ we conclude that $e_k \leqq \mathscr{E}(t_k, \tilde{u}, \tilde{z}) + \mathscr{D}(z_k, \tilde{z})$, which proves $(\mathrm{S})_{\mathrm{discr}}$.

The upper estimate follows from the minimality of (u_k, z_k) with $(u, z) = (u_{k-1}, z_{k-1})$ as competitor:

$$e_k + \delta_k \leqq \mathscr{E}(t_k, u_{k-1}, z_{k-1}) + \mathscr{D}(z_{k-1}, z_{k-1}) = e_{k-1} + \int_{t_{k-1}}^{t_k} \partial_t \mathscr{E}(s, u_{k-1}, z_{k-1})\,\mathrm{d}s.$$

The lower estimate is obtained from $(u_{k-1}, z_{k-1}) \in \mathscr{S}(t_{k-1})$ when testing with $(\tilde{u}, \tilde{z}) = (u_k, z_k)$:

$$e_{k-1} \leqq \mathscr{E}(t_{k-1}, u_k, z_k) + \delta_k = e_k + \delta_k - \int_{t_{k-1}}^{t_k} \partial_t \mathscr{E}(s, u_k, z_k)\,\mathrm{d}s.$$

This proves $(\mathrm{E})_{\mathrm{discr}}$.

Starting from the upper estimate in $(\mathrm{E})_{\mathrm{discr}}$, inserting (3.1) and then using (3.2) under the integral provides

$$\begin{aligned} e_k + \delta_k &\leqq e_{k-1} + (c_E^{(0)} + e_{k-1})(\mathrm{e}^{c_E^{(1)}(t_k - t_{k-1})} - 1) \\ &= (c_E^{(0)} + e_{k-1})\,\mathrm{e}^{c_E^{(1)}(t_k - t_{k-1})} - c_E^{(0)}. \end{aligned} \tag{3.6}$$

Using $\delta_k \geqq 0$, induction over $j = 1, \dots, k$ gives

$$c_E^{(0)} + e_k \leqq (c_E^{(0)} + e_0)\prod_{j=1}^{k} \mathrm{e}^{c_E^{(1)}(t_j - t_{j-1})} = (c_E^{(0)} + e_0)\mathrm{e}^{c_E^{(1)} t_k} \quad \text{for } k = 1, \dots, N.$$

Moreover, since $c_E^{(0)} + e_k \geqq 0$, we estimate the dissipated energy via

$$\sum_{j=1}^{k} \delta_j \leqq e_0 - e_k + \sum_{j=1}^{k} (c_E^{(0)} + e_{j-1})(\mathrm{e}^{c_E^{(1)}(t_j - t_{j-1})} - 1)$$

$$\leqq_{(3.6)} (c_E^{(0)} + e_0) - (c_E^{(0)} + e_k) + (c_E^{(0)} + e_0) \sum_{1}^{k} (\mathrm{e}^{c_E^{(1)} t_j} - \mathrm{e}^{c_E^{(1)} t_{j-1}})$$

$$\leqq (c_E^{(0)} + e_0) - 0 + (c_E^{(0)} + e_0)(\mathrm{e}^{c_E^{(1)} t_k} - 1) = (c_E^{(0)} + e_0)\, \mathrm{e}^{c_E^{(1)} t_k}.$$

The proof is complete. □

So far, we have assumed lower semicontinuity of $\mathscr{E}(t, \cdot)$ and $\mathscr{D}$. The existence of incremental solutions can be guaranteed by assuming in addition that

$$\text{(3.7)} \qquad \forall t \in [0, T]\ \forall E \in \mathbb{R}: \quad \text{the sublevels } \{(u, z) \in \mathscr{F} \times \mathscr{Z} \mid \mathscr{E}(t, u, z) \leqq E\} \text{ are compact.}$$

Recall that lower semicontinuity is equivalent to the property that all sublevels are closed. Hence, the sublevels of the sum of $\mathscr{E}(t, \cdot)$ and $\mathscr{D}(z_{k-1}, \cdot)$ are also closed. From $\mathscr{D} \geqq 0$ we conclude that these sublevels are in fact contained in the corresponding compact sublevel of $\mathscr{E}$ alone. Since closed subsets of compact sets are compact, we conclude by Weierstraß' extremum principle that the desired minimizers in (IP) exist.

Let X_1 and $X_2 \subset X_3$ be Banach spaces such that X_2 is compactly embedded in X_3. The set $\mathscr{F}$ of admissible deformations is a weakly closed subset of X_1 and the set $\mathscr{Z}$ of internal states a weakly closed subset of X_2. Both sets are equipped with the weak topology. Then, $\mathscr{E}$ is called uniformly coercive on $[0, T] \times X_1 \times X_2$ if

$$\mathscr{E}(t_j, u_j, z_j) \to \infty \quad \text{implies} \quad \|u_j\|_{X_1} + \|z_j\|_{X_2} \to \infty$$

while the dissipation distance is called X_3-*coercive* if there exists $c_{\mathscr{D}} > 0$ with

$$\text{(3.8)} \qquad \mathscr{D}(z_0, z_1) \geqq c_{\mathscr{D}} \|z_1 - z_0\|_{X_3} \quad \text{for all } z_0, z_1 \in \mathscr{Z}.$$

The above a priori estimates, together with coercivity, imply that the piecewise constant interpolants

$$(u^N, z^N): \begin{cases} [0, T] \to \mathscr{F} \times \mathscr{Z}, \\ t \mapsto \begin{cases} (u_j, z_j) & \text{for } t \in [t_j, t_{j+1}) \text{ with } j = 0, 1, \ldots, N-1, \\ (u_N, z_N) & \text{for } t = t_N = T, \end{cases} \end{cases}$$

satisfy the bounds

$$\text{(3.9)} \qquad \begin{cases} \mathscr{E}\big(t, u^N(t), z^N(t)\big) \leqq \big(\mathscr{E}(0, u_0, z_0) + c_E^{(0)}\big)\mathrm{e}^{c_E^{(1)} t} - c_E^{(0)}, \\ \mathrm{Diss}_{\mathscr{D}}(z^N; [0, T]) = \sum_{1}^{N} \mathscr{D}(z_{j-1}, z_j) \leqq \big(\mathscr{E}(0, u_0, z_0) + c_E^{(0)}\big)\mathrm{e}^{c_E^{(1)} T} =: E_*. \end{cases}$$

Indeed, for the first estimate, it suffices to consider $t \in [t_j, t_{j+1})$, to apply (3.1) and (3.2) to

$$\mathscr{E}\big(t, u^N(t), z^N(t)\big) = \mathscr{E}(t, u_j, z_j) = \mathscr{E}(t_j, u_j, z_j) + \int_{t_j}^{t} \partial_s \mathscr{E}(s, u_j, z_j)\,\mathrm{d}s,$$

and to recall (3.5). The second estimate follows directly from (3.5).

Thus, we obtain

$$\|u^N\|_{\mathrm{L}^\infty((0,T),X_1)} \leqq C_*, \quad \|z^N\|_{\mathrm{L}^\infty((0,T),X_2)} \leqq C_*, \quad \mathrm{Var}_{X_3}(z^N, [0,T]) \leqq C_*,$$

where C_* is independent of the partition and of the incremental solution.

By adding the inequalities $(\mathrm{E})_{\mathrm{discr}}$ from Theorem 3.2 we immediately obtain that for $0 \leqq n \leqq k \leqq N$ we have the upper energy estimate

$$\begin{aligned}\mathscr{E}\big(t_k, u^N(t_k), z^N(t_k)\big) &+ \mathrm{Diss}_{\mathscr{D}}(z^N; [t_n, t_k]) \\ &\leqq \mathscr{E}\big(t_n, u^N(t_n), z^N(t_n)\big) + \int_{t_n}^{t_k} \theta^N(s)\,\mathrm{d}s\end{aligned} \tag{3.10}$$

with $\theta^N(t) = \partial_t \mathscr{E}\big(t, u^N(t), z^N(t)\big)$.

Step 2. Selection of subsequences. We now choose a sequence

$$\Pi^N = \{0 = t_0^N < \cdots < t_k^N < \cdots < t_N^N = T\}$$

of nested partitions (i.e., $\Pi^N \subset \Pi^{N+1}$) whose fineness

$$\Delta_N = \Delta(\Pi^N) := \max\{t_j^N - t_{j-1}^N \mid j = 1, \ldots, N\}$$

tends to 0 and obtain the associated approximations (u^N, z^N). Using a suitable version of Helly's selection principle (cf. [5], [37] and for a very general form [25]) it is possible to find a subsequence $(z^{N_k})_{k\in\mathbb{N}}$ and a limit function $z \in \mathrm{L}^\infty\big((0,T),X_2\big) \cap \mathrm{BV}([0,T],X_3)$ such that

$$\forall t \in [0,T]: \quad z^{N_k}(t) \to z(t) \text{ strongly in } X_3 \text{ and weakly in } X_2.$$

Here one uses that X_2 is compactly embedded in X_3. Moreover, we may also assume that for all $t \in [0,T]$ the limit $\delta(t) := \lim_{k\to\infty} \mathrm{Diss}(z^{N_k}, [0,t])$ exists.

Further, note that the functions θ^{N_k} with $\theta^{N_k}(t) = \partial_t \mathscr{E}\big(t, u^{N_k}(t), z^{N_k}(t)\big)$ form a bounded sequence in $\mathrm{L}^\infty\big((0,T)\big)$. This follows immediately from (3.1), together with the first estimate in (3.9). Employing (3.1) once again we find

$$|\theta^N(t)| = \big|\partial_t \mathscr{E}\big(t, u^N(t), z^N(t)\big)\big| \leqq E_1$$

for a suitable constant.

Choosing a further subsequence if necessary, we may assume

$$\theta^{N_k} \overset{*}{\rightharpoonup} \theta_* \quad \text{in } \mathrm{L}^\infty\big((0,T)\big). \tag{3.11}$$

Moreover, we define $\theta : [0, T] \to \mathbb{R}$ via

$$\theta(t) = \limsup_{k\to\infty} \theta^{N_k}(t),$$

and note that $\theta_*(t) \leqq \theta(t)$ a.e. by application of Fatou's Lemma.

For fixed $t \in [0, T]$ we now choose a t-dependent subsequence $(N_l^t)_{l\in\mathbb{N}}$ of $(N_k)_{k\in\mathbb{N}}$ such that

$$\theta^{N_l^t}(t) \to \theta(t) \quad \text{for } l \to \infty,$$

$$u^{N_l^t}(t) \rightharpoonup u(t) \quad \text{for } l \to \infty \text{ in } X_1.$$

This defines the (possibly non-measurable) function $u : [0, T] \to \mathscr{F} \subset X_1$, i.e., $u \in \mathrm{B}([0, T], X_1)$, the space of bounded functions from $[0, T]$ into X_1 (like $\mathrm{L}^\infty([0, T], X_1)$ but without measurability). It remains to show that $(u, z) : [0, T] \to \mathscr{F} \times \mathscr{Z}$ is a solution.

Step 3. Stability of the limit process. The stability of $\big(u(t), z(t)\big)$ is obtained by showing that $\mathscr{S}_{[0,T]}$ is closed in the weak topology of $X_1 \times X_2$. This closedness is a major ingredient to the theory and has to be proved using specific properties of the problem. Note that

$$\big(u^{N_l^t}(t), z^{N_l^t}(t)\big) \rightharpoonup \big(u(t), z(t)\big) \quad \text{in } X_1 \times X_2$$

and $\big(u^{N_l^t}(t), z^{N_l^t}(t)\big) \in \mathscr{S}(\tau_l^t)$ for $\tau_l^t = \max\{\hat{t} \in \Pi^{N_l^t} \mid \hat{t} \leqq t\}$ with $\tau_l^t \nearrow t$ for $l \to \infty$.

In this work the closedness of $\mathscr{S}_{[0,T]}$ is obtained since $\mathscr{D}$ is strongly continuous on X_3 and hence weakly continuous on X_2 by the compact embedding of X_2 into X_3. For $(t_j, u_j, z_j) \in \mathscr{S}_{[0,T]}$ with $t_j \to t$, $u_j \rightharpoonup u$ in X_1 and $z_j \rightharpoonup z$ in X_2, we have, for all $(\tilde{u}, \tilde{z}) \in \mathscr{F} \times \mathscr{Z}$,

$$\begin{aligned} \mathscr{E}(t, u, z) &\leqq_{\mathscr{E}\ \mathrm{lsc}} \liminf_{j\to\infty} \mathscr{E}(t_j, u_j, z_j) \\ &\leqq_{(\mathrm{S})} \liminf_{j\to\infty} \mathscr{E}(t_j, \tilde{u}, \tilde{z}) + \mathscr{D}(z_j, \tilde{z}) = \mathscr{E}(t, \tilde{u}, \tilde{z}) + \mathscr{D}(z, \tilde{z}), \end{aligned}$$

which is the desired stability of (u, z).

Step 4. Upper energy estimate. We first show that $\mathscr{E}$ converges along the approximation sequence, i.e.

$$\mathscr{E}\big(t, u(t), z(t)\big) = \lim_{l\to\infty} \mathscr{E}\big(t, u^{N_l^t}(t), z^{N_l^t}(t)\big). \tag{3.12}$$

By lower semi-continuity of $\mathscr{E}(t, \cdot, \cdot)$ we have the lower estimate. The upper estimate follows by weak continuity of $\mathscr{D}$ and stability, namely

$$\begin{aligned} \mathscr{E}\big(t, u(t), z(t)\big) &= \lim_{l\to\infty} \mathscr{E}\big(t, u(t), z(t)\big) + \mathscr{D}\big(z^{N_l^t}(t), z(t)\big) \\ &\geqq \limsup_{l\to\infty} \mathscr{E}\big(t, u^{N_l^t}(t), z^{N_l^t}(t)\big). \end{aligned}$$

Next, we have to show that the power of the external forces converges, namely

$$\partial_t\mathscr{E}\big(t, u(t), z(t)\big) = \lim_{l\to\infty} \partial_t\mathscr{E}\big(t, u^{N_l^l}(t), z^{N_l^l}(t)\big).$$

If $\partial_t\mathscr{E}(t, u, z) = -\langle \ell(t), u\rangle$ for some $\ell \in \mathrm{C}^1([0, T], X_1^*)$ (which is the case for time-independent Dirichlet data), then this is easily confirmed. However, for time-dependent Dirichlet data this method does not work. This problem was first solved in [11], Lemma 4.11, where it was shown that, for quasiconvex functionals, weak convergence of (u_m, z_m) and convergence of the energy implies convergence of the stresses. Here, we present an abstract version of that result.

Proposition 3.3. *If $\mathscr{E}$ is weakly lower semicontinuous and satisfies condition* (3.3), *then for all $t \in (0, T)$ the following implication holds*:

$$\left.\begin{array}{l}(u_m, z_m) \rightharpoonup (u, z) \textit{ in } X_1 \times X_2 \\ \textit{and } \mathscr{E}(t, u_m, z_m) \to \mathscr{E}(t, u, z) < \infty\end{array}\right\} \;\Rightarrow\; \partial_t\mathscr{E}(t, u_m, z_m) \to \partial_t\mathscr{E}(t, u, z). \tag{3.13}$$

For our proof we only need the one-sided estimate $\limsup_{m\to\infty} \partial_t\mathscr{E}(t, u_m, z_m) \leqq \partial_t\mathscr{E}(t, u, z)$.

Proof. Let $E_0, h > 0$ be such that $t \pm h \in [0, T]$ and $\mathscr{E}(t, u_m, z_m), \mathscr{E}(t, u, z) \leqq E_0$ for sufficiently large m. Then, condition (3.3) implies

$$\left|\frac{1}{h}[\mathscr{E}(t \pm h, u_m, z_m) - \mathscr{E}(t, u_m, z_m)] \mp \partial_t\mathscr{E}(t, u_m, z_m)\right| \leqq \omega_{E_0}(h), \tag{3.14}$$

since the difference quotient can be replaced by a derivative at an intermediate value. The same estimate also holds for $(u_\infty, z_\infty) = (u, z)$.

The lower semicontinuity of $\mathscr{E}(t, \cdot)$ and the assumed convergence at time t imply that

$$\liminf_{m\to\infty} \frac{1}{h}\big(\mathscr{E}(t \pm h, u_m, z_m) - \mathscr{E}(t, u_m, z_m)\big) \geqq \frac{1}{h}\big(\mathscr{E}(t \pm h, u, z) - \mathscr{E}(t, u, z)\big).$$

Combining the case "$-$" with the case "$-$" in (3.14) we find

$$\begin{aligned}
&\limsup_{m\to\infty} \partial_t\mathscr{E}(t, u_m, z_m) \\
&\qquad \leqq \limsup_{m\to\infty} \frac{1}{h}\big(\mathscr{E}(t, u_m, z_m) - \mathscr{E}(t - h, u_m, z_m)\big) + \omega_{E_0}(h) \\
&\qquad = \omega_{E_0}(h) - \liminf_{m\to\infty} \frac{1}{h}\big(\mathscr{E}(t - h, u_m, z_m) - \mathscr{E}(t, u_m, z_m)\big) \\
&\qquad \leqq \omega_{E_0}(h) - \frac{1}{h}\big(\mathscr{E}(t - h, u, z) - \mathscr{E}(t, u, z)\big) \leqq \partial_t\mathscr{E}(t, u, z) + 2\omega_{E_0}(h).
\end{aligned}$$

Similarly, the case "+" gives $\liminf_{m\to\infty} \partial_t\mathscr{E}(t, u_m, z_m) \geqq \partial_t\mathscr{E}(t, u, z) - 2\omega_{E_0}(h)$. Since h can be made arbitrarily small, the result is proved. $\square$

Recalling Step 2 and (3.12), we thus conclude that

$$\theta(t) = \partial_t \mathscr{E}\big(t, u(t), z(t)\big).$$

The upper energy estimate on $[0, t]$ now follows from the discrete upper estimate for (u^{N_k}, z^{N_k}) in Step 1. We use

$$\big(u^{N_k}(t), z^{N_k}(t)\big) = \big(u^{N_k}(\tau_k), u^{N_k}(\tau_k)\big) \quad \text{with } 0 \leqq t - \tau_k \leqq \Delta(\Pi^{N_k}) =: \Delta_k \to 0 \quad \text{for } k \to \infty$$

and obtain, thanks to (3.1), (3.2), and (3.10),

$$\begin{aligned}
&\mathscr{E}\big(t, u^{N_k}(t), z^{N_k}(t)\big) + \mathrm{Diss}_{\mathscr{D}}(z^{N_k}; [0, t]) \\
&\quad \leqq \mathscr{E}\big(\tau_k, u^{N_k}(\tau_k), z^{N_k}(\tau_k)\big) + \mathrm{Diss}_{\mathscr{D}}(z^{N_k}; [0, \tau_k]) + C\Delta_k \\
&\quad \leqq \mathscr{E}\big(0, u(0), z(0)\big) + \int_0^{\tau_k} \theta^{N_k}(s)\,\mathrm{d}s + C\Delta_k \\
&\quad \leqq \mathscr{E}\big(0, u(0), z(0)\big) + \int_0^{t} \theta^{N_k}(s)\,\mathrm{d}s + 2C\Delta_k.
\end{aligned}$$

The weak convergence $\theta^{N_k} \rightharpoonup \theta_*$ from (3.11) and $\delta(t) = \lim_{k\to\infty} \mathrm{Diss}_{\mathscr{D}}(z^{N_k}; [0, t])$ give in the limit

$$\mathscr{E}\big(t, u(t), z(t)\big) + \delta(t) \leqq \mathscr{E}\big(0, u(0), z(0)\big) + \int_0^t \theta_*(s)\,\mathrm{d}s.$$

Using the lower semi-continuity of the dissipation (i.e., $\mathrm{Diss}_{\mathscr{D}}(z, [0, t]) \leqq \delta(t)$) and $\theta_*(t) \leqq \theta(t) = \partial_t \mathscr{E}\big(t, u(t), z(t)\big)$ we obtain the desired upper energy estimate

$$\mathscr{E}\big(t, u(t), z(t)\big) + \mathrm{Diss}_{\mathscr{D}}(z; [0, t]) \leqq \mathscr{E}\big(0, u(0), z(0)\big) + \int_0^t \partial_t \mathscr{E}\big(s, u(s), z(s)\big)\,\mathrm{d}s.$$

Step 5. Lower energy estimate. The lower estimate for the energy balance is a direct consequence of stability, which was observed first in [36] and generalized in [25]. Let $s = \tau_0 < \tau_1 < \cdots < \tau_K = t$ be any partition of $[s, t]$, which is completely independent of the partitions used in Step 2. Then stability of $\big(u(\tau_{j-1}), z(\tau_{j-1})\big)$ under testing with $(\tilde{u}, \tilde{z}) = \big(u(\tau_j), z(\tau_j)\big)$ gives

$$\begin{aligned}
&\mathscr{E}\big(\tau_{j-1}, u(\tau_{j-1}), z(\tau_{j-1})\big) \\
&\quad \leqq \mathscr{E}\big(\tau_{j-1}, u(\tau_j), z(\tau_j)\big) + \mathscr{D}\big(z(\tau_{j-1}), z(\tau_j)\big) \\
&\quad \leqq \mathscr{E}\big(\tau_j, u(\tau_j), z(\tau_j)\big) + \mathscr{D}\big(z(\tau_{j-1}), z(\tau_j)\big) - \int_{\tau_{j-1}}^{\tau_j} \partial_t \mathscr{E}\big(s, u(\tau_j), z(\tau_j)\big)\,\mathrm{d}s.
\end{aligned}$$

After summation over $j = 1, \ldots, K$ we find

$$\begin{aligned}
&\mathscr{E}\big(t,u(t),z(t)\big)+\mathrm{Diss}_{\mathscr{D}}(z,[s,t])-\mathscr{E}\big(s,u(s),z(s)\big)\\
&\quad\geqq\sum_{j=1}^{K}\int_{\tau_{j-1}}^{\tau_j}\partial_t\mathscr{E}\big(s,u(\tau_j),z(\tau_j)\big)\,\mathrm{d}s\\
&\quad=\sum_{j=1}^{K}\partial_t\mathscr{E}\big(\tau_j,u(\tau_j),z(\tau_j)\big)(\tau_j-\tau_{j-1})-\sum_{j=1}^{K}(\tau_j-\tau_{j-1})\rho_j
\end{aligned}$$

with $\rho_j=\frac{1}{\tau_j-\tau_{j-1}}\int_{\tau_{j-1}}^{\tau_j}\big[\partial_t\mathscr{E}\big(s,u(\tau_j),z(\tau_j)\big)-\partial_t\mathscr{E}\big(\tau_j,u(\tau_j),z(\tau_j)\big)\big]\,\mathrm{d}s$. Condition (3.3) gives the uniform bound $|\rho_j|\leqq\omega_{E_*}(\tau_j-\tau_{j-1})\leqq\omega_{E_*}\big(\Delta(\Pi_K)\big)$, which allows us to estimate the last sum by $\omega_{E_*}\big(\Delta(\Pi_K)\big)T\to 0$ for $\Delta_K\to 0$. Using a general result of approximation of Lebesgue integrals via Riemann sums (see [11], Sect. 4.4, or Lemma 4.5 for general Banach-space valued functions) gives in the limit the lower energy estimate

$$\mathscr{E}\big(t,u(t),z(t)\big)+\mathrm{Diss}_{\mathscr{D}}(z,[s,t])-\mathscr{E}\big(s,u(s),z(s)\big)\geqq\int_s^t\partial_t\mathscr{E}\big(\tau,u(\tau)z(\tau)\big)\,\mathrm{d}\tau.$$

Thus, we have shown that $(u,z):[0,T]\to\mathscr{F}\times\mathscr{Z}$ is a solution.

Step 6. Improved convergence. In fact, the lower and upper energy estimate imply, with $e(t)=\mathscr{E}\big(t,u(t),z(t)\big)$,

$$\begin{aligned}
e(0)+\int_0^t\theta(s)\,\mathrm{d}s&\leqq e(t)+\mathrm{Diss}_{\mathscr{D}}(z,[0,t])\leqq e(t)+\delta(t)\\
&\leqq e(0)+\int_0^t\theta_*(s)\,\mathrm{d}s\leqq e(0)+\int_0^t\theta(s)\,\mathrm{d}s.
\end{aligned}$$

Hence, all inequalities are in fact equalities and we conclude $\theta_*=\theta$ a.e. in $[0,T]$ and $\mathrm{Diss}_{\mathscr{D}}(z,[0,t])=\delta(t)$. Applying Lemma 3.5, given at the end of this section, we conclude from $\theta^{N_k}\rightharpoonup\theta_*=\theta$ that in fact $\theta^{N_k}\to\theta$ in $\mathrm{L}^1\big((0,T);\mathbb{R}\big)$. Thus, after choosing a further subsequence $n_l=N_{k_l}$ we also have the following convergences:

(i) for all $t\in[0,T]:z^{n_l}(t)\rightharpoonup z(t)$ in X_2 for $l\to\infty$;

(ii) for all $t\in[0,T]:\mathscr{E}\big(t,u^{n_l}(t),z^{n_l}(t)\big)\to\mathscr{E}\big(t,u(t),z(t)\big)$ for $l\to\infty$;

(iii) for all $t\in[0,T]:\mathrm{Diss}_{\mathscr{D}}(z^{n_l},[0,t])\to\mathrm{Diss}_{\mathscr{D}}(z,[0,t])$ for $l\to\infty$;

(iv) for a.e. $t\in[0,T]:\theta^{n_l}(t)\to\theta(t)$ for $l\to\infty$.

The latter convergence does not require the further t-dependent subsequences $\big(\theta^{N_l^t}(t)\big)$ for which convergence was initially obtained.

Thus, we have proved the following abstract existence result.

Theorem 3.4. *Let the Banach spaces X_1 and X_2 be reflexive and such that X_2 is compactly embedded into the Banach space X_3. $\mathscr{F}\subset X_1$ and $\mathscr{Z}\subset X_2$ are weakly closed subsets. For all $t\in[0,T]$ the energy functional $\mathscr{E}(t,\cdot):\mathscr{F}\times\mathscr{Z}\to\mathbb{R}_\infty$ has weakly compact* *sublevels* $\{(u,z) \mid \mathscr{E}(t,u,z) \leqq E_*\} \subset X_1 \times X_2$ *and satisfies* (3.1) *and* (3.3). *The dissipation* $\mathscr{D} : X_3 \times X_3 \to [0,\infty)$ *satisfies the triangle inequality* (3.4), *is continuous and* X_3*-coercive* (*see* (3.8)).

Then, for each $(u_0, z_0) \in \mathscr{S}(0)$ *there exists an energetic solution* $(u,z) : [0,T] \to \mathscr{F} \times \mathscr{Z}$ *of* (S) *and* (E) *with* $\big(u(0), z(0)\big) = (u_0, z_0)$ *and*

$$u \in \mathrm{B}([0,T], X_1) \quad \text{and} \quad z \in \mathrm{L}^\infty([0,T], X_2) \cap \mathrm{BV}([0,T], X_3).$$

Here we do not state that $u : [0,T] \to X_1$ is measurable. However, it should be possible to adapt the method in [24], where techniques from set-valued analysis are used to find a measurable selection.

We finally provide the lemma which was used in Step 6.

Lemma 3.5. *Let* $(f_k)_{k\in\mathbb{N}}$ *be a bounded sequence in* $\mathrm{L}^\infty\big((0,T);\mathbb{R}\big)$ *with* $f_k \overset{*}{\rightharpoonup} f_*$ *and* $f_{\sup}(t) = \limsup_{k\to\infty} f_k(t)$. *If* $f_* = f_{\sup}$ *a.e. in* $(0,T)$, *then* $\|f_k - f_*\|_{\mathrm{L}^1((0,T);\mathbb{R})} \to 0$.

Proof. Without loss of generality we may assume $f_{\sup} = f_* \equiv 0$. Letting $g_n(t) = \sup\{f_k(t) \mid k \geqq n\}$ we have $g_n(t) \searrow f_{\sup}(t) = 0$ for $n \to \infty$. For $f_k^+ := \max\{0, f_k\}$ this implies $0 \leqq f_k^+ \leqq g_k \searrow 0$ and hence $\|f_k^+\|_{\mathrm{L}^1((0,T);\mathbb{R})} \to 0$.

With the trivial identity $|f_k| = -f_k + 2f_k^+$ we find

$$\|f_k\|_{\mathrm{L}^1((0,T);\mathbb{R})} = -\int_0^T f_k(t)\,\mathrm{d}t + 2\|f_k^+\|_{\mathrm{L}^1((0,T);\mathbb{R})} \to 0$$

since the first integral converges to $0 = \int_0^T f_*\,\mathrm{d}t$. □

Unfortunately, we cannot conclude $f_k(t) \to f_*(t)$ for a.a. $t \in (0,T)$ which is seen by the simple counterexample $f_k = -\chi_{(a_k,b_k)}$ on $(0,2)$ where $a_k = \sqrt{k} \bmod 1$ and $b_k = a_k + 1/k$. Obviously, $f_k \to 0$ in $\mathrm{L}^1\big((0,2);\mathbb{R}\big)$ and $f_{\sup} = 0$, but $\liminf_{k\to\infty} f_k(t) = -1$ for $t \in (0,1)$. However, there exists a subsequence such that a.e. on $(0,2)$ we have $f_{k_l}(t) \to 0$ for $l \to \infty$.

4. Energy densities with restricted growth

Consider an elastic body occupying a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary and denote by $\varphi : \Omega \to \mathbb{R}^d$ the elastic deformation field. The phase variable z belongs a.e. to a compact subset Z of $\mathbb{R}^m$. For example, in the case of a two-phase process one chooses $Z := [0,1]$ (cf. [36]); for magnetism the appropriate choice is $Z = \mathbb{S}^{d-1}$ (cf. [31], Sect. 7.4]). We are not concerned in this study with the most general loading process, but will demonstrate the method for a reasonably wide variety of loads consisting, in the notation of Section 2, of a time-varying load $l(t)$ and of a time-varying boundary displacement load $\varphi_{\mathrm{Dir}}(t)$ on the part Γ_{Dir} of the boundary of Ω. Throughout we assume that $\varphi_{\mathrm{Dir}}(t)$ is extended to all of $\bar{\Omega}$. Specifically, we assume the following regularity on the data:

$$(4.1) \qquad l \in \mathrm{W}^{1,1}\big(0,T;\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)^*\big) \quad \text{and} \quad \varphi_{\mathrm{Dir}} \in \mathrm{W}^{1,1}\big(0,T;\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)\big),$$

where $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)^*$ is the dual of $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$. Since every $\varphi \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ can be identified with a pair $(\varphi_0, \varphi_{\partial\Omega}) \in \mathrm{W}_0^{1,p}(\Omega;\mathbb{R}^d) \times \mathrm{W}^{1-\frac{1}{p},p}(\partial\Omega;\mathbb{R}^d)$, we may identify $l \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)^*$ with a pair

$$(f,g) \in \mathrm{W}^{-1,p'}(\Omega;\mathbb{R}^d) \times \mathrm{W}^{-\frac{1}{p'},p'}(\partial\Omega;\mathbb{R}^d), \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1,$$

by using the duality pairing (cf. also (2.4))

$$\langle l, \varphi \rangle := \langle f, \varphi_0 \rangle_{\mathrm{W}^{-1,p'},\mathrm{W}_0^{1,p}} + \langle g, \varphi_{\partial\Omega} \rangle_{\mathrm{W}^{-\frac{1}{p'},p'},\mathrm{W}^{\frac{1}{p'},p}}.$$

We define the stored energy $\tilde{\mathscr{E}}$ as

$$(4.2) \qquad \tilde{\mathscr{E}}(t,\varphi,z) = \int_\Omega W\big(x,\nabla\varphi(x),z(x)\big) + \frac{\sigma}{\alpha}|\nabla z(x)|^\alpha \,\mathrm{d}x - \langle l(t), \varphi(t) \rangle,$$

with $1 < \alpha < \infty$ and $\sigma > 0$. (The case $\alpha = 1$, where the space X_2 is $\mathrm{BV}(\Omega)$, is treated in [24].) The kinematically admissible deformation fields at time t lie in

$$\mathscr{F}(t) := \{\varphi \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^d) \,|\, \varphi = \varphi_{\mathrm{Dir}}(t) \text{ on } \Gamma_{\mathrm{Dir}}\},$$

while the admissible phases at time t lie in

$$\mathscr{Z} = \{z \in \mathrm{W}^{1,\alpha}(\Omega;\mathbb{R}^m) \,|\, z(x) \in Z \text{ for a.e. } x \in \Omega\}.$$

The stored-energy density $W : \mathbb{R}^{d\times d} \times Z \mapsto \mathbb{R}$ is assumed to be continuous and C^1 in its first variable and to further satisfy, for some $0 < c < C < \infty$, for some $1 < p < \infty$, all $F \in \mathbb{R}^{d\times d}$, $z \in Z$,

$$(4.3) \qquad \begin{array}{ll} \forall (F,z) \in \mathbb{R}^{d\times d} \times Z\colon & c|F|^p - C \leqq W(F,z) \leqq C(1+|F|^p), \\ \forall z \in Z\colon & W(\cdot\,,z) : \mathbb{R}^{d\times d} \to [0,\infty[\text{ is quasiconvex.} \end{array}$$

Note that quasiconvexity implies separate convexity; the following estimate holds true for some constant C_p depending only upon p, see [10]:

$$(4.4) \qquad |\mathrm{D}W(F,z)| \leqq C_p(1+|F|^{p-1}).$$

In addition, we introduce the dissipation as a continuous function $\hat{d} : Z \times Z \to [0,\infty[$ that satisfies the coercivity estimate

$$(4.5) \qquad \frac{1}{C}|z_1 - z_0| \leqq \hat{d}(z_0,z_1) \leqq C|z_1 - z_0| \quad \text{for all } z_0, z_1 \in Z,$$

and the triangle inequality

$$\hat{d}(z_0,z_2) \leqq \hat{d}(z_0,z_1) + \hat{d}(z_1,z_2) \quad \text{for all } z_0, z_1, z_2 \in Z.$$

We allow for the unsymmetry $\hat{d}(z_0, z_1) \neq \hat{d}(z_1, z_0)$, so that $\hat{d}$ could be called an unsymmetric metric. The global dissipation distance $\mathscr{D}$ now defines the following unsymmetric metric on $L^1(\Omega, Z)$:

$$\mathscr{D}(z_0, z_1) = \int_\Omega \hat{d}\big(z_0(x), z_1(x)\big)\,dx,$$

which, according to (4.5), further satisfies

$$\frac{1}{C}\|z_1 - z_0\|_{X_3} \leqq \mathscr{D}(z_0, z_1) \leqq C\|z_1 - z_0\|_{X_3} \quad \text{for all } z_0, z_1 \in \mathscr{Z} \subset X_2, \tag{4.6}$$

where, in the notation of Section 3,

$$X_2 := W^{1,\alpha}(\Omega; \mathbb{R}^m), \quad X_3 := L^1(\Omega; \mathbb{R}^m).$$

This is clearly satisfied in the special case of Section 2.

The dissipation along a path $z : [0, T] \to \mathscr{Z}$ is then defined as the following total variation:

$$\mathrm{Diss}_{\mathscr{D}}(z; [s, t]) = \sup\left\{ \sum_{j=1}^{N} \mathscr{D}\big(z(\tau_{j-1}), z(\tau_j)\big) \,\middle|\, N \in \mathbb{N}, s = \tau_0 < \tau_1 < \cdots < \tau_N \leqq t \right\}.$$

In this section, we establish the following existence theorem for a phase transformation evolution:

Theorem 4.1. *Given* $(\varphi_0, z_0) \in \mathscr{F}(0) \times \mathscr{Z}$ *which minimizes*

$$(\varphi, z) \mapsto \tilde{\mathscr{E}}(0, \varphi, z) + \mathscr{D}(z_0, z) \quad \text{on } \mathscr{F}(0) \times \mathscr{Z},$$

there exists a function $(\varphi, z) : [0, T] \to \mathscr{F}(t) \times \mathscr{Z}$ *satisfying* $\big(\varphi(0), z(0)\big) = (\varphi_0, z_0)$ *and such that, for all* $t \in [0, T]$, *the conditions* $(\mathrm{S})_t$ *and* $(\mathrm{E})_t$ *below hold true*:

$(\mathrm{S})_t$ *Stability*: *For all* $(\tilde{\varphi}, \tilde{z}) \in \mathscr{F}(t) \times \mathscr{Z}$:

$$\tilde{\mathscr{E}}\big(t, \varphi(t), z(t)\big) \leqq \tilde{\mathscr{E}}(t, \tilde{\varphi}, \tilde{z}) + \mathscr{D}\big(z(t), \tilde{z}\big).$$

$(\mathrm{E})_t$ *Energy conservation*:

$$\begin{aligned} &\tilde{\mathscr{E}}\big(t, \varphi(t), z(t)\big) + \mathrm{Diss}_{\mathscr{D}}(z; [0, t]) \\ &\quad = \tilde{\mathscr{E}}(0, \varphi_0, z_0) + \int_0^t \bigg[\int_{\partial\Omega} \mathrm{D}W\big(\nabla\varphi(s, x)\big) n . \dot{\varphi}_{\mathrm{Dir}}(s, x)\, d\mathscr{H}^{N-1} \\ &\qquad\qquad - \langle \dot{l}(s), \varphi(s) \rangle - \langle l(s), \dot{\varphi}_{\mathrm{Dir}}(s) \rangle \bigg]\, ds. \end{aligned}$$

Furthermore, $z \in \mathrm{BV}(0, T; X_3)$.

In the theorem above it is implicit that the integrands in the right-hand side of the energy conservation identity are integrable in time, although $t \mapsto \varphi(t) \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^d)$ might not a priori be even measurable.

Remark 4.2. A simple variation permits to recover (2.1) from $(\mathrm{S})_t$, and also (2.3), provided that $z(t)$ is smooth enough in time. Under smoothness of z in time, we thus conclude the existence of a solution for that evolution problem. Further, thanks to $(\mathrm{E})_t$, that evolution satisfies the weak form (2.5) of the energy conservation, in spite of the possible lack of regularity of the field $\varphi(t)$.

We define

$$\mathscr{F} := X_1 := \mathrm{W}_0^{1,p}(\Omega; \mathbb{R}^d),$$

and translate φ by φ_{Dir} by introducing $u = \varphi - \varphi_{\mathrm{Dir}}(t) \in \mathscr{F}$. Moreover, we set

$$(4.7) \qquad \mathscr{E}(t,u,z) = \tilde{\mathscr{E}}\big(t, u + \varphi_{\mathrm{Dir}}(t), z\big) = \int_\Omega W\big(\nabla u(x) + \nabla \varphi_{\mathrm{Dir}}(t,x), z(x)\big)\,\mathrm{d}x + \frac{\sigma}{\alpha}\int_\Omega |\nabla z|^\alpha\,\mathrm{d}x - \langle l(t), u + \varphi_{\mathrm{Dir}}(t)\rangle.$$

The method for proving the evolution theorem follows the abstract framework developed in Section 3. As mentioned in the introduction, the proof puts the new ideas of a similar existence result for quasi-static brittle fracture evolution [11], [14] into the more general context of abstract rate-independent models, as studied in [25]. Note, however, that in this section the abstract assumptions (3.1) and (3.3) do not hold, since we have chosen (4.1) quite generally and have not imposed any kind of uniform continuity on $\mathrm{D}W$. Thus we have to repeat the main steps of the proof.

Consider partitions $\Pi_n := \{0 = t_0^n < t_1^n < \cdots < t_{k(n)}^n = T\}$ of the interval $[0,T]$ with fineness $\Delta_n := \max_{i=1,\ldots,k(n)} (t_i^n - t_{i-1}^n)$. The associated time-incremental minimization problems are, for $i \geqq 1$,

$(\mathrm{IP})_{n,i}$ Find (u_i^n, z_i^n) which minimizes $\mathscr{E}(t_i^n, u, z) + \mathscr{D}(z_{i-1}^n, z)$ over $\mathscr{F} \times \mathscr{Z}$,

with $z_0^n := z_0$ and $u_0^n = \varphi_0 - \varphi_{\mathrm{Dir}}(0)$. Existence of solutions to $(\mathrm{IP})_{n,i}$ immediately follows from classical coercivity and weak lower semicontinuity arguments.

Lemma 4.3. *$\mathscr{E}(t,\cdot,\cdot) : \mathscr{F} \times \mathscr{Z} \to \mathbb{R}$ is weakly lower semicontinuous and coercive, while $\mathscr{D} : \mathscr{Z} \times \mathscr{Z} \to [0,\infty[$ is weakly continuous and* L^1*-coercive.*

Proof. The result is classical (cf. [1]), since W is quasiconvex in its first argument and continuous in its second argument, while Z is closed, hence stable under strong L^1-convergence. Also, u_0^n and z_0^n are actually independent of n and equal, by definition, to $u(0)$ and $z(0)$, respectively. □

Define the piecewise constant functions $(u^n, z^n) : [0,T] \to \mathscr{Z} \times \mathscr{Z}$ via

$$\big(u^n(t), z^n(t)\big) := (u_i^n, z_i^n) \quad \text{for } t \in [t_i^n, t_{i+1}^n) \text{ and } \big(u^n(T), z^n(T)\big) := (u_n^n, z_n^n).$$

Step 1. A priori estimates. A priori estimates on $u^n(t)$, $z^n(t)$ are obtained as in [11], [25], [34], [35], [36]. To this effect, we first test $(\mathrm{IP})_{n,i}$ with $(0,0)$ and obtain, for some constant $C > 0$,

$$\mathscr{E}(t_i^n, u_i^n, z_i^n) + \mathscr{D}(z_{i-1}^n, z_i^n) \leqq C\big(1 + \|\nabla(\varphi_{\mathrm{Dir}})_i^n\|_{\mathrm{L}^p}^p\big) + \|l_i^n\|_{[\mathrm{W}^{1,p}]*}\|(\varphi_{\mathrm{Dir}})_i^n\|_{\mathrm{W}^{1,p}},$$

where $l_i^n := l(t_i^n)$, $(\varphi_{\mathrm{Dir}})_i^n := \varphi_{\mathrm{Dir}}(t_i^n)$.

Then, in view of (4.1), the above estimate yields

$$(4.8) \qquad (u^n)_n \text{ is bounded in } \mathrm{L}^\infty(0,T;X_1) \quad \text{and} \quad (z^n)_n \text{ is bounded in } \mathrm{L}^\infty(0,T;X_2).$$

Next, we test, for $i \geqq 1$, $(\mathrm{IP})_{n,i}$ with (u_{i-1}^n, z_{i-1}^n); since $\Phi \mapsto \int_\Omega W(\Phi, z_{i-1}^n)\,\mathrm{d}x$ is a C^1-map from $\mathrm{L}^p(\Omega;\mathbb{R}^{d\times d})$ into $\mathbb{R}$ with differential $\Psi \mapsto \int_\Omega \mathrm{D}W(\Phi, z_{i-1}^n).\Psi\,\mathrm{d}x$, we obtain

$$\begin{aligned}\mathscr{E}(t_i^n, u_i^n, z_i^n) + \mathscr{D}(z_{i-1}^n, z_i^n) &\leqq \mathscr{E}(t_i^n, u_{i-1}^n, z_{i-1}^n)\\ &= \mathscr{E}(t_{i-1}^n, u_{i-1}^n, z_{i-1}^n) + \big(\mathscr{E}(t_i^n, u_{i-1}^n, z_{i-1}^n) - \mathscr{E}(t_{i-1}^n, u_{i-1}^n, z_{i-1}^n)\big)\\ &= \mathscr{E}(t_{i-1}^n, u_{i-1}^n, z_{i-1}^n) + \int_{t_{i-1}^n}^{t_i^n} \partial_t\mathscr{E}(s, u_{i-1}^n, z_{i-1}^n)\,\mathrm{d}s\end{aligned}$$

with

$$(4.9) \qquad \begin{aligned}\partial_t\mathscr{E}(t,u,z) = {}&\int_\Omega \mathrm{D}W\big(\nabla[u + \varphi_{\mathrm{Dir}}(t)], z\big).\nabla\dot\varphi_{\mathrm{Dir}}(t)\,\mathrm{d}x\\ &- \langle \dot l(t), u + \varphi_{\mathrm{Dir}}(t)\rangle - \langle l(t), \dot\varphi_{\mathrm{Dir}}(t)\rangle.\end{aligned}$$

Summing the previous inequality for $i = 1, \ldots, k$, we obtain, for

$$\tau^n(t) := t_k^n \leqq t < t_{k+1}^n,$$

$$(4.10) \qquad \begin{aligned}&\mathscr{E}\big(\tau^n(t), u^n(t), z^n(t)\big) + \mathrm{Diss}(z^n, [0,t])\\ &\quad \leqq \mathscr{E}\big(0, u(0), z(0)\big) + \int_0^{\tau^n(t)} \partial_t\mathscr{E}\big(s, u^n(s), z^n(s)\big)\,\mathrm{d}s.\end{aligned}$$

Here $\tau^n(t)$ denotes the greatest time in Π_n below t.

In view of (4.1), the bound from below (4.6) on $\mathscr{D}$ and (4.8), inspection of (4.9) yields that

$$(4.11) \qquad \big|\partial_t\mathscr{E}\big(t, u^n(t), z^n(t)\big)\big| \leqq h(t) \quad \text{with } h \in L^1(0,T),$$

so that (4.10) implies in particular that

$$z^n \text{ is bounded in } \mathrm{BV}(0,T;X_3),$$

hence, with (4.8), that

$$\text{(4.12)} \qquad z^n \text{ is bounded in } \mathrm{BV}\big((0,T)\times\Omega;\mathbb{R}^d\big).$$

Step 2. Selection of subsequences. By compactness of BV into L^1, we deduce from (4.12) the existence of a limit phase $z \in \mathrm{BV}\big((0,T)\times\Omega;\mathbb{R}^d\big) \cap L^1(0,T;\mathscr{Z})$ such that, for a *t-independent* subsequence of $\{n\}$ still denoted by $\{n\}$,

$$\text{(4.13)} \qquad \begin{cases} z^n(t) \to z(t) & \text{strongly in } X_3 \text{ for a.e. } t \in [0,T], \\ z^n \to z & \text{strongly in } L^1(0,T;X_3). \end{cases}$$

Further, thanks to (4.8), we are at liberty to assume that $z \in L^\infty(0,T;X_2)$ and that

$$\text{(4.14)} \qquad z^n(t) \rightharpoonup z(t) \quad \text{weakly in } X_2 \quad \text{for a.e. } t \in [0,T].$$

As in Section 3, we now set for a.e. $t \in [0,T]$

$$\text{(4.15)} \qquad \theta^n(t) := \partial_t \mathscr{E}\big(t, u^n(t), z^n(t)\big) \quad \text{and} \quad \theta(t) := \limsup_{n\to\infty} \theta^n(t).$$

In view of (4.11), Fatou's lemma immediately implies that $\theta \in L^1(0,T)$ and that

$$\text{(4.16)} \qquad \limsup_{n\to\infty} \int_0^{\tau^n(t)} \theta^n(s)\,\mathrm{d}s \leqq \int_0^t \theta(s)\,\mathrm{d}s.$$

Furthermore, we are at liberty to extract, for a.e. $t \in [0,T]$, a t-dependent subsequence of $\{n\}$, denoted by $\{n_t\}$, such that

$$\text{(4.17)} \qquad \theta(t) = \lim_{n_t} \theta^{n_t}(t) = \lim_{n_t}\bigg(\int_\Omega \mathrm{D}W\big(\nabla u^{n_t}(t) + \nabla\varphi_{\mathrm{Dir}}(t), z^{n_t}(t)\big).\nabla\dot{\varphi}_{\mathrm{Dir}}(t)\,\mathrm{d}x \\ - \langle l(t), \dot{\varphi}_{\mathrm{Dir}}(t)\rangle - \langle \dot{l}(t), u^{n_t}(t) + \varphi_{\mathrm{Dir}}(t)\rangle\bigg).$$

In turn, (4.8) finally yields another *t-dependent* subsequence u^{n_t} (still indexed by n_t), such that

$$\text{(4.18)} \qquad u^{n_t}(t) \rightharpoonup u(t) \quad \text{weakly in } X_1.$$

It is not clear that $u(t)$ so defined enjoys any kind of measurability property in t.

Step 3. Stability of the limit function. First, we check the stability of the pair $\big(u(t), z(t)\big)$ defined through (4.18), (4.13). To this effect, recalling $(\mathrm{IP})_{n_t i}$, we first note that, in view of the triangle inequality for $\hat{d}$, for any $(v,\zeta) \in \mathscr{F}\times\mathscr{Z}$,

$$\mathscr{E}\big(\tau^{n_t}(t), u^{n_t}(t), z^{n_t}(t)\big) + \mathscr{D}\big(z^{n_t}\big(\tau^{n_t}(t) - \Delta_{n_t}\big), z^{n_t}(t)\big) \\ \leqq \mathscr{E}\big(\tau^{n_t}(t), v, \zeta\big) + \mathscr{D}\big(z^{n_t}(t), \zeta\big) + \mathscr{D}\big(z^{n_t}\big(\tau^{n_t}(t) - \Delta_{n_t}\big), z^{n_t}(t)\big),$$

hence

$$(4.19) \qquad \mathscr{E}\big(\tau^{n_t}(t), u^{n_t}(t), z^{n_t}(t)\big) \leqq \mathscr{E}\big(\tau^{n_t}(t), v, \zeta\big) + \mathscr{D}\big(z^{n_t}(t), \zeta\big).$$

The regularity assumptions on φ_{Dir} in (4.1) imply that, for all $t \in [0, T]$, we have $\varphi_{\mathrm{Dir}}\big(\tau^n(t)\big) \to \varphi_{\mathrm{Dir}}(t)$ strongly in $\mathrm{W}^{1,p}(\Omega; \mathbb{R}^d)$ and $l\big(\tau^n(t)\big) \to l(t)$ strongly in $\mathrm{W}^{1,p}(\Omega; \mathbb{R}^d)^*$. Using sequential weak lower-semicontinuity, together with (4.13), (4.14), (4.18), it is then straightforward to pass to the limit in the left-hand side of the above inequality. We obtain

$$(4.20) \qquad \mathscr{E}\big(t, u(t), z(t)\big) \leqq \liminf_{n_t} \mathscr{E}\big(\tau^{n_t}(t), u^{n_t}(t), z^{n_t}(t)\big).$$

Recalling that W is continuous in its arguments and has p-growth allows us to pass to the limit in the first term in the right-hand side of inequality (4.19). The limit in the second term is obtained from (4.13) and from the growth assumption (4.6) on $\mathscr{D}$. We thus conclude that $\big(u(t), z(t)\big)$ satisfy the stability principle $(\mathrm{S})_t$.

Step 4. Upper energy estimates. We were now at liberty to test $(\mathrm{IP})_{n_t i}$ with $\big(u(t), z(t)\big)$. Computing the limit of the right-hand side in (4.19) as before, we obtain $\limsup_{n_t} \mathscr{E}\big(\tau^{n_t}(t), u^{n_t}(t), z^{n_t}(t)\big) \leqq \mathscr{E}\big(t, u(t), z(t)\big)$. Hence, with (4.20), we find

$$(4.21) \qquad \lim_{n_t} \mathscr{E}\big(\tau^{n_t}(t), u^{n_t}(t), z^{n_t}(t)\big) = \mathscr{E}\big(t, u(t), z(t)\big).$$

The following result provides exactly the same result as Proposition 3.3 in the abstract part. However, in this version the assumptions are quite different. Instead of using the uniform continuity of $\partial_t \mathscr{E}$ on sublevels, we use that the Gateaux derivative $\mathrm{D}_u \mathscr{E}$ exists and satisfies a certain uniform continuity property in balls of $\mathscr{F} \times \mathscr{Z}$. Here we follow closely the arguments in [11], Lemma 4.11.

Proposition 4.4. *Under assumptions* (4.1), (4.3), *we have*

$$(4.22) \qquad \left.\begin{aligned} &(u^m, z^m) \rightharpoonup (u, z) \text{ in } X_1 \times X_2 \\ &\text{and } \mathscr{E}(t, u^m, z^m) \to \mathscr{E}(t, u, z) \end{aligned}\right\} \Rightarrow \begin{cases} \mathrm{D}W\big(\nabla u^m + \nabla\varphi_{\mathrm{Dir}}(t), z^m\big) \\ \rightharpoonup \mathrm{D}W\big(\nabla u + \nabla\varphi_{\mathrm{Dir}}(t), z\big) \\ \text{weakly in } \mathrm{L}^{p'}(\Omega; \mathbb{R}^d). \end{cases}$$

Proof. It is enough to prove that

$$(4.23) \qquad \int_\Omega \mathrm{D}W\big(\nabla u(t) + \nabla\varphi_{\mathrm{Dir}}(t), z(t)\big).\Psi \,\mathrm{d}x \leqq \liminf_m \int_\Omega \mathrm{D}W\big(\nabla u^m(t) + \nabla\varphi_{\mathrm{Dir}}(t), z^m\big).\Psi \,\mathrm{d}x,$$

for every $\Psi \in \mathrm{L}^p(\Omega; \mathbb{R}^{d\times d})$. Let η be a sequence of positive numbers converging to 0. The sequential weak lower semi-continuity of

$$X_1 \times X_2 \ni (v, \zeta) \mapsto \int_\Omega W\big(\nabla v + \nabla\varphi_{\mathrm{Dir}}(x, t), \zeta\big)\,\mathrm{d}x + \frac{\sigma}{\alpha}\int_\Omega |\zeta|^\alpha \,\mathrm{d}x$$

(see e.g. Lemma 4.3), the assumed convergence of the energy and the continuity of the term $v \mapsto \langle l(t), v + \varphi_{\mathrm{Dir}}(t)\rangle$ easily yield

$$\frac{1}{\eta}\int_\Omega \big(W\big(\nabla u+\nabla\varphi_{\mathrm{Dir}}(t)+\eta\Psi,z\big)-W\big(\nabla u+\varphi_{\mathrm{Dir}}(t),z\big)\big)\,\mathrm{d}x$$

$$\leqq \liminf_m \frac{1}{\eta}\int_\Omega \big(W\big(\nabla u^m+\nabla\varphi_{\mathrm{Dir}}(t)+\eta\Psi,z^m\big)-W\big(\nabla u^m+\nabla\varphi_{\mathrm{Dir}}(t),z^m\big)\big)\,\mathrm{d}x.$$

Thus, there exists a sequence $\varepsilon_m \searrow 0$, such that

$$\lim_m \frac{1}{\varepsilon_m}\int_\Omega \big(W\big(\nabla u+\nabla\varphi_{\mathrm{Dir}}(t)+\varepsilon_m\Psi,z\big)-W\big(\nabla u+\varphi_{\mathrm{Dir}}(t),z\big)\big)\,\mathrm{d}x$$

$$\leqq \liminf_m \frac{1}{\varepsilon_m}\int_\Omega \big(W\big(\nabla u^m+\nabla\varphi_{\mathrm{Dir}}(t)+\varepsilon_m\Psi,z^m\big)-W\big(\nabla u^m+\nabla\varphi_{\mathrm{Dir}}(t),z^m\big)\big)\,\mathrm{d}x.$$

Since $W(F,z)$ is of class C^1 with respect to F, we have

$$\int_\Omega \mathrm{D}W\big(\nabla u+\nabla\varphi_{\mathrm{Dir}}(t),z\big).\Psi\,\mathrm{d}x$$

$$=\lim_{\varepsilon_m}\frac{1}{\varepsilon_m}\int_\Omega \big(W\big(\nabla u+\nabla\varphi_{\mathrm{Dir}}(t)+\varepsilon_m\Psi,z\big)-W\big(\nabla u(t)+\nabla\varphi_{\mathrm{Dir}}(t),z\big)\big)\,\mathrm{d}x.$$

In addition, for some $\tau_m \in [0,\varepsilon_m]$ we find

$$\frac{1}{\varepsilon_m}\int_\Omega \big(W\big(\nabla u^m+\nabla\varphi_{\mathrm{Dir}}(t)+\varepsilon_m\Psi,z^m(t)\big)-W\big(\nabla u^m+\nabla\varphi_{\mathrm{Dir}}(t),z^m\big)\big)\,\mathrm{d}x$$

$$=\int_\Omega \mathrm{D}W\big(\nabla u^m(t)+\nabla\varphi_{\mathrm{Dir}}(t)+\tau_m\Psi,z^m\big).\Psi\,\mathrm{d}x.$$

Thus, we obtain that

$$\int_\Omega \mathrm{D}W\big(\nabla u+\nabla\varphi_{\mathrm{Dir}}(t),z\big).\Psi\,\mathrm{d}x \leqq \liminf_m \int_\Omega \mathrm{D}W\big(\nabla u^m+\nabla\varphi_{\mathrm{Dir}}(t)+\tau_m\Psi,z^m\big).\Psi\,\mathrm{d}x.$$

We now apply [11], Lemma 4.9. That lemma, which uses a simple argument based on the uniform continuity of $\mathrm{D}W$ on compact sets, together with the already uniform bounds on ∇u^m and z^m (following from the weak convergence) permits us to drop the term $\tau_m\Psi$ from the integrand in the previous inequality. This establishes (4.23), and (4.22) follows since Ψ was arbitrary. □

In view of (4.22) and of the continuous character of the remaining terms in the expression (4.17) for $\theta^n(t)$, we obtain

$$\theta(t)=\partial_t\mathscr{E}\big(t,u(t),z(t)\big)=\int_\Omega \mathrm{D}W\big(\nabla u(t)+\nabla\varphi_{\mathrm{Dir}}(t),z(t)\big).\nabla\dot\varphi_{\mathrm{Dir}}(t)\,\mathrm{d}x$$
$$-\langle \dot l(t),u(t)+\varphi_{\mathrm{Dir}}(t)\rangle-\langle l(t),\dot\varphi_{\mathrm{Dir}}(t)\rangle. \tag{4.24}$$

We now address energy conservation. Recall (4.10), (4.15). In view of (4.16), (4.24), the lim sup of the right-hand side of (4.10) is less than

$$\mathscr{E}\big(0,u(0),z(0)\big)+\int_0^t\int_\Omega\big(\mathrm{D}W\big(\nabla u(s)+\nabla\varphi_{\mathrm{Dir}}(s),z(s)\big).\nabla\dot\varphi_{\mathrm{Dir}}(s)\,\mathrm{d}x$$
$$-\langle l(s),\dot\varphi_{\mathrm{Dir}}(s)\rangle-\langle\dot l(s),u(s)+\varphi_{\mathrm{Dir}}(s)\rangle\big)\,\mathrm{d}s.$$

As for the left-hand side of (4.10) written for $n=n_t$, we already know, by virtue of (4.20), how to bound from below the lim inf of the first term. But, in view of the continuity and boundedness properties (4.5) of $\hat d$, the total variation is sequentially weakly lower semi-continuous for weak-$\star$ convergence in $\mathrm{BV}(0,T;X_3)$, so that

$$\mathrm{Diss}(z;[0,t])\leqq\liminf_{n_t}\ \mathrm{Diss}(z^{n_t},[0,t]).$$

Finally, we obtain the following upper bound on the sum of the energy and the dissipation at t:

$$\tag{4.25}\begin{aligned}&\mathscr{E}\big(t,u(t),z(t)\big)+\mathrm{Diss}(z;[0,t])\\&\leqq\mathscr{E}\big(0,u(0),z(0)\big)+\int_0^t\bigg(\int_\Omega\mathrm{D}W\big(\nabla u(s)+\nabla\varphi_{\mathrm{Dir}}(s),z(s)\big).\nabla\dot\varphi_{\mathrm{Dir}}(s)\,\mathrm{d}x\\&\qquad-\langle l(s),\dot\varphi_{\mathrm{Dir}}(s)\rangle-\langle\dot l(s),u(s)+\varphi_{\mathrm{Dir}}(s)\rangle\bigg)\,\mathrm{d}s.\end{aligned}$$

Step 5. Lower energy estimates. The lower bound on that sum is obtained from the stability criterion $(\mathrm{S})_t$ as follows. For $s<t$, test $(\mathrm{S})_s$ by $\big(u(t),z(t)\big)$; we get

$$\tag{4.26}\mathscr{E}\big(s,u(s),z(s)\big)\leqq\mathscr{E}\big(s,u(t),z(t)\big)+\mathscr{D}\big(z(s),z(t)\big).$$

Since

$$\mathrm{Diss}(z;[0,s])+\mathscr{D}\big(z(s),z(t)\big)\leqq\mathrm{Diss}(z;[0,t]),$$

(4.26) is immediately seen to imply that, for some $\rho(s,t)\in[s,t]$,

$$\tag{4.27}\begin{aligned}&\mathscr{E}\big(t,u(t),z(t)\big)+\mathrm{Diss}(z;[0,t])-\big\{\mathscr{E}\big(s,u(s),z(s)\big)+\mathrm{Diss}(z;[0,s])\big\}\\&\geqq\mathscr{E}\big(t,u(t),z(t)\big)-\mathscr{E}\big(s,u(t),z(t)\big)\\&=\int_\Omega\mathrm{DW}\bigg(\nabla u(t)+\nabla\varphi_{\mathrm{Dir}}(t)-\rho(s,t)\int_s^t\nabla\dot\varphi_{\mathrm{Dir}}(\sigma)\,\mathrm{d}\sigma,z(t)\bigg)\\&\quad\cdot\bigg(\int_s^t\nabla\dot\varphi_{\mathrm{Dir}}(\sigma)\,\mathrm{d}\sigma\bigg)\,\mathrm{d}x-\bigg\langle l(t),\int_s^t\dot\varphi_{\mathrm{Dir}}(s)(\sigma)\,\mathrm{d}\sigma\bigg\rangle\\&\quad-\bigg\langle\int_s^t\dot l(\sigma)\,\mathrm{d}\sigma,u(t)+\varphi_{\mathrm{Dir}}(t)\bigg\rangle+\bigg\langle\int_s^t\dot l(\sigma)\,\mathrm{d}\sigma,\int_s^t\dot\varphi_{\mathrm{Dir}}(\sigma)\,\mathrm{d}\sigma\bigg\rangle.\end{aligned}$$

Consider a partition $0:=s_0^n\leqq s_1^n\leqq\cdots\leqq s_{k(n)}^n=t$ such that

$$\tag{4.28}\lim_{n\to\infty}S_n=0,\quad\text{where }S_n:=\max_{1\leqq i\leqq k(n)}(s_i^n-s_{i-1}^n),$$

define

$$u_n(s) := u(s_{i+1}^n), \quad g_n(s) := \varphi_{\mathrm{Dir}}(s_{i+1}^n), \quad z_n(s) := z(s_{i+1}^n), \quad l_n(s) := l(s_{i+1}^n),$$

and

$$X_n(s) := -\rho(s_i^n, s_{i+1}^n) \int_{s_i^n}^{s_{i+1}^n} \nabla\dot{\varphi}_{\mathrm{Dir}}(\tau)\,\mathrm{d}\tau,$$

for $s \in (s_i^n, s_{i+1}^n]$, and note that, since $\varphi_{\mathrm{Dir}} \in \mathrm{W}^{1,1}\big((0,t); \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)\big)$,

$$\|X_n(s)\|_{\mathrm{L}^p(\Omega;\mathbb{R}^d)} \to 0, \quad \text{uniformly on } [0,t]. \tag{4.29}$$

We apply (4.27) for $s = s_i^n$ and $t = s_{i+1}^n$, and sum the result for $i = 0, \ldots, k(n)-1$. Since $\dot{l} \in \mathrm{L}^1\big(0,T;\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)^*\big)$, and $\dot{\varphi}_{\mathrm{Dir}}(s) \in \mathrm{L}^1\big(0,T;\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)\big)$, we obtain

$$\begin{aligned}
&\mathscr{E}\big(t,u(t),z(t)\big) + \mathrm{Diss}(z;[0,t]) - \mathscr{E}\big(0,u(0),z(0)\big)\\
&\quad \geqq \int_0^t \big(\mathrm{D}W\big(\nabla u_n(s) + \nabla g_n(s) + X_n(s), z_n(s)\big).\nabla\dot{\varphi}_{\mathrm{Dir}}(s)\,\mathrm{d}x\\
&\qquad - \langle l_n(s), \dot{\varphi}_{\mathrm{Dir}}(s)\rangle - \langle \dot{l}(s), u_n(s) + g_n(s)\rangle\big)\,\mathrm{d}s + O(\Delta_n).
\end{aligned}$$

Recalling (4.29), we apply, once again, [11], Lemma 4.9, and conclude that, for a.e. $s \in [0,t]$,

$$\begin{aligned}
&\int_\Omega \mathrm{D}W\big(\nabla u_n(s) + \nabla g_n(s) + X_n(s), z_n(s)\big).\nabla\dot{\varphi}_{\mathrm{Dir}}(s)\,\mathrm{d}x\\
&\qquad - \int_\Omega \mathrm{D}W\big(\nabla u_n(s) + \nabla g_n(s), z_n(s)\big).\nabla\dot{\varphi}_{\mathrm{Dir}}(s)\,\mathrm{d}x \to 0.
\end{aligned}$$

The growth property of DW, together with the uniform $\mathrm{L}^p(\Omega;\mathbb{R}^{d\times d})$-bound on $\nabla\varphi_n$ and (4.29), imply that

$$\begin{aligned}
&\int_0^t \bigg(\int_\Omega \mathrm{D}W\big(\nabla u_n(s) + \nabla g_n(s) + X_n(s), z_n(s)\big).\nabla\dot{\varphi}_{\mathrm{Dir}}(s)\,\mathrm{d}x\\
&\qquad - \int_\Omega \mathrm{D}W\big(\nabla u_n(s) + \nabla g_n(s), z_n(s)\big).\nabla\dot{\varphi}_{\mathrm{Dir}}(s)\,\mathrm{d}x\bigg)\,\mathrm{d}s \to 0,
\end{aligned}$$

so that

$$\begin{aligned}
&\mathscr{E}\big(t,u(t),z(t)\big) + \mathrm{Diss}(z;[0,t]) - \mathscr{E}\big(0,u(0),z(0)\big)\\
&\quad \geqq \int_0^t \bigg(\int_\Omega \mathrm{D}W\big(\nabla u_n(s) + \nabla g_n(s), z_n(s)\big).\nabla\dot{\varphi}_{\mathrm{Dir}}(s)\,\mathrm{d}x\\
&\qquad - \langle l_n(s), \dot{\varphi}_{\mathrm{Dir}}(s)\rangle - \langle \dot{l}(s), u_n(s) + g_n(s)\rangle\bigg)\,\mathrm{d}s + O(\Delta_n).
\end{aligned} \tag{4.30}$$

The proof is completed by appealing to a measure theoretic result, [11], Lemma 4.12, which essentially states that Lebesgue integrals can be approximated by Riemann sums, *albeit* for a carefully chosen sequence of partitions, and which we recall for the reader's convenience.

Lemma 4.5. *Let X be a Banach space and $F \in \mathrm{L}^1\big((0,t);X\big)$. Then, there exists a sequence of partitions* $0 = s_0^n \leqq s_1^n \leqq \cdots \leqq s_{k(n)}^n = t$, *satisfying* (4.28), *such that*

$$\lim_n \sum_{i=1}^{k(n)} \left\| (s_i^n - s_{i-1}^n)F(s_i^n) - \int_{s_{i-1}^n}^{s_i^n} F(t)\,\mathrm{d}t \right\|_X = 0.$$

We apply this lemma to

$$F := (\dot\varphi_{\mathrm{Dir}}, \nabla\dot\varphi_{\mathrm{Dir}}, l, \dot l, \theta) \in \mathrm{L}^1\big(0,t;\mathrm{W}^{1,p}(\Omega;\mathbb{R}^{d\times d}) \times \mathrm{L}^p(\Omega;\mathbb{R}^{d\times d}) \times \big(\mathrm{W}^{1,p}(\Omega;\mathbb{R}^{d\times d})^*\big)^2 \times \mathbb{R}\big),$$

which allows us to find a sequence of subdivisions $0 = s_0^n \leqq s_1^n \leqq \cdots \leqq s_{k(n)}^n = t$, so that first $\dot\varphi_{\mathrm{Dir}}(s)$, $\nabla\dot\varphi_{\mathrm{Dir}}(s)$, $\dot l(s)$ are replaced by

$$H_n(s) := \varphi_{\mathrm{Dir}}(s_i^n), \quad G_n(s) := \nabla\dot\varphi_{\mathrm{Dir}}(s_i^n), \quad \dot l_n(s) := \dot l(s_i^n), \quad s_{i-1}^n < s \leqq s_i^n$$

in (4.30), and also so that

$$\int_0^t \bigg(\int_\Omega \mathrm{D}W\big(\nabla u_n(s) + \nabla g_n(s), z_n(s)\big) \cdot G_n(s)\,\mathrm{d}x - \langle \dot l_n(s), u_n(s) + g_n(s)\rangle$$
$$- \langle l_n(s), H_n(s)\rangle \bigg)\,\mathrm{d}s \to \int_0^t \theta(s)\,\mathrm{d}s.$$

In view of the expression (4.24) for $\theta(s)$, we get the opposite inequality in (4.25) and recover the conservation of energy $(\mathrm{E})_t$.

Remark 4.6. The conclusions of Step 6 in Section 3 apply verbatim in the current context.

5. Finite-strain elasticity

As in the previous sections we consider an elastic body $\Omega \subset \mathbb{R}^d$ given as a bounded domain with Lipschitz boundary. By $\varphi : \Omega \to \mathbb{R}^d$ we denote the elastic deformation and $F = \mathrm{D}\varphi$ denotes the strain tensor. In addition we have an internal variable $z : \Omega \to Z$. Here again, we assume that Z is a compact subset of $\mathbb{R}^m$. The dissipation distance $\hat d : Z \times Z \to [0,\infty)$ and the integrated version $\mathscr{D} : \mathscr{Z} \times \mathscr{Z} \to [0,\infty)$ are exactly as in Section 4. In particular, we have the triangle inequality and the equivalence to the norm in $\mathrm{L}^1(\Omega;\mathbb{R}^m)$, see (4.6).

The stored energy $\tilde{\mathscr{E}}$ takes the form

$$\tilde{\mathscr{E}}(t,\varphi,z) = \int_\Omega W\big(\mathrm{D}\varphi(x), z(x)\big) + \frac{\sigma}{\alpha}|\mathrm{D}z(x)|^\alpha\,\mathrm{d}x - \langle l(t),\varphi\rangle.$$

Here $\varphi : \Omega \to \mathbb{R}^d$ varies in the possibly time-dependent set $\hat{\mathscr{F}}(t)$ of admissible deformations

$$\tilde{\mathscr{F}}(t) = \{\varphi \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^d) \mid (\varphi - \varphi_{\mathrm{Dir}}(t))\big|_{\Gamma_{\mathrm{Dir}}} = 0\},$$

where $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$ has positive surface measure and $t \mapsto \varphi_{\mathrm{Dir}}(t) \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^d)$ are given time-dependent boundary data.

In finite-strain elasticity, natural physical requirements for the stored-energy density W as a function of (F, z) are (i) frame indifference and (ii) local non-interpenetration:

$$\text{(5.1)} \qquad \begin{aligned} &\text{(i)} \quad \forall Q \in \mathrm{SO}(\mathbb{R}^d)\colon \quad W(QF, z) = W(F, z);\\ &\text{(ii)} \quad W(F, z) = +\infty \quad \text{if } \det F \leqq 0. \end{aligned}$$

Recall that W may also depend explicitly on $x \in \Omega$, but we drop this dependence for notational simplicity. These conditions are not compatible with convexity. Moreover, there is no well-developed theory for quasiconvex functions taking the value $+\infty$. Hence, the class of polyconvex functions is better suited to our purpose, since it allows us to construct energy densities which satisfy (5.1) and generate a weakly lower semi-continuous functional.

To be more precise, we additionally assume that W is coercive, i.e., that, for some $c, C > 0$ we have

$$\text{(5.2)} \qquad \forall (F, z) \in \mathbb{R}^{d\times d} \times Z\colon \quad W(F, z) \geqq c|F|^p - C,$$

and that $W : \Omega \to \mathbb{R}^{d\times d} \times Z \to \mathbb{R}_\infty$ is such that the functional

$$\text{(5.3)} \qquad \hat{\mathscr{I}}_W : \mathrm{W}^{1,p}(\Omega; \mathbb{R}^d) \times \mathrm{W}^{1,\alpha}(\Omega; \mathbb{R}^m) \to \mathbb{R}_\infty \text{ is weakly lower semi-continuous,}$$

where $\hat{\mathscr{I}}_W(\varphi, z) = \int_\Omega W(\nabla\varphi(x), z(x)) + \frac{\sigma}{\alpha}|\nabla z(x)|^\alpha \, \mathrm{d}x.$

Remark 5.1. We give here examples of nontrivial functions W which satisfy the above assumptions (5.1)–(5.2). We consider the case $p > d$ and

$$W(F, z) = W_0(F) + W_1(F, z) \quad \text{with } W_0(F) \geqq c|F|^p - C.$$

Moreover, W_0 is assumed to be polyconvex (cf. [2], [10]) with $W_0(F) = \infty$ for $\det F \leqq 0$. For instance, we may consider Ogden materials in the form

$$W_0(F) = \begin{cases} b_1|F|^{\beta_1} + b_2(\det F)^{-\beta_2} & \text{for } F \in \mathrm{GL}_+(\mathbb{R}^d) = \{F \in \mathbb{R}^{d\times d} \mid \det F > 0\},\\ \infty & \text{else,} \end{cases}$$

with $\beta_1 \geqq p$ and $b_1, b_2, \beta_2 > 0$. The function W_1 should satisfy exactly the same conditions as W in Section 4, see (4.3) and above. The additive split of W gives the additive split $\hat{\mathscr{I}}_W = \mathscr{I}_0 + \mathscr{I}_1$ with $\mathscr{I}_j = \hat{\mathscr{I}}_{W_j}$. Now $\mathscr{I}_1$ is weakly lower semi-continuous according to Lemma 4.3 and $\mathscr{I}_0$ is so by polyconvexity. Hence the sum $\hat{\mathscr{I}}_W = \mathscr{I}_0 + \mathscr{I}_1$ satisfies (5.3). The coercivity (5.2) of W follows from coercivity of each W_j. Condition (ii) in (5.1) is true and (i) follows if $W_1(QF, z) = W_1(F, z)$ since the same holds for W_0.

The main differences between this and the previous section are that we do not have an upper bound on W in the form $W(F,z) \leqq C(1+|F|^p)$ and also that the estimate $|\mathrm{D}_F W(F,z)| \leqq C(1+|F|^{p-1})$ (cf. (4.4)) is no longer available. However, these estimates are only needed when using the additive split $\varphi = u + \varphi_{\mathrm{Dir}}(t)$ for adjusting to time-dependent Dirichlet data. In the finite-strain case this split is no longer appropriate. Instead we will use a multiplicative split in the sense of compositions of functions, namely $\varphi = \varphi_{\mathrm{Dir}}(t,\cdot) \circ \psi$, where ψ now satisfies time-independent Dirichlet data. We will see that the following estimate, which is fully compatible with finite-strain elasticity (i.e., with (5.1) to (5.3)), works well together with this multiplicative split. This estimate is based on the Kirchhoff stress tensor $K(F,z) = \mathrm{D}_F W(F,z) F^{\mathsf{T}}$:

$$(5.4) \qquad \exists c_W^{(0)}, c_W^{(1)} > 0 \ \forall (F,z) \in \mathrm{GL}_+(\mathbb{R}^d) \times Z: \quad |K(F,z)| \leqq c_W^{(1)}\big(c_W^{(0)} + W(F,z)\big).$$

This condition will enable us to bound the power of the Dirichlet data in terms of the energy, see (3.1). We will also need a further condition to guarantee the uniform continuity on sublevels, as imposed in the abstract part in (3.3):

$$(5.5) \qquad \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall z \in Z \ \forall F, G \in \mathrm{GL}_+(\mathbb{R}^d):$$
$$|G - I| \leqq \delta \Rightarrow |K(GF,z) - K(F,z)| \leqq \varepsilon\big(c_W^{(0)} + W(F,z)\big).$$

The Kirchhoff stress occurs as a multiplicative derivative of W. We have

$$K(F,z) : H = \mathrm{D}_F W(F,z) : (HF) = \mathrm{D}_F W(F,z)[HF] = \frac{\mathrm{d}}{\mathrm{d}s} W\big((I+sH)F,z\big)\Big|_{s=0}.$$

Thus, the Kirchhoff stress is particularly suited for situations in which the multiplicative character of matrices in $\mathrm{GL}_+(\mathbb{R}^d)$ is important.

Parts (a) and (b) of the following proposition are proved in [3].

Proposition 5.2. *Let W satisfy* (5.4).

(a) *There exists $\gamma > 0$ such that*

$$|G - I| \leqq \gamma \text{ and } G \in \mathrm{GL}_+(\mathbb{R}^d) \Rightarrow \begin{cases} W(GF,z) + c_W^{(0)} \leqq \dfrac{d}{d-1}\big(W(F,z) + c_W^{(0)}\big) \text{ and} \\ |\mathrm{D}_F W(GF) F^{\mathsf{T}}| \leqq 2d c_W^{(1)}\big(W(F,z) + c_W^{(0)}\big). \end{cases}$$

(b) *There exists $c_W^{(2)} > 0$ such that with $s = d c_W^{(1)}$ we have*

$$W(F,z) \leqq c_W^{(2)}(|F|^s + |F^{-1}|^s) \quad \text{for all } (F,z) \in \mathrm{GL}_+(\mathbb{R}^d) \times Z.$$

(c) *If $K(F,z)$ is differentiable in F and there exists $c_W^{(3)} > 0$ with*

$$|\mathrm{D}_F K(F,z)[HF]| \leqq c_W^{(3)}\big(c_W^{(0)} + W(F,z)\big)|H| \quad \text{for } H \in \mathbb{R}^{d\times d} \text{ and } (F,z) \in \mathrm{GL}_+(\mathbb{R}^d) \times Z,$$

then K satisfies condition (5.5).

Proof. For showing part (c) we let $G_\theta = (1-\theta)I + \theta G$, such that $G_0 = I$, $G_1 = G$ and for all $\theta \in [0,1]$ we have $\det G_\theta > 0$. Thus, we obtain

$$K(GF,z) - K(F,z) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} K(G_\theta F,z)\,\mathrm{d}\theta = \int_0^1 \mathrm{D}_F K(G_\theta F,z)[(G-I)F]\,\mathrm{d}\theta.$$

Using the postulated estimate from (c) we find

$$|K(GF,z) - K(F,z)| \leqq \int_0^1 c_W^{(3)}\big(c_W^{(0)} + W(G_\theta F,z)\big)|G-I|\,\mathrm{d}\theta.$$

Employing now the result from part (a) we see that (5.4) holds if we choose δ smaller than $\min\{\gamma, \varepsilon(d-1)/(dc_W^{(3)})\}$. □

Remark 5.3. The functions W_0 with $W_0(F) = b_1|F|^{\beta_1} + b_2(\det F)^{-\beta_2}$ for $\det F > 0$ and $W_0(F) = \infty$ otherwise, used in Remark 5.1, satisfy (5.4) and (5.5). Just use that the derivative of the determinant is given by the cofactor matrix $\operatorname{cof} F$ and that $(\operatorname{cof} F)F^\top = (\det F)I$. This gives

$$K(F) = \mathrm{D}_F W(F)F^\top = b_1\beta_1|F|^{\beta_1-2}FF^\top - b_2\beta_2(\det F)^{-\beta_2}I$$

and

$$\begin{aligned}\mathrm{D}_F K(F)[HF] = b_1\beta_1\big((\beta_1-2)|F|^{\beta_1-4}(FF^\top : H)FF^\top + |F|^{\beta_1-2}(HFF^\top + FF^\top H^\top)\big)\\ + b_2\beta_2^2(\det F)^{-\beta_2}(\operatorname{tr} H)I.\end{aligned}$$

Thus, for $b_1, b_2, \beta_2 > 0$ and $\beta_1 \geqq 1$ both conditions hold.

For the sum $W = W_0 + W_1$ the conditions (5.4) and (5.5) also hold if W_1 satisfies (4.3), is twice differentiable and if $\mathrm{D}K_1(F,z)F^\top$ can be bounded from above by a constant times $c_W^{(0)} + W_0$.

Finally, we consider the time-dependent external loading l satisfying

$$l \in \mathrm{C}^1\big([0,T], \mathrm{W}^{1,p}(\Omega,\mathbb{R}^d)^*\big). \tag{5.6}$$

More difficult is the forcing of the body through the time-dependent Dirichlet data on the subset $\Gamma_{\mathrm{Dir}} \neq \emptyset$ of $\partial\Omega$ given via $t \mapsto \varphi_{\mathrm{Dir}}(t)$. We assume that the Dirichlet data $\varphi_{\mathrm{Dir}} \in \mathrm{C}^1([0,T],\Gamma_{\mathrm{Dir}})$ can be extended smoothly to all of $\mathbb{R}^d$ as follows:

$$\begin{aligned}&\varphi_{\mathrm{Dir}} \in \mathrm{C}^1([0,T]\times\mathbb{R}^d;\mathbb{R}^d), \quad \nabla\varphi_{\mathrm{Dir}} \in \mathrm{BC}^1\big([0,T]\times\mathbb{R}^d, \mathrm{Lin}(\mathbb{R}^d;\mathbb{R}^d)\big)\\ &\text{and} \quad |\nabla\varphi_{\mathrm{Dir}}(t,x)^{-1}| \leqq C \quad \text{for all } (t,x) \in [0,T]\times\mathbb{R}^d,\end{aligned} \tag{5.7}$$

where "BC" stands for bounded and continuous. Thus, for each $t \in [0,T]$, the mapping $\varphi_{\mathrm{Dir}}(t,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is a global diffeomorphism.

We now look for $\varphi(t) : \Omega \to \mathbb{R}^d$ in the form $\varphi(t) = \varphi_{\mathrm{Dir}}(t) \circ \psi$ with

$$\psi \in \mathscr{F} := \{\psi \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d) \mid \psi|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id}|_{\Gamma_{\mathrm{Dir}}}\},$$

then $\varphi(t,x) = \varphi_{\mathrm{Dir}}(t,x)$ for $(t,x) \in [0,T] \times \Gamma_{\mathrm{Dir}}$ as desired. Here we always assume $p > d$, so that $\psi \in \mathscr{F}$ is continuous. Moreover, $\mathscr{F}$ is nonempty, since $\psi = \mathrm{id} \in \mathscr{F}$.

The energy functional $\tilde{\mathscr{E}}(t,\varphi,z) = \int_\Omega W(\nabla\varphi,z) + \frac{\sigma}{\alpha}|\nabla z|^\alpha \,\mathrm{d}s - \langle l(t),\varphi\rangle$ is now transformed into

$$\mathscr{E}(t,\psi,z) := \int_\Omega W\big(\nabla\varphi_{\mathrm{Dir}}\big(t,\psi(x)\big)\nabla\psi(x), z(x)\big) + \frac{\sigma}{\alpha}|\nabla z(x)|^\alpha \,\mathrm{d}x - \langle l(t), \varphi_{\mathrm{Dir}}(t) \circ \psi\rangle.$$

Lemma 5.4. *Let $p > d$ and assume that* (5.2), (5.3), (5.6) *and* (5.7) *hold, then for all $t \in [0,T]$ the functional $\mathscr{E}(t,\cdot) : \mathscr{F} \times \mathscr{Z} \to \mathbb{R}_\infty$ is weakly lower semicontinuous with respect to the standard topology of* $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d) \times \mathrm{W}^{1,\alpha}(\Omega;\mathbb{R}^m)$. *Moreover, $\mathscr{E}$ is coercive, i.e., there exist constants $c, C > 0$ such that*

$$\mathscr{E}(t,\psi,z) \geqq c\|\psi\|^p_{\mathrm{W}^{1,p}} + c\|z\|^\alpha_{\mathrm{W}^{1,\alpha}} - C \quad \textit{for all } (t,\psi,z) \in [0,T] \times \mathscr{F} \times \mathscr{Z}.$$

Proof. To establish coercivity we use the following estimates on φ_{Dir}:

$$\forall y \in \mathbb{R}^d: \quad |\varphi_{\mathrm{Dir}}(t,y)| \leqq C(1+|y|), \quad \big|\big(\nabla\varphi_{\mathrm{Dir}}(t,y)\big)\big| \leqq C, \quad \big|\big(\nabla\varphi_{\mathrm{Dir}}(t,y)\big)^{-1}\big| \leqq C.$$

For $F = \nabla\psi$ this implies the lower estimate $|\nabla\varphi_{\mathrm{Dir}}F| \geqq |F|/|(\nabla\varphi_{\mathrm{Dir}})^{-1}| \geqq |F|/C$. With this and the coercivity of W it is standard to obtain the lower estimate for $\mathscr{E}(t,\psi,z)$ as given above.

To show weak lower semi-continuity take a sequence with $(t_k,\psi_k,z_k) \rightharpoonup (t,\psi,z)$. Because of $p > d$, the embedding of $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ into $\mathrm{C}(\bar{\Omega};\mathbb{R}^d)$ is compact. Hence, we conclude $\psi_k \to \psi$ uniformly in Ω. Let $\varphi_k = \varphi_{\mathrm{Dir}}(t_k) \circ \psi_k$ and $\varphi = \varphi_{\mathrm{Dir}}(t) \circ \psi$, then (5.7) implies $\varphi_k \to \varphi$ uniformly in Ω. Moreover, for $F_k(x) := \nabla\varphi_k = \nabla\varphi_{\mathrm{Dir}}\big(t_k,\psi_k(x)\big)\nabla\psi_k(x)$ and $F(x) := \nabla\varphi(x) = \varphi_{\mathrm{Dir}}\big(t,\psi(x)\big)\nabla\psi(x)$ we obtain

$$F_k = G_k\nabla\psi_k \rightharpoonup F = G\nabla\psi \quad \text{in } \mathrm{L}^p(\Omega;\mathbb{R}^{d\times d}),$$

since $G_k = \nabla\varphi_{\mathrm{Dir}}\big(t_k,\psi_k(\cdot)\big)$ converges to $G = \nabla\varphi_{\mathrm{Dir}}\big(t,\psi(\cdot)\big)$ uniformly and $\nabla\psi_k$ to $\nabla\psi$ weakly. Thus, we have shown that φ_k converges weakly to φ in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$.

Now we use the transformation rule $\mathscr{E}(t_k,\psi_k,z_k) = \tilde{\mathscr{E}}(t_k,\varphi_k,z_k)$ and the assumption (5.3) which guarantees that $\tilde{\mathscr{E}}$ is weakly lower semi-continuous. Hence we conclude

$$\mathscr{E}(t,\psi,z) = \tilde{\mathscr{E}}(t,\varphi,z) \leqq \liminf_{k\to\infty} \tilde{\mathscr{E}}(t_k,\varphi_k,z_k) = \liminf_{k\to\infty} \mathscr{E}(t_k,\psi_k,z_k),$$

which is the desired result. □

The next result shows that the power of the external forces given via $\partial_t\mathscr{E}(t,\psi,z)$ satisfies the abstract condition (3.3).

Lemma 5.5. *Let the assumptions of Lemma* 5.4 *as well as* (5.4) *hold, then for all* $(\psi, z) \in \mathcal{F} \times \mathcal{Z}$ *with* $\mathcal{E}(0, \psi, z) < \infty$ *the function* $t \mapsto \mathcal{E}(t, \psi, z)$ *is continuously differentiable with the derivative*

$$\partial_t \mathcal{E}(t, \psi, z) = \int_\Omega \mathrm{D}_F W(\nabla\varphi_{\mathrm{Dir}} \nabla\psi, z)(\nabla\varphi_{\mathrm{Dir}} \nabla\psi)^{\mathsf{T}} : [(\nabla\varphi_{\mathrm{Dir}})^{-1} \nabla\dot{\varphi}_{\mathrm{Dir}}] \,\mathrm{d}x$$

$$- \langle \dot{l}(t), \varphi_{\mathrm{Dir}} \rangle - \langle l(t), \dot{\varphi}_{\mathrm{Dir}} \rangle,$$

where $\nabla\varphi_{\mathrm{Dir}}$, φ_{Dir} *and* $\dot{\varphi}_{\mathrm{Dir}} = \partial_t \varphi_{\mathrm{Dir}}$ *are evaluated at* $(t, \psi(x))$. *Moreover, there exist constants* $c_E^{(1)}, c_E^{(0)} > 0$ *such that* $|\partial_t \mathcal{E}(t, \psi, z)| \leqq c_E^{(1)} (c_E^{(0)} + \mathcal{E}(t, \psi, z))$ *for all* $t \in [0, T]$.

If additionally condition (5.5) *holds, then the abstract condition* (3.3) *of uniform continuity on sublevels of* $\mathcal{E}$ *also holds.*

Proof. In this lemma (ψ, z) are fixed throughout such that $\mathcal{E}(t, \psi, z) < \infty$. Hence, we know by Lemma 5.4 that $\psi \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^d)$. We write

$$\mathcal{E}(t, \psi, z) = \hat{\mathcal{E}}(t, \psi, z) - \langle l(t), \varphi_{\mathrm{Dir}}(t, \psi(\cdot)) \rangle$$

and, using the assumptions on φ_{Dir} and l, we easily obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle l(t), \varphi_{\mathrm{Dir}}(t, \psi(\cdot)) \rangle = \langle \dot{l}(t), \varphi_{\mathrm{Dir}}(t) \circ \psi \rangle + \langle l(t), \dot{\varphi}_{\mathrm{Dir}}(t) \circ \psi \rangle.$$

Thus, it remains to consider $\hat{\mathcal{E}}$ which is the integral over $W(\nabla\varphi_{\mathrm{Dir}} \nabla\psi, z)$. To simplify the notation we omit the argument $z(x)$, since it is fixed throughout and all the constants are independent of z. For fixed t and $\hat{\mathcal{E}}(t, \psi) < \infty$ we consider the different quotient

$$\frac{1}{h} (\hat{\mathcal{E}}(t+h, \psi) - \hat{\mathcal{E}}(t, \psi)) = \int_\Omega \frac{1}{h} [W(C_h(x)F(x)) - W(F(x))] \,\mathrm{d}x$$

where $F(x) = \nabla\varphi_{\mathrm{Dir}}(t, \psi(x)) \nabla\psi(x)$ and $C_h(x) = \nabla\varphi_{\mathrm{Dir}}(t+h, \psi(x)) [\nabla\varphi_{\mathrm{Dir}}(t, \psi(x))]^{-1}$. Applying the mean-value theorem to $[0,1] \ni s \mapsto W(C_{sh}(x)F(x))$ we find $\theta(h, x) \in [0,1]$ such that

$$\frac{1}{h} [W(C_h(x)F(x)) - W(F(x))] = \mathrm{D}_F W(C_{\theta(h,x)h}(x)F(x)) F(x)^{\mathsf{T}} : \frac{1}{h} (C_h(x) - I).$$

We now apply Part (a) in Proposition 5.2. Using (5.7) and making h sufficiently small, we have $|C_{\theta(h,x)h} - I| < \gamma$ and thus the estimate in Part (a) supplies an h-independent integrable majorant, i.e.,

$$\frac{1}{h} |W(C_h(x)F(x)) - W(F(x))| \leqq 2\, d c_W^{(1)} (c_W^{(0)} + W(F(x))) C_{\varphi_{\mathrm{Dir}}},$$

where $C_{\varphi_{\mathrm{Dir}}}$ is the supremum over $\frac{1}{h}(C_h - I)$ for $x \in \Omega$ and $|h| \leqq h_0$. Note that $\mathcal{E}(t, \psi) < \infty$ implies $W(F(\cdot)) \in \mathrm{L}^1(\Omega)$.

Finally, use that $\frac{1}{h}(C_h - I)$ converges to $\nabla\dot{\varphi}_{\mathrm{Dir}}[\nabla\varphi_{\mathrm{Dir}}]^{-1}$ pointwise. Thus, Lebesgue's theorem of dominated convergence gives

$$\partial_t\hat{\mathscr{E}}(t,\psi) = \lim_{h\to 0}\frac{1}{h}\big(\hat{\mathscr{E}}(t+h,\psi) - \hat{\mathscr{E}}(t,\psi)\big) = \int_\Omega \mathrm{D}_F W(F)F^{\mathsf{T}} : \nabla\dot{\varphi}_{\mathrm{Dir}}[\nabla\varphi_{\mathrm{Dir}}]^{-1}\,\mathrm{d}x,$$

where $F = \nabla\varphi_{\mathrm{Dir}}\nabla\psi$. Hence, the existence and the formula for $\partial_t\mathscr{E}(t,\psi,z)$ is established.

The continuity in $t \in [0,T]$ follows similarly by employing Part (a) of Proposition 5.2 once again and by Lebesgue's theorem.

The construction of the explicit constants $c_E^{(0)}$ and $c_E^{(1)}$ now works as follows. We use constants $c, C, \ldots$ which may differ in each inequality, but they are fixed and independent of $E = \mathscr{E}(t,\psi,z)$. With Lemma 5.4 we find $\|\psi\|^p_{\mathrm{W}^{1,p}} \leqq (E+C)/c$ and, using $p > d$, we conclude $\|\psi\|_\infty \leqq CE^{1/p} + C$. Using the estimates (5.7) for φ_{Dir} and (5.6) for l we obtain

$$\text{(5.8)}\qquad \begin{aligned} &\text{(i)}\quad |\langle \ell(t), \varphi_{\mathrm{Dir}}(t)\circ\psi\rangle| \leqq CE^{1/p} + C,\\ &\text{(ii)}\quad |\langle \dot{l}(t), \varphi_{\mathrm{Dir}}(t)\circ\psi\rangle + \langle l(t), \dot{\varphi}_{\mathrm{Dir}}(t)\circ\psi\rangle| \leqq CE^{1/p} + C. \end{aligned}$$

Using (ii) the volume integral in $\mathscr{E}(t,\psi,z)$ can be estimated via

$$\int_\Omega W(\nabla\varphi_{\mathrm{Dir}}\nabla\psi, z)\,\mathrm{d}x = \mathscr{E}(t,\psi,z) + \langle \ell(t), \varphi_{\mathrm{Dir}}(t)\circ\psi\rangle \leqq E + CE^{1/p} + C \leqq 2E + C_2.$$

Now $\partial_t\mathscr{E}$ can be estimated on the basis of the explicit formula derived above. The term from the loading is dominated by $E + C_3$ according to (5.8)(ii). The volume integral involves the product of $K(\nabla\varphi_{\mathrm{Dir}}\nabla\psi, z)$ and $\nabla\dot{\varphi}_{\mathrm{Dir}}[\nabla\varphi_{\mathrm{Dir}}]^{-1}$. The first term can be estimated via W evaluated at the same arguments according to assumption (5.4). The second term is uniformly bounded by C_4 because of (5.7). Together we find

$$\begin{aligned} |\partial_t\mathscr{E}(t,\psi,z)| &\leqq \int_\Omega |K(\nabla\varphi_{\mathrm{Dir}}\nabla\psi, z)|\,|\nabla\dot{\varphi}_{\mathrm{Dir}}[\nabla\varphi_{\mathrm{Dir}}]^{-1}|\,\mathrm{d}x + EC_3\\ &\leqq \int_\Omega c_W^{(1)}\big(c_W^{(0)} + W(\ldots)\big)C_4\,\mathrm{d}x + E + C_3\\ &\leqq C_4 c_W^{(1)}\int_\Omega W(\ldots)\,\mathrm{d}x + E + |\Omega| c_W^{(1)} c_W^{(0)} C_4 + C_3\\ &\leqq C_4 c_W^{(1)}(2E + C_2) + E + |\Omega| c_W^{(1)} c_W^{(0)} C_4 + C_3. \end{aligned}$$

Using $E = \mathscr{E}(t,\psi,z)$ this is the desired result (3.1) with $c_E^{(1)} = 1 + 2C_4 c_W^{(1)}$ and suitable $c_E^{(0)}$.

The uniform continuity of $\partial_t\mathscr{E}(t,\psi,z)$ on a given sublevel

$$S(E_0) := \{(t,\psi,z) \in [0,T]\times\mathscr{F}\times\mathscr{Z} \mid \mathscr{E}(t,\psi,z) \leqq E_0\}$$

is now obtained by the uniform continuity of the continuous functions l, $\dot{l}$ and φ_{Dir}, $\dot{\varphi}_{\mathrm{Dir}}$, $\nabla\varphi_{\mathrm{Dir}}$ and $\nabla\dot{\varphi}_{\mathrm{Dir}}$ on the time interval $[0,T]$. For this use that by the above arguments we see that all values of ψ on $[0, T] \times \bar{\Omega}$ lie in a compact subset of $\mathbb{R}^d$ which only depends on E_0. Thus, the uniform continuity of the power of the loading through l (see (5.8)(ii)) is easily seen to be uniformly continuous.

It remains to estimate $\partial_t \hat{\mathscr{E}}(t, \psi)$, where we used the splitting from above and where we again dropped the variable z. The above estimates can now be written as

$$\forall (t, \psi, z) \in S(E_0): \quad \int_\Omega W(\nabla\varphi_{\mathrm{Dir}} \nabla\psi, z)\, \mathrm{d}x \leqq 2E_0 + C_2.$$

Our aim is now to estimate $|\partial_t \hat{\mathscr{E}}(t, \psi) - \partial_t \hat{\mathscr{E}}(s, \psi)|$ for $(t, \psi, z) \in S(E_0)$. Note that now the constants are allowed to depend on E_0 as well, but all constants will be independent of $(t, \psi, z) \in S(E_0)$. We introduce the abbreviations

$$F_t(x) = \nabla\varphi_{\mathrm{Dir}}\big(t, \psi(x)\big)\nabla\psi(x) = \Phi_t(x)\nabla\psi$$

and

$$L_t(x) = \nabla\dot{\varphi}_{\mathrm{Dir}}\big(t, \psi(x)\big)\big[\nabla\varphi_{\mathrm{Dir}}\big(t, \psi(x)\big)\big]^{-1},$$

such that we have $\partial_t \hat{\mathscr{E}}(t, \psi) = \int_\Omega K(F_t) : L_t \, \mathrm{d}x$.

Assumption (5.7) guarantees that $t \mapsto L_t \in \mathrm{C}^0(\bar{\Omega}; \mathbb{R}^{d\times d})$ and $t \to \Phi_t \in \mathrm{C}^0(\bar{\Omega}; \mathbb{R}^{d\times d})$ are Lipschitz continuous and their Lipschitz constants can be bounded in terms of E_0. Since $F_s F_t^{-1} = \Phi_s \Phi_t^{-1}$ we also find $C > 0$ with

$$|F_s(x)F_t(x)^{-1} - I| \leqq C|t - s| \quad \text{for all } t, s \in [0, T] \text{ and } x \in \Omega.$$

Employing the estimate (5.5) for K we now estimate

$$\begin{aligned}
|\partial_t \mathscr{E}(s, \psi) - \partial_t \mathscr{E}(t, \psi)| &\leqq \int_\Omega |K(F_s) : L_s - K(F_t) : L_t| \, \mathrm{d}x \\
&\leqq \int_\Omega |K(F_s) - K(F_t)|\,|L_s| \, \mathrm{d}x + \int_\Omega |K(F_t)|\,|L_s - L_t| \, \mathrm{d}x \\
&\leqq \int_\Omega |K([F_s F_t^{-1}]F_t) - K(F_t)| C \, \mathrm{d}x + \int_\Omega |K(F_t)| C|t - s| \, \mathrm{d}x \\
&\leqq_{(*)} \int_\Omega \varepsilon\big(c_W^{(0)} + W(F_t)\big) C \, \mathrm{d}x + C|t - s| \int_\Omega c_W^{(1)}\big(c_W^{(0)} + W(F_t)\big) \, \mathrm{d}x \\
&\leqq C(\varepsilon + |t - s|) =: \hat{\varepsilon},
\end{aligned}$$

where in the estimate $\leqq_{(*)}$ we have used that $|t - s|$ can be chosen less than $\hat{\delta} > 0$ such that $|F_s F_t^{-1} - I| \leqq C\hat{\delta} < \delta$ with the $\delta > 0$ needed in (5.5) to obtain ε.

Thus, uniform continuity is shown with the modulus of continuity $\omega_{E_0}(\hat{\delta}) = \hat{\varepsilon}$. □

Having these two lemmas it is easy to see that all six steps of the abstract construction of the solutions can be performed and we have thus proved the following existence result:

Theorem 5.6. *Under the assumptions of Lemma* 5.5 *the rate-independent evolution problem* (S) *and* (E) *associated with* $\mathscr{E}$ *and* $\mathscr{D}$ *has for each* $(\psi_0, z_0) \in \mathscr{S}(0)$ *a solution* $(\psi, z) : [0, T] \to \mathscr{F} \times \mathscr{Z}$.

Remark 5.7. If the Dirichlet data φ_{Dir} do not depend on the time variable $t \in [0, T]$ the assumptions of the above result can be simplified considerably. In fact, (5.4) and (5.5) are no longer needed. The set $\tilde{\mathscr{F}} = \{\varphi \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^d) \mid (\varphi - \varphi_{\mathrm{Dir}})|_{\Gamma_{\mathrm{Dir}}} = 0\}$ is an affine subspace and time independent. Thus we may set $\mathscr{E}(t, u, z) = \tilde{\mathscr{E}}(t, u + \varphi_{\mathrm{Dir}}, z)$ and obtain

$$\partial_t \mathscr{E}(t, u, z) = -\langle \dot{l}(t), u + \varphi_{\mathrm{Dir}} \rangle.$$

However, this term is linear in u and hence weakly continuous. Thus, the abstract conditions (3.1) and (3.3) hold by employing the coercivity of $\mathscr{E}(t, \cdot, z)$ in $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^d)$ and the uniform continuity of $t \mapsto \dot{l}(t)$.

Also Proposition 3.3 can be avoided, since its conclusion is obvious in this case.

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Laboratoire P.M.T.M., Institut Galilée, Université Paris 13,
Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France

Weierstraß Institut für Angewandte Analysis und Stochastik,
Mohrenstr. 39, 10117 Berlin, Germany
e-mail: mielke@wias-berlin.de

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