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Dedicated to M. I. Vishik on the occasion of his 80*th birthday*

Infinite-dimensional trajectory attractors of elliptic boundary-value problems in cylindrical domains

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Abstract. This paper is a study of an abstract model of a second-order non-linear elliptic boundary-value problem in a cylindrical domain by the methods of the theory of dynamical systems. It is shown that, under some natural conditions, the essential solutions of the problem in question are described by means of the global attractor of the corresponding trajectory dynamical system, and this attractor can have infinite fractal dimension and infinite topological entropy. Moreover, sharp upper and lower bounds are obtained for the Kolmogorov ε-entropy of these attractors.

Contents

§ 0. Introduction

Spatial dynamical systems arise in the study of non-linear elliptic boundary-value problems in cylindrical domains in which the spatial coordinate corresponding to the axis of the cylinder plays the role of time. The use of the theory of dynamical systems in the study of these problems was initiated by [20], where a local centre manifold for a quasi-linear elliptic equation in a strip was constructed. This method of reducing an elliptic boundary-value problem to a dynamical system on the spatial centre manifold, which was later called the Kirchgässner reduction, was subsequently developed in [23], [24], [18], [17], [16], [5], [26], where it was used to study diverse problems in mathematical physics arising, in particular, in hydrodynamics and non-linear elasticity theory. A special case of elliptic variational problems was studied in [25], where the existence of a Hamiltonian structure was proved for the reduced dynamical system on the spatial centre manifold.

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Methods using the qualitative theory of dynamical systems to study the global structure of the set of bounded solutions for elliptic boundary-value problems in cylindrical domains were developed in parallel, starting from [6], [27], [28]. The main idea of these methods is to introduce an auxiliary elliptic problem in the semicylinder $\Omega_+ := (0,\infty)\times\omega$ ($(t,x)\in\Omega_+$) with an additional boundary condition $u\big|_{t=0} = u_0$ on the base of the cylinder, and to investigate from the dynamical point of view the 'evolution' operator

$$S_t\colon u(0,x)\to u(t,x), \tag{0.1}$$

where $u(t,x)$ stands for a *bounded* solution of this problem. As is known, in this case if a global attractor of the operator (0.1) exists, then it is generated by the essential solutions of the original boundary-value problem in the entire cylinder $\Omega=\mathbb{R}\times\omega$, that is, by the solutions that are defined and bounded on Ω. This relationship enables one to study the 'dynamics' of the essential solutions by studying the dynamical properties of the evolution operator (0.1) on its attractor.

Unfortunately, a bounded solution of the above auxiliary problem is not unique as a rule, and therefore the semigroup (0.1) can be correctly defined only as a semigroup of set-valued maps. One can avoid the appearance of set-valued maps by using the so-called trajectory approach under which the set $\mathcal{K}^+$ of all bounded solutions of the auxiliary problem ($\mathcal{K}^+$ is endowed with an appropriate topology) is regarded as the (trajectory) phase space for the dynamical system generated by the semigroup of positive shifts $(\mathcal{T}_h)_{h\geqslant 0}$ along the axis of the cylinder, where the semigroup is defined by the formula

$$(\mathcal{T}_h u)(t,x)=u(t+h,x)\quad\text{for any}\quad (t,x)\in\Omega_+,\quad h\geqslant 0. \tag{0.2}$$

If this semigroup admits a global attractor, then it is called the *trajectory attractor* of the original problem. If the trajectory attractor can be embedded in a finite-dimensional invariant manifold, then this manifold is said to be an *essential manifold*, because it contains all essential solutions of the problem in question; see [28], [36], [7], [33]. (The above trajectory approach is also used to study diverse *evolution* equations of mathematical physics such that the uniqueness problem for their solutions is still open, for instance, for the three-dimensional Navier–Stokes system of equations, non-linear wave equations with rapidly growing non-linear terms, and so on; see [8].)

For second-order elliptic boundary-value problems there is another way to avoid set-valued maps, based on replacement of the operator (0.1) by the following evolution operator:

$$\mathbb{S}_t\colon (u(0),\partial_t u(0))\to (u(t),\partial_t u(t)),\qquad (u(0),\partial_t u(0))\in\mathbb{K}^+, \tag{0.3}$$

where $\mathbb{K}^+$ is the set of all 'initial data' $(u(0),\partial_t u(0))$ such that the auxiliary problem has a bounded solution. Then this solution is unique under some additional assumptions, and hence the formula (0.3) correctly defines a continuous semigroup in the phase space $\mathbb{K}^+$ [6]. However, we note that this semigroup turns out to be homeomorphic to the semigroup of shifts (0.2) defined on the trajectory phase space $\mathcal{K}^+$ (see, for instance, § 2 below).

Another approach related to the direct investigation of the evolution operator (0.1) by using the corresponding generalization of the notion of global attractor to the case of semigroups of set-valued maps was suggested in [2].

Other ideas and methods of the qualitative theory of dynamical systems have also been applied to the study of elliptic boundary-value problems. For example, exponential dichotomies were constructed in [32] to study bifurcations of solitary waves. A Floquet-type theory in a neighbourhood of a spatially periodic solution was developed in [27], [12]. The Conley index was used in [15] to prove the existence of non-trivial heteroclinic solutions. For the case in which the auxiliary problem has a unique solution and the original elliptic system admits a global Lyapunov function, a description of the structure of the attractor was obtained in [37].

However, very little is known about the Hausdorff dimension and the fractal dimension of attractors of elliptic equations, despite the existence of a well-developed theory for estimating these dimensions for evolution equations (see, for instance, [34]). In fact, at present we know of only two quite narrow classes of elliptic boundary-value problems whose attractors are finite-dimensional. The first is the case in which the auxiliary problem in the semicylinder is uniquely soluble, and thus the elliptic problem in the semicylinder can be reduced to some evolution equation on the section of the semicylinder [37], and the second is the case in which an essential manifold exists, and hence the dimension of the attractor is naturally majorized by the dimension of the essential manifold (see [28], [3]).

In the present paper we show that the dimension of the attractor can be infinite for elliptic boundary-value problems not belonging to the above classes, and we give a quantitative description of the 'thickness' of such attractors in terms of the Kolmogorov ε-entropy.

We consider the following abstract quasi-linear elliptic problem:

$$\begin{cases} \ddot{u} - \gamma\dot{u} - Au = F(u, \dot{u}) \quad \text{for} \quad t > 0, \\ u\big|_{t=0} = u_0. \end{cases} \tag{0.4}$$

Here $u(t)$ stands for an element of some Hilbert space H with inner product $\langle\,\cdot\,,\cdot\,\rangle$. The linear operator $A\colon D(A) \to H$ is assumed to be self-adjoint and positive definite ($\langle Au, u\rangle \geqslant \lambda_0\|u\|^2$ for some $\lambda_0 \geqslant 1$), and to have a compact inverse (the operator A^{-1} exists and is compact). Let us introduce a scale of Hilbert spaces $(H^s)_{s\in\mathbb{R}}$ generated by the operator A by the formula $H^s = D(A^{s/2})$ with $\|\cdot\|_s \equiv \|\cdot\|_{H^s} = \|A^{s/2}\cdot\|$. We also introduce the spaces $\mathbb{H}^s = H^s \times H^{s-1}$ with the natural induced Hilbert structure.

Moreover, it is assumed that γ is a bounded symmetric operator on H and that the non-linear function F satisfies the following conditions: there are positive constants C and δ ($\delta \ll 1$) and monotone functions $Q_\mu\colon \mathbb{R}^+ \to \mathbb{R}^+$ defined for any $\mu > 0$ and such that

$$\begin{cases} \text{(a)} \quad F \in C^1(H^{3/2-\delta} \times H^{1/2-\delta}, H), \\ \text{(b)} \quad \mathrm{D}_u F(u, v) \geqslant -C - \tfrac{1}{2}A, \\ \text{(c)} \quad \langle F(u, v), u\rangle \geqslant -C - \tfrac{1}{2}(\|u\|_1^2 + \|v\|^2), \\ \text{(d)} \quad \|F(u, v)\|^2 \leqslant Q_\mu(\|u\|_{1/2}) + \mu\|u\|_2^2 + C(\|u\|_1^2 + \|v\|^2). \end{cases} \tag{0.5}$$

The choice of the specific form of the equation (0.4) is motivated by the following elliptic boundary-value problem in a cylindrical domain $\Omega_+ = \mathbb{R}^+ \times \omega$ (ω is a bounded domain in $\mathbb{R}^n$):

$$\begin{cases} \ddot{u} - \gamma\dot{u} + \Delta_x u = f(u,\dot{u}) + g(x) \quad \text{for} \quad (t,x) \in \Omega_+, \\ u\big|_{\mathbb{R}^+\times\partial\omega} = 0, \quad u\big|_{t=0} = u_0, \end{cases} \tag{0.6}$$

where $u = (u^1,\dots,u^k) \in \mathbb{R}^k$, $\gamma = \gamma^* \in \mathcal{L}(\mathbb{R}^k,\mathbb{R}^k)$, and $g \in L^2(\omega)$. Such a problem arises, for instance, when studying solutions in the form of a traveling wave for the corresponding evolution equation in an unbounded cylindrical domain $\Omega = \mathbb{R} \times \omega$ (see, for instance, [6], [2], or [36]).

The existence of a bounded solution $u(t)$, $t \geqslant 0$, of the equation (0.4) for any initial condition $u_0 \in H^{3/2}$ is proven in § 1. Moreover, in § 1 we derive a dissipative estimate for the *bounded* solutions $u \in W^2_{\mathrm{bd}}(\mathbb{R}^+)$ (see Definition 1.1) of (0.4), which plays the basic role in the trajectory approach described above.

In § 2 we prove that the abstract equation (0.4) has a trajectory attractor $\mathcal{A} = \mathcal{A}^{\mathrm{traj}}$, that is, the semigroup (0.2) defined on the space $\mathcal{K}^+$ of all bounded solutions $u \in W^2_{\mathrm{bd}}(\mathbb{R}^+)$ has a global attractor $\mathcal{A}$ that is generated by the essential solutions of the equation (0.4), namely, $\mathcal{A}^{\mathrm{traj}} = \Pi_+\mathcal{K}$, where $\mathcal{K} \subset W^2_{\mathrm{bd}}(\mathbb{R})$ is the set of all essential solutions of (0.4) and Π_+ is the operator restricting functions to the semi-axis $\mathbb{R}^+$.

Moreover, if the non-linear function F satisfies the additional condition

$$\| \mathrm{D}_u F(u,v)\|_{H^1\to H} + \| \mathrm{D}_v F(u,v)\|_{H\to H} \leqslant Q(\|u\|_{3/2} + \|v\|_{1/2}) \tag{0.7}$$

(where Q is some monotone function), then we prove that any bounded solution $u(t)$ of (0.4) is uniquely determined by a pair of initial values $(u(0), \partial_t u(0))$ and thus the semigroup (0.3) is correctly defined on the space $\mathbb{K}^+$. This result is based on estimates (similar to logarithmic convexity) for abstract elliptic equations; see [1] and [6]. In this section we also show that the natural projection $\Pi_0 \colon \mathcal{K}^+ \to \mathbb{K}^+$ defined by the formula $\Pi_0 u := (u(0), \partial_t u(0))$ is a Hölder-continuous homeomorphism, and hence the semigroup (0.3) can be defined by the formula

$$\mathbb{S}_h := \Pi_0 \mathcal{T}_h (\Pi_0)^{-1}. \tag{0.8}$$

This result ensures the existence of a global attractor $\mathbb{A} \subset \mathbb{K}^+$ for the semigroup (0.3) and also proves the relation

$$\mathbb{A} = \Pi_0 \mathcal{A}^{\mathrm{traj}}. \tag{0.9}$$

In § 3 we use the notion of Kolmogorov ε-entropy to study quantitative characteristics of the attractor $\mathcal{A}^{\mathrm{traj}}$ constructed in the previous section for the elliptic problem (0.4). For a detailed exposition of the Kolmogorov entropy, see, for instance, [21]; applications of this notion to evolution equations in mathematical physics are treated in [9], [10], [38], [40], [11]. The main result of § 3 is the following upper bound for the Kolmogorov ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A}^{\mathrm{traj}}|_{(0,T)})$ of the restriction of the attractor $\mathcal{A}^{\mathrm{traj}}$ to an arbitrary finite interval $(0,T)$:

$$\mathbf{H}_\varepsilon(\mathcal{A}^{\mathrm{traj}}|_{(0,T)}) \leqslant C\left[T + \log_+ \frac{R_0}{\varepsilon}\right] \log_+ \frac{R_0}{\varepsilon}\,, \tag{0.10}$$

where C and R_0 are positive constants independent of $\varepsilon > 0$ and $T \geqslant 0$, and $\log_+ z := \max\{\log z, 0\}$.

The estimate (0.10) is not sufficient to conclude that the fractal dimension $\dim_{\mathrm{fract}}(\mathbb{A})$ of the attractor is finite. Moreover, as is shown in § 4, this dimension can really be infinite. In fact, we construct an example of an operator A and a non-linear map F satisfying the conditions (0.5) and such that the Kolmogorov entropy of the corresponding attractor admits the lower estimate

$$\mathbf{H}_\varepsilon(\mathcal{A}^{\mathrm{traj}}|_{(0,T)}) \geqslant C'T \log_+ \frac{R_0'}{\varepsilon} \tag{0.11}$$

for some $C' > 0$ and $R_0' > 0$ independent of $T \geqslant 1$ and $\varepsilon > 0$. Moreover, it follows from (0.11) (by arguments based on logarithmic convexity) that

$$\mathbf{H}_\varepsilon(\mathbb{A}) \geqslant C'' \left(\log_+ \frac{R_0''}{\varepsilon}\right)^{3/2}, \tag{0.12}$$

which shows that the following dimensions are infinite:

$$\dim_{\mathrm{fract}}(\mathcal{A}^{\mathrm{traj}}|_{(0,T)}) = \dim_{\mathrm{fract}}(\mathbb{A}) = \infty.$$

Our example is based on the counterexample in [12] to the analogue of the Floquet theory for linear elliptic equations with periodic coefficients. This counterexample is of the form

$$\ddot{v} - Av = L_1(t)v + L_2(t)\dot{v},$$

where L_1 and L_2 are periodic with respect to $t \in \mathbb{R}$ and chosen in such a way that there is a non-trivial solution $v\colon \mathbb{R} \to H^2$ decreasing more rapidly than any exponential function as $t \to \pm\infty$ ($\|v(t)\|_2 \leqslant ce^{-t^2}$). To construct this counterexample, we need the conditions

$$L_1 \in C_{\mathrm{per}}(\mathbb{R}, \mathcal{L}(H^{s+r_1}, H^s)) \quad \text{and} \quad L_2 \in C_{\mathrm{per}}(\mathbb{R}, \mathcal{L}(H^{s+r_2}, H^s))$$

for some $r_1 \geqslant 1$, $r_2 \geqslant 0$, and $s \geqslant 0$. On the other hand, in our existence theorem for a trajectory attractor (see (0.5) and (0.7)) it is assumed that $r_1 \leqslant 1$ and $r_2 \leqslant 0$. Thus, our example is exactly at the boundary of the domain of admissible values of the parameters r_1 and r_2.

In § 5 we give a more detailed investigation of the dynamical properties of the trajectory dynamical system corresponding to the example constructed in the previous section. In particular, we show that, in contrast to the dynamical systems generated by ordinary differential equations and by most of the natural evolution partial differential equations in bounded domains, the dynamical system in question has infinite topological entropy. We describe the chaotic nature of this dynamical system by means of a homeomorphic embedding of a Bernoulli scheme with infinitely many symbols. We also note that this type of chaotic behaviour turns out to be very close to the behaviour of dynamical systems generated by evolution partial differential equations in unbounded domains; see [39], [40].

The elliptic boundary-value problem (0.6) on the entire cylinder $\Omega = \mathbb{R} \times \omega$ can be interpreted, in particular, as an equation for finding the equilibria of the corresponding system of reaction-diffusion equations in Ω:

$$\partial_\eta u = \ddot{u} - \gamma\dot{u} + \Delta_x u - f(u, \dot{u}) - g(x), \ \ (t, x) \in \Omega, \ \ \eta > 0, \quad u\big|_{\eta=0} = u^0, \quad (0.13)$$

where t remains a spatial variable, and the role of physical time is played by the variable η. As is known (see [4], [31], [14]), under natural conditions on the non-linear function f and the external force g, this equation has a global attractor $\mathcal{A}^{\text{glob}} \subset W^2_{\text{bd}}(\mathbb{R})$. Moreover, we have the obvious embedding

$$\mathcal{K} \subset \mathcal{A}^{\text{glob}}, \quad (0.14)$$

where $\mathcal{K}$ is the set of essential solutions of the elliptic boundary-value problem (0.6). An analogue of the bounds in (0.10) and (0.11) for the ε-entropy of the global attractor $\mathcal{A}^{\text{glob}}$ was obtained in [10], [38], [14]; for a more detailed discussion of this analogy, see also § 5.

In the next paper we shall study the problem of additional conditions ensuring that the fractal dimension $\dim_{\text{fract}}(\mathbb{A})$ and/or the topological entropy $h_{\text{top}}(\mathbb{S}_h, \mathbb{A})$ are finite. At present this fact is known only if the spectrum of the operator A has a gap ensuring the existence of an essential manifold [28] or if the condition $\gamma \gg \text{id}$ holds, which ensures the uniqueness of a solution of the problem (0.4); see [7], [37], and Remark 2.3 below.

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§ 1. *A priori* estimates. Existence of solutions

In this section we derive some *a priori* estimates for solutions of the problem (0.4) and, using these estimates, we prove the existence of bounded solutions of the problem. To this end, we need the following function spaces.

Definition 1.1. For any $-\infty \leqslant T_1 < T_2 \leqslant +\infty$ and $l \in \mathbb{R}^+$ we define the space

$$W^l(T_1, T_2) \equiv L^2((T_1, T_2), H^l) \cap W^{l,2}((T_1, T_2), H). \quad (1.1)$$

For simplicity, we write $W^l(T)$ below instead of $W^l(T, T+1)$.

We denote by $W^l_{\text{loc}}(\mathbb{R}^+)$ the Fréchet space generated by the seminorms $\|\cdot\|_{W^l(T)}$, $T \in \mathbb{R}^+$. Moreover, we also introduce the space

$$W^l_{\text{bd}}(\mathbb{R}^+) \equiv \Big\{u \in W^l_{\text{loc}}(\mathbb{R}^+) : \|u\|_{l,b} \equiv \sup_{T \in \mathbb{R}^+} \|u\|_{W^l(T)} < \infty\Big\}. \quad (1.2)$$

The spaces $W^l_{\text{loc}}(\mathbb{R})$ and $W^l_{\text{bd}}(\mathbb{R})$ are defined similarly by replacing $T \in \mathbb{R}^+$ by $T \in \mathbb{R}$.

The main result of this section is the following theorem.

Theorem 1.2. *Let the conditions* (0.5) *be satisfied. Then for any* $u_0 \in H^{3/2}$ *there is at least one solution* $u \in W^2_{\mathrm{bd}}(\mathbb{R}^+)$ *of the problem* (0.4). *Moreover, there are constants* $C_*, \alpha > 0$ *and a monotone function* $Q\colon \mathbb{R}^+ \to \mathbb{R}^+$ *such that any solution* $u \in W^2_{\mathrm{bd}}(\mathbb{R}^+)$ *of* (0.4) *satisfies the estimate*

$$\|u\|_{W^2_{\mathrm{bd}}(T)} \leqslant Q(\|u_0\|_{3/2})e^{-\alpha T} + C_* \quad \textit{for} \quad T \geqslant 0. \tag{1.3}$$

Proof. Let us first derive the *a priori* estimate (1.3). We then prove the existence of a solution on the basis of this estimate.

Following [37], we take the inner product in H of the equation (0.4) and a function $\rho(t)u(t)$ and integrate over $t \in [\tau, +\infty)$. Here $\rho\colon \mathbb{R}^+ \to (0,\infty)$ is a weight function such that $\int_0^\infty \rho(t)\,dt < \infty$ and $|\dot\rho(t)| \leqslant \varepsilon\rho(t)$ for any $t \geqslant 0$. Integrating the resulting equality by parts and using the fact that γ is symmetric, we see that

$$\begin{aligned}\int_\tau^\infty \bigl[\|\dot u\|^2 + \|u\|_1^2\bigr]\rho\,dt + \biggl[\frac{\rho}{2}\partial_t\|u\|^2 - \frac{\rho}{2}\langle\gamma u, u\rangle\biggr]_{t=\tau} \\ = \int_\tau^\infty \bigl[-\langle u, F(u,\dot u)\rangle\rho - \langle u, \dot u\rangle\dot\rho + \langle\gamma u, u\rangle\dot\rho/2\bigr]\,dt \\ \leqslant \int_\tau^\infty \biggl[C + \frac12(\|u\|_1^2 + \|\dot u\|^2) + \varepsilon\|u\|\,\|\dot u\| + \varepsilon\|\gamma\|\,\|u\|^2\biggr]\rho\,dt.\end{aligned} \tag{1.4}$$

Here we also used the condition (0.5)(c). By setting $\rho(t) := e^{\varepsilon(t-\tau)}$ and choosing a sufficiently small $\varepsilon > 0$, we get that

$$\frac{d}{d\tau}\|u(\tau)\|^2 \leqslant C' + C'\|u(\tau)\|^2 \quad \text{for} \quad \tau \geqslant 0. \tag{1.5}$$

The Gronwall inequality now implies the first *a priori* estimate for the H-norm of the solution,

$$\|u(t)\|^2 \leqslant e^{C't}\|u_0\|^2 + e^{C't} - 1 \quad \text{for} \quad t \geqslant 0. \tag{1.6}$$

Of course, this bound is useful only for small t, because the right-hand side of (1.6) increases exponentially as $t \to \infty$.

As the next step, we prove an analogue of the estimate (1.3) (for the $W^1(T)$-norm of a solution) in which the right-hand side does not grow as $t \to \infty$. To this end, it is necessary to pay special attention to the initial condition $u_0 = u\big|_{t=0}$. Indeed, by the abstract trace theorem, there is a bounded linear operator $\mathbb{T}\colon H^{3/2} \to W^2(\mathbb{R}^+)$ such that $v = \mathbb{T}u_0$ satisfies the conditions

$$v(0) = u_0, \quad \operatorname{supp} v \subset [0,1], \quad \text{and} \quad \|v\|_{W^2(0)} \leqslant C\|u_0\|_{3/2}. \tag{1.7}$$

Rewriting the equation (0.4) with respect to the new unknown function $w = u - v$, we see that

$$\begin{cases} \ddot w - \gamma\dot w - Aw = F(w+v, \dot w + \dot v) - h(t), \\ w\big|_{t=0} = 0, \end{cases} \tag{1.8}$$

where $h = \ddot{v} - \gamma\dot{v} - Av$. Hence,

$$\operatorname{supp} h \in [0,1], \qquad \|h\|_{L^2(0)} \leqslant C_1\|u_0\|_{3/2}. \tag{1.9}$$

Taking the inner product in H of the equation (1.8) and the function $\rho(t)w(t)$, integrating with respect to $t \in \mathbb{R}^+$, using the boundary condition $w(0) = 0$, and arguing as in the proof of (1.4), we get that

$$\begin{aligned}\int_0^\infty &\big[\|\dot{w}\|^2 + \|w\|_1^2\big]\rho\,dt \\ &= \int_0^\infty \bigg[-\langle w, F(v+w, \dot{v}+\dot{w})\rangle - \langle w, h\rangle + \frac{\dot{\rho}}{2\rho}\big[\langle \gamma w, w\rangle - 2\langle w, \dot{w}\rangle\big]\bigg]\rho\,dt. \end{aligned}\tag{1.10}$$

The first summand on the right-hand side of (1.10), which contains the non-linear function F, requires the most complicated estimate. To estimate it, we recall that the functions $v(t)$ and h are non-zero only for $t \in [0,1]$. Therefore, for $t \in [0,1]$ we must use the weaker estimates (0.5)(b)+(d) of the non-linear function, whereas for $t \geqslant 1$ we can apply the sharper bound in (0.5)(c). Thus, for non-zero v we get that

$$\begin{aligned}-\langle w, F(v+w, \dot{v}+\dot{w})\rangle &= -\langle w, F(v+w, \dot{v}+\dot{w}) - F(v, \dot{v}+\dot{w})\rangle - \langle w, F(v, \dot{v}+\dot{w})\rangle \\ &\leqslant C\|w\|^2 + \frac{1}{2}\|w\|_1^2 + \frac{1}{4\alpha}\|w\|^2 + \alpha\|F(v, \dot{v}+\dot{w})\|^2 \\ &\leqslant C(\alpha)\|w\|^2 + \frac{1}{2}\|w\|_1^2 + \alpha Q_\mu(\|v\|_{1/2}) \\ &\qquad + \alpha\mu\|v\|_2^2 + \alpha C(\|v\|_1^2 + \|\dot{v}+\dot{w}\|^2) \\ &\leqslant C(\alpha)\|w\|^2 + \frac{1}{2}\|w\|_1^2 + Q_1(\alpha)(\|u_0\|_{3/2}) + 2\alpha C\|\dot{w}\|^2.\end{aligned}$$

For $t \geqslant 1$ we can use the sharper bound (0.5)(c), because $v \equiv 0$ in this case, and thus

$$-\langle w, F(v+w, \dot{v}+\dot{w})\rangle = -\langle F(w, \dot{w}), w\rangle \leqslant C_2 + \frac{1}{2}(\|w\|_1^2 + \|\dot{w}\|^2).$$

Substituting these estimates into (1.10), choosing sufficiently small constants α and $\varepsilon := \sup|\dot{\rho}|/\rho$, and using the inequality (1.6) to estimate the expression $C(\alpha)\|w(t)\|^2$ for $t \in [0,1]$, we derive the estimate

$$\int_0^\infty \big[\|\dot{w}\|^2 + \|w\|_1^2\big]\rho\,dt \leqslant \int_0^1 Q_2(\|u_0\|_{3/2})\rho\,dt + \int_0^\infty \bigg[C_2 + \frac{3}{4}\big[\|\dot{w}\|^2 + \|w\|_1^2\big]\bigg]\rho\,dt.$$

Setting $\rho(t) := e^{-\varepsilon|t-T|}$, $T \geqslant 0$, for the weight function in this inequality, we have

$$\begin{aligned}\|w\|_{W^1(T)}^2 = \int_T^{T+1} \big[\|\dot{w}\|^2 + \|w\|_1^2\big]\,dt &\leqslant e^\varepsilon \int_0^\infty \big[\|\dot{w}\|^2 + \|w\|_1^2\big]e^{-\varepsilon|t-T|}\,dt \\ &\leqslant 4e^\varepsilon \sup_{t\in[0,1]} e^{-\varepsilon|t-T|}\,Q_2(\|u_0\|_{3/2}) + e^\varepsilon\frac{8}{\varepsilon}C_2 \leqslant 5Q_2(\|u_0\|_{3/2})e^{-\varepsilon T} + \frac{10C_2}{\varepsilon}\,.\end{aligned}$$

Returning to the variable $u=v+w$ and using the inequality (1.7) for the function v, we prove the following estimate:

$$\|u\|_{W^1(T)} \leqslant Q_3(\|u_0\|_{3/2})e^{-\varepsilon T} + C_3. \tag{1.11}$$

We can now complete the proof of the *a priori* estimate for a solution in the space $W^2(T)$. To this end, we use the regularity of a solution of the equation $\ddot{z}-Az = f$ with Dirichlet conditions on the boundary, an abstract elliptic boundary-value problem. Let us consider the interval $J_T = (\max\{0, T-1\}, T+1)$ and introduce a truncating function $\psi\colon \mathbb{R} \to [0,1]$ such that $\psi(t) = 1$ for $t \in [0,1]$ and $\psi(t) = 0$ for $t \notin [-1,2]$. We also set $\psi_T(t) = \psi(t-T)$ and $w_T = \psi_T w$. Then by (1.8), the function w_T satisfies the equation

$$\begin{cases} \ddot{w}_T - Aw_T = f_T \equiv \psi_T(\gamma\dot{w} + F(u,\dot{u}) + h) + 2\dot{\psi}_T\dot{w} + \ddot{\psi}_T w \quad \text{for} \quad t \in J_T, \\ w_T = 0 \quad \text{for} \quad t \in \partial J_T. \end{cases} \tag{1.12}$$

Applying the theorem on regularity of solutions to the linear elliptic equation (1.12) and using the condition (0.5), we get that

$$\begin{aligned} \|w_T\|^2_{W^2(J_T)} &\leqslant C\big[\|w\|^2_{W^1(J_T)} + \|h\|^2_{L^2(J_T)} + \|\psi_T F(u,\dot{u})\|^2_{L^2(J_T)}\big] \\ &\leqslant Q_4(\|u_0\|_{3/2})e^{-\varepsilon T} + C_4 + \int_{J_T} C\big[Q_\mu(\|u(t)\|_{1/2}) + \mu\psi_T(t)^2\|w(t)\|_2^2\big]\,dt \\ &\leqslant Q_5(\|u_0\|_{3/2})e^{-\varepsilon T} + C_5 + C\mu\|w_T\|^2_{W^2(J_T)}. \end{aligned} \tag{1.13}$$

Here we used the condition (0.5)(d) and the estimate (1.11). Choosing a sufficiently small constant μ in the estimate (1.13), we get that

$$\|w\|_{W^2(T)} \leqslant C\|w_T\|_{W^2(J_T)} \leqslant 2Q_5(\|u_0\|_{3/2})e^{-\varepsilon T} + 2C_5. \tag{1.14}$$

Thus, the *a priori* estimate (1.3) is proved. To complete the proof of Theorem 1.2, it remains to verify the existence of a solution $u \in W^2_{\mathrm{bd}}(\mathbb{R}^+)$ of the problem under consideration. To this end, we shall first construct for any $N \in \mathbb{N}$ a solution $u_N(t)$ of the following auxiliary problem of the form (0.4) on a finite interval:

$$\begin{cases} \ddot{u}_N - \gamma\dot{u}_N - Au = F(u_N, \dot{u}_N) \quad \text{for} \quad t \in (0,N), \\ u_N\big|_{t=0} = u_0, \quad u_N\big|_{t=N} = 0, \end{cases} \tag{1.15}$$

after which we shall obtain a solution of the original problem by passing to the limit as $N \to \infty$.

Literally repeating the proof of the estimate (1.3), we can see that any solution $u_N \in W^2(0,T)$ of the problem (1.15) admits the estimate

$$\|u_N\|_{W^2(T)} \leqslant Q(\|u_0\|_{H^{3/2}})e^{-\alpha T} + C_* \quad \text{for} \quad T \in [0, N-1], \tag{1.16}$$

in which the function Q and the constants C_* and α are as in the estimate (1.3), and hence do not depend on $N \in \mathbb{N}$. We also note that, according to (0.5)(a) and the abstract trace theorem, the non-linear operator $F\colon u\to F(u,\dot u)$ is a compact and continuous map from $W^2(0,N)$ into $W^0(0,N)$ for any *finite* N. Thus, the existence of a solution u_N of the problem (1.15) can be derived in the standard way from the *a priori* estimate (1.16) by means of the Leray–Schauder fixed point theorem (see, for instance, [36], [37]).

To construct a solution u of the original problem (0.4), we note that, by (1.16), the sequence u_N is uniformly bounded in $W^2(0,T)$ for any $T>0$. Therefore, since the space $W^2(0,T)$ is reflexive, we can invoke Cantor's diagonal procedure and assume without loss of generality that u_N converges weakly in the space $W^2_{\mathrm{loc}}(\mathbb{R}_+)$ to some function $u\in W^2_{\mathrm{bd}}(\mathbb{R}^+)$. Passing to the limit as $N\to\infty$ in the equation (1.15), we see that u is a solution of (0.4). Indeed, the passage to the limit in the linear terms of (1.15) is obvious, and the non-linear function presents no additional difficulties due to the condition (0.5)(a) and the strong convergence

$$(u_N,\dot u_N)\to(u,\dot u)\quad\text{in the space}\quad C_{\mathrm{loc}}(H^{3/2-\delta}\times H^{1/2-\delta})$$

for any $\delta>0$. This proves Theorem 1.2.

§ 2. The attractor

This section is devoted to the behaviour of solutions of the problem (0.4) as $t\to\infty$. We note that the conditions (0.5) ensure only the existence of a solution u and not its uniqueness, and hence we cannot use the standard way of constructing a dynamical system corresponding to (0.4). To avoid this difficulty, we use the trajectory approach developed in [8] for the case of evolution problems without the uniqueness property and in [36] for elliptic boundary-value problems.

Let us first define the phase space $\mathcal{K}^+$ of the desired dynamical system as the set of all solutions of (0.4) that are defined on $\mathbb{R}^+$ and bounded as $t\to\infty$, that is,

$$\mathcal{K}^+\equiv\bigl\{u\in W^2_{\mathrm{bd}}(\mathbb{R}^+) : u \text{ is a solution of (0.4) for some } u_0\in H^{3/2}\bigr\}.$$

Since our equation does not depend explicitly on t, it follows that the semigroup $(\mathcal{T}_h)_{h\geqslant0}$ of shifts along the t axis acts on $\mathcal{K}^+$:

$$\mathcal{T}_h\colon \mathcal{K}^+\to\mathcal{K}^+,\quad (\mathcal{T}_hu)(t)\equiv u(t+h),\qquad h\geqslant0. \tag{2.1}$$

Let us endow the set $\mathcal{K}^+$ with the *local* topology induced by embedding $\mathcal{K}^+$ in the Fréchet space $W^2_{\mathrm{loc}}(\mathbb{R}^+)$. Since $W^2_{\mathrm{loc}}(\mathbb{R}^+)$ is metrizable, it follows that $\mathcal{K}^+$ is also a metrizable topological space.

Definition 2.1. The set $\mathcal{K}^+$ endowed with the *local* topology is called the *trajectory phase space* of the problem (0.4), the semigroup $(\mathcal{T}_h)_{h\geqslant0}$ defined in (2.1) is called the *trajectory dynamical system* generated by (0.4), and the global attractor $\mathcal{A}$ of the semigroup $(\mathcal{T}_h)_{h\geqslant0}$ acting on $\mathcal{K}^+$ is called the *trajectory attractor* of (0.4) and is denoted by $\mathcal{A}^{\mathrm{traj}}$.

Remark 2.2. We recall that, by definition, the global attractor $\mathcal{A}$ of the semigroup $\mathcal{T}_h$ in $\mathcal{K}^+$ must attract *bounded* subsets of $\mathcal{K}^+$, although *boundedness* is a metric property and can be *a priori* dependent on the choice of the metric in $W^2_{\mathrm{loc}}(\mathbb{R}^+)$.

However, as can readily be seen by using the estimate (1.3), a set $B \subset \mathcal{K}^+$ is bounded in $W^2_{\mathrm{loc}}(\mathbb{R})$ in our case if and only if B is bounded in $W^2_{\mathrm{bd}}(\mathbb{R}^+)$, and hence the 'bounded' subsets of $\mathcal{K}^+$ are well defined.

Remark 2.3. It should also be noted that the topology in $\mathcal{K}^+$ can be chosen in such a way that if the problem (0.4) has a unique solution that depends continuously on the initial conditions u_0 (for sufficient conditions, see [7] and [37]), then the semigroup $\mathcal{T}_h$ coincides up to a homeomorphism (and even up to a C^1-diffeomorphism under the assumptions of [37]) with the 'ordinary' semigroup $S_h\colon H^{3/2} \to H^{3/2}$, $S_h u_0 = u(h)$.

To state the next theorem, we need the notion of *essential solution*. By definition, this is a solution of (0.4) that is defined for all $t \in \mathbb{R}$ and belongs to the space $W^2_{\mathrm{bd}}(\mathbb{R})$. We also denote by $\mathcal{K} \subset W^2_{\mathrm{bd}}(\mathbb{R})$ the *essential set* of (0.4) consisting of all essential solutions (see [28]).

Theorem 2.4. *Under the above assumptions the equation* (0.4) *has a trajectory attractor* $\mathcal{A} = \mathcal{A}^{\mathrm{traj}}$ *that admits the following description*:

$$\mathcal{A} = \Pi_+\mathcal{K}, \tag{2.2}$$

where $\mathcal{K} \subset W^2_{\mathrm{bd}}(\mathbb{R})$ *stands for the essential set of* (0.4) *and* Π_+ *for the operator restricting a function to the semiaxis* $\mathbb{R}^+$.

Proof. According to the existence theorem for an attractor of an abstract semigroup (see, for instance, [4]), it suffices to show that:

(i) the set $\mathcal{K}^+$ is a complete metric space;
(ii) the operator $\mathcal{T}_h\colon \mathcal{K}^+ \to \mathcal{K}^+$ is continuous for any fixed h;
(iii) the semigroup $\mathcal{T}_h$ has a precompact absorbing set B_0 in $\mathcal{K}^+$, that is, a set such that for any bounded $B \subset \mathcal{K}^+$ there is a number $\tau = \tau(B)$ such that $\mathcal{T}_h B \subset B_0$ for $h \geqslant \tau$.

Let us verify these conditions. Indeed, since the space $W^2_{\mathrm{loc}}(\mathbb{R}^+)$ is complete, the first condition holds because $\mathcal{K}^+$ is closed in $W^2_{\mathrm{loc}}(\mathbb{R}^+)$. This property of $\mathcal{K}^+$ is obvious, because a limit of solutions of (0.4) is also a solution with the corresponding initial condition u_0 (see the end of the proof of Theorem 1.2). It is also clear that the operator $\mathcal{T}_h$ is continuous, because $\mathcal{T}_h$ is a shift along the t axis.

Thus, it remains to construct a precompact absorbing set $B_0 \subset \mathcal{K}^+$. It follows from the bound (1.3) that

$$B_* = \bigl\{u \in W^2_{\mathrm{loc}}(\mathbb{R}^+) : \|u\|_{W^2_{\mathrm{bd}}(\mathbb{R}^+)} \leqslant 2C_*\bigr\} \cap \mathcal{K}^+ \neq \varnothing \tag{2.3}$$

is an absorbing set of $\mathcal{T}_h$ in $\mathcal{K}^+$. Hence, $B_0 = \mathcal{T}_1 B_*$ is also an absorbing set of the semigroup $\mathcal{T}_h$. Let us show that this set is precompact in $W^2_{\mathrm{loc}}(\mathbb{R}^+)$. To this end, we use the regularity of solutions of elliptic equations.

According to Cantor's diagonal procedure, it suffices to show that $B_*|_{[T,T+1]} = \{u|_{[T,T+1]} : u \in B_*\}$ is a precompact set in $W^2(T)$ for any $T \geqslant 1$. Let $(u_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in B_*. Let us define the functions $z_{T,n} = \psi_T u_n$, where the truncating function ψ_T is as in the formula (1.12). Then the function $z_{T,n}$ is a solution of the equation

$$\begin{cases} \ddot{z}_{T,n} - Az_{T,n} = h_{T,n} \equiv \psi_T(\gamma\dot{u}_n + F(u_n, \dot{u}_n)) + 2\dot{\psi}_T\dot{u}_n + \ddot{\psi}_T u_n, \\ z_{T,n}(T-1) = z_{T,n}(T+2) = 0. \end{cases} \tag{2.4}$$

Since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^2_{\mathrm{bd}}(\mathbb{R}^+)$, it follows from the condition (0.5)(a) and the fact that the embedding $W^2((T-1,T+2)) \subset W^{2-\delta}((T-1,T+2))$ is compact that $h_{T,n} \to h_T$ in the space $L^2((T-1,T+2))$ (after passing to a subsequence if necessary). Applying the regularity theorem now to the equation (2.4), we see that $z_{T,n} = \psi_T u_n \to u_T$ in $W^2((T-1,T+2))$. Since $\psi_T(t) = 1$ for $t \in [T, T+1]$, it follows that $u_n \to u$ in $W^2(T)$. This proves Theorem 2.4.

In the second part of this section we give another interpretation of the dynamical system generated by the equation (0.4); this interpretation clarifies the nature of non-uniqueness for solutions of (0.4). To this end, we need an additional condition (0.7) concerning the non-linear function F and some estimates for the difference $v(t) := u_1(t) - u_2(t)$ between arbitrary solutions $u_1, u_2 \in W^2_{\mathrm{bd}}(\mathbb{R}^+)$ of (0.4), which satisfies the abstract linear elliptic equation

$$\ddot{v} - Av = L_1(t)v + L_2(t)\dot{v}, \tag{2.5}$$

where the operators L_j are given by the formulae

$$L_1(t) := \int_0^1 \mathrm{D}_u\, F\big(u_1(t) + sv(t), \dot{u}_1(t) + s\dot{v}(t)\big)\, ds,$$
$$L_2(t) := \gamma + \int_0^1 \mathrm{D}_v\, F\big(u_1(t) + sv(t), \dot{u}_1(t) + s\dot{v}(t)\big)\, ds.$$

Moreover, it follows from the conditions (0.7) and Theorem 1.2 that

$$\|L_1(t)\|_{H^1\to H} + \|L_2(t)\|_{H\to H} \leqslant M, \qquad t \in \mathbb{R}^+, \tag{2.6}$$

where the constant M depends only on $\|u_1(0)\|_{H^{3/2}}$ and $\|u_2(0)\|_{H^{3/2}}$, and hence is uniformly bounded on bounded subsets of $\mathcal{K}^+$.

The following theorem holds for the solutions of (2.5).

Theorem 2.5. *Let $J = (0,T)$ and let $v \in W^2(J)$ satisfy* (2.5). *Suppose that there is an $M > 0$ such that*

$$\sup_{t\in J} \|L_1(t)\|_{H^1\to H} \leqslant M, \qquad \sup_{t\in J} \|L_2(t)\|_{H\to H} \leqslant M. \tag{2.7}$$

(Recall that $\|L_1\|_{H^1\to H} = \|LA^{-1/2}\|_{H\to H}$.) *Let $y(t) := \|v(t)\|_1^2 + \|\dot{v}(t)\|^2$. Then $y\colon J \to \mathbb{R}$ satisfies the following two estimates for any $t \in J$:*

$$y(t) \geqslant y(0)e^{-2M^2t^2 - bt}, \quad \textit{where} \quad b = 4M - 4\langle A^{3/4}v(0), A^{1/4}\dot{v}(0)\rangle / y(0); \tag{2.8}$$

$$y(t) \leqslant [y(0)]^{1-t/T}[y(T)]^{t/T} e^{2M(M+4/T)t(T-t)}. \tag{2.9}$$

The existence of an embedding $W^2((0,T)) \to C([0,T],H^{3/2}) \cap C^1([0,T],H^{1/2})$ ensures that the constant b in (2.8) is correctly defined.

Proof. The proof of this theorem is based on estimates (similar to logarithmic convexity) obtained in [1]. Indeed, let us introduce the function

$$\xi(t) = \begin{pmatrix} \dot v(t) + A^{1/2}v(t) \\ \dot v(t) - A^{1/2}v(t) \end{pmatrix}, \qquad \xi \in W^1_{\mathrm{bd}}(J)^2 \subset C_{\mathrm{bd}}(J, H^{1/2} \times H^{1/2}). \tag{2.10}$$

Then it follows from (2.5) that ξ is a solution of the linear equation

$$\dot\xi(t) - \mathcal{B}\xi(t) = \mathcal{C}(t)\xi(t), \tag{2.11}$$

where

$$\mathcal{B} = \begin{pmatrix} -A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix}, \quad \mathcal{C}(t) = \frac{1}{2}\begin{pmatrix} \mathcal{C}_1(t) & \mathcal{C}_2(t) \\ \mathcal{C}_1(t) & \mathcal{C}_2(t) \end{pmatrix},$$

with $\mathcal{C}_1 = L_2 + L_1A^{-1/2}$ and $\mathcal{C}_2 = L_2 - L_1A^{-1/2}$. Elementary manipulations show that $2y(t) = \|\xi(t)\|_*^2 \equiv \|\xi_1(t)\|^2 + \|\xi_2(t)\|^2$. Moreover, it follows from the condition (2.7) that the operator $\mathcal{C}(t)$ satisfies the estimate

$$\|\mathcal{C}(t)\xi\|_* \leqslant 2M\|\xi\|_* \quad \text{for any} \quad t \in J. \tag{2.12}$$

Suppose that $y(\tau) > 0$ for some $\tau \in J$ (otherwise there is nothing to prove). Then it follows from (2.10) that there is a maximal relatively open interval J_τ in $[0,T]$ for which $\tau \in J_\tau$ and $y(t) > 0$ if $t \in J_\tau$. Let us define a function $\alpha\colon J_\tau \to \mathbb{R}$ by the formula

$$\alpha(t) = \log y(t) - \int_\tau^t \Phi(s)\,ds, \quad \text{where} \quad \Phi(t) = \frac{\langle \mathcal{C}(t)\xi(t), \xi(t)\rangle_*}{y(t)}, \tag{2.13}$$

with $\langle\cdot,\cdot\rangle_*$ the inner product in $H \times H$.

Differentiating (2.13) with respect to t and using the definition of the function $\Phi(t)$ and the equation (2.11), we get after simple algebra that

$$\dot\alpha = \langle \mathcal{B}\xi, \xi\rangle_*/y, \qquad \ddot\alpha = \frac{2}{y}\big[\langle \eta,\eta\rangle_* + \langle \eta, \mathcal{C}\xi\rangle_*\big], \quad \text{where} \quad \eta = \mathcal{B}\xi - \frac{\langle \mathcal{B}\xi,\xi\rangle_*}{2y}\xi.$$

It now follows from the Cauchy–Schwarz inequality and the estimate (2.12) that

$$\ddot\alpha(t) \geqslant -\frac{\|\mathcal{C}(t)\xi(t)\|_*^2}{2y(t)} \geqslant -4M^2 \quad \text{for} \quad t \in J_\tau.$$

This inequality shows that the function $J_\tau \ni t \mapsto \widetilde\alpha(t) = \alpha(t) + 2M^2t^2$ is convex. The convexity, together with the assumption that $y(\tau) > 0$, shows immediately that the interval J_τ coincides with $J = [0,T]$.

Choosing $\tau = 0$ in the definition (2.13) and exponentiating the inequality $\widetilde\alpha(t) \geqslant \widetilde\alpha(0) + \dot{\widetilde\alpha}(0)t$, we get that

$$y(t) \geqslant y(0)e^{-2M^2t^2 + \dot\alpha(0)t + \int_0^t \Phi(s)\,ds}.$$

This inequality, together with the obvious estimates $|\Phi(s)| \leqslant 4M$ and $\dot\alpha(0) \geqslant -b$, proves the estimate (2.8). The estimate (2.9) can be derived similarly from the inequality $\widetilde\alpha(t) \leqslant (1-t/T)\widetilde\alpha(0) + (t/T)\widetilde\alpha(T)$. This completes the proof of Theorem 2.5.

We now recall that $\mathbb{H}^s = H^s \times H^{s-1}$, introduce a linear trace map by the formula

$$\Pi_0 \colon W^2_{\mathrm{loc}}(\mathbb{R}^+) \to \mathbb{H}^{3/2}, \qquad \Pi_0 u = (u(0), \dot u(0)), \tag{2.14}$$

and define the set $\mathbb{K}^+ = \Pi_0 \mathcal{K}^+ \subset \mathbb{H}^{3/2}$. It is clear that the set $\mathbb{K}^+$ is closed. By Theorem 2.5, the map $\Pi_0\big|_{\mathcal{K}^+} \colon \mathcal{K}^+ \to \mathbb{K}^+$ is a bijection. This means that, if the conditions of Theorem 2.5 are satisfied, then the Cauchy problem

$$\begin{cases} \ddot u - \gamma \dot u - Au = F(u, \dot u), \\ u(0) = u_0, \quad \dot u(0) = v_0 \end{cases} \tag{2.15}$$

has a unique solution $u \in W^2_{\mathrm{bd}}(\mathbb{R}^+)$ for any $(u_0, v_0) \in \mathbb{K}^+$, and hence the solving semigroup

$$\mathbb{S}_h \colon \mathbb{K}^+ \to \mathbb{K}^+, \qquad \mathbb{S}_h(u(0), \dot u(0)) = (u(h), \dot u(h)), \tag{2.16}$$

or, equivalently, $\mathbb{S}_h = \Pi_0 \mathcal{T}_h (\Pi_0)^{-1}$, is correctly defined. Moreover, the following corollary to Theorem 2.5 shows that the semigroup (2.16) is Hölder continuous on $\mathbb{K}^+$.

Corollary 2.6. *Let the conditions of Theorem* 1.2 *hold together with the condition* (0.7). *Then the semigroup* (2.16) *is Hölder continuous with arbitrary Hölder exponent α belonging to the interval* $(0,1)$, *that is,*

$$\|\mathbb{S}_h z_1 - \mathbb{S}_h z_2\|_{\mathbb{H}^{3/2}} \leqslant C_\alpha e^{M_\alpha h^2} \|z_1 - z_2\|^{1-\alpha}_{\mathbb{H}^{3/2}}, \qquad z_1, z_2 \in \mathbb{K}^+, \tag{2.17}$$

where the constants C_α and M_α depend on α and $\|z_i\|_{\mathbb{H}^{3/2}}$, $i = 1, 2$.

Proof. Indeed, by setting $t = h$ and $T = h/\alpha$ in the estimate (2.9) and using the fact that the solutions u_1 and u_2 are bounded as $t \to \infty$, we get that

$$\|\mathbb{S}_h z_1 - \mathbb{S}_h z_2\|_{\mathbb{H}^1} \leqslant C'_\alpha e^{M'_\alpha h^2} \|z_1 - z_2\|^{1-\alpha}_{\mathbb{H}^1}, \qquad z_1, z_2 \in \mathbb{K}^+. \tag{2.18}$$

To derive the estimate (2.17) from (2.18), it suffices to note that the abstract regularity theorem applied to the equation (2.5), together with the trace theorem, implies the estimate

$$\|v(h)\|_{H^{3/2}} + \|\dot v(h)\|_{H^{1/2}} \leqslant C\|v\|_{W^2(h)} \leqslant C_1\|v(0)\|_{H^{3/2}} + C_2\|v\|_{W^1(h,h+2)}, \tag{2.19}$$

in which the constants C and C_1 depend only on the operator A, and the constant C_2 depends also on the constant M defined in (2.6). Combining the estimates (2.18) and (2.19), we obtain the inequality (2.17). This proves Corollary 2.6.

Our next problem is to show that the dynamical system $(\mathbb{S}_h)_{h\geqslant 0}$ on the space $\mathbb{K}^+$ and the trajectory dynamical system $(\mathcal{T}_h)_{h\geqslant 0}$ on the space $\mathcal{K}^+$ are topologically conjugate by means of the homeomorphism Π_0. To this end, it suffices of course to prove that $(\Pi_0)^{-1}$ is continuous. To verify the last assertion, we introduce the following metric in the space $\mathcal{K}^+$:

$$d(u_1,u_2) := \sup_{T\in\mathbb{R}^+} e^{-T^4}\|u_1-u_2\|_{W^2(T)}. \tag{2.20}$$

On the one hand, one can readily see that the topology induced in $\mathcal{K}^+$ by the metric (2.20) coincides with the topology induced by the embedding $\mathcal{K}^+ \subset W^2_{\mathrm{loc}}(\mathbb{R}^+)$ (because $\mathcal{K}^+$ is bounded in $W^2_{\mathrm{bd}}(\mathbb{R}^+)$), and on the other hand, it follows from the estimate (2.17) that

$$d(u_1,u_2) \leqslant C''_\alpha\bigl(\|u_1(0)-u_2(0)\|_{H^{3/2}} + \|\dot u_1(0)-\dot u_2(0)\|_{H^{1/2}}\bigr)^{1-\alpha} \tag{2.21}$$

for any $u_1,u_2\in\mathcal{K}^+$ and any $0<\alpha<1$. Here we also used the estimate

$$\|v\|_{W^2(h)} \leqslant C\|v\|_{C(h,h+1;H^{3/2})\cap C^1(h,h+1;H^{1/2})} \tag{2.22}$$

for the solutions of (2.5); this is an elementary consequence of the abstract regularity theorem and the estimate (2.6).

Thus, we have verified that the semigroups (2.16) and (2.1) are indeed topologically conjugate. Therefore, Theorem 2.4 gives the following result.

Theorem 2.7. *Let the conditions of Theorem* 1.2 *hold together with the condition* (0.7). *Then the semigroup $\mathbb{S}_h$ has a global attractor $\mathbb{A}$ in $\mathbb{K}^+$ that can be described as follows*:

$$\mathbb{A} = \Pi_0\mathcal{A}^{\mathrm{traj}}. \tag{2.23}$$

§ 3. Kolmogorov entropy of an attractor and upper bounds for it

In this section we use the notion of Kolmogorov ε-entropy to study the trajectory attractor of (0.4) constructed in the previous section. We begin by recalling the definitions of ε-entropy and fractal dimension. For a detailed exposition of these problems, see [21]. Applications of the notion of Kolmogorov entropy to evolution equations of mathematical physics can be found in [9], [10], [38], [41], [42].

Definition 3.1. Let M be a metric space and let K be a precompact set in M. For any $\varepsilon>0$ we denote by $N_\varepsilon(K)=N_\varepsilon(K,M)$ the least number of ε-balls in M needed to cover the set K (this number is obviously finite, because K is precompact). By definition, the Kolmogorov ε-entropy of the set K in M is the number

$$\mathbf{H}_\varepsilon(K) = \mathbf{H}_\varepsilon(K,M) \equiv \log N_\varepsilon(K). \tag{3.1}$$

Example 3.2. Let K be a compact n-dimensional Lipschitz submanifold of M. Then it is obvious that $C_1(1/\varepsilon)^n \leqslant N_\varepsilon(K) \leqslant C_2(1/\varepsilon)^n$, and hence $\mathbf{H}_\varepsilon(K) = (n+o(1)_{\varepsilon\to0})\log(1/\varepsilon)$.

This example justifies the following definition.

Definition 3.3. By the *fractal* (*entropy*) *dimension* of a set $K \Subset M$ we mean the number

$$\dim_{\text{fract}}(K) = \dim_{\text{fract}}(K, M) = \limsup_{\varepsilon\to 0} \frac{\mathbf{H}_\varepsilon(K)}{\log(1/\varepsilon)}\,. \tag{3.2}$$

We note that the attractor $\mathcal{A}$ of (0.4) constructed above is compact only in the local topology of the space $W^2_{\text{loc}}(\mathbb{R}^+)$, and therefore we consider the ε-entropy of the restriction $\mathcal{A}\big|_{[0,T]}$ of this attractor to a finite interval $[0,T]$ and study the dependence of the entropy on the two parameters T and ε. To this end, we need weighted analogues of the spaces $W^l_{\text{bd}}(\mathbb{R}^+)$.

Definition 3.4. For a given weight function $\phi \in C(\mathbb{R},(0,\infty))$ we define the space

$$W^l_{\text{bd},\phi}(\mathbb{R}^+) \equiv \Big\{ u \in W^l_{\text{loc}}(\mathbb{R}^+) : \|u\|_{l,\text{bd},\phi} \equiv \sup_{T\in\mathbb{R}^+} \{\phi(T)\|u\|_{W^l(T)}\} < \infty \Big\}. \tag{3.3}$$

Following [31], [29], [13], [30], [42], we also introduce a class of admissible weight functions.

Definition 3.5. A function $\phi \in L^\infty_{\text{loc}}(\mathbb{R})$ is said to be a *weight* function of growth order $\mu \geqslant 0$ if there is a $C > 0$ such that

$$\phi(x+y) \leqslant Ce^{\mu|x|}\phi(y), \quad \phi(x) > 0 \tag{3.4}$$

for any $x, y \in \mathbb{R}$.

It can readily be seen that the conditions (3.4) also imply the inequality $\phi(x+y) \geqslant C^{-1}e^{-\mu|x|}\phi(y)$ for any $x, y \in \mathbb{R}$.

The main result of this section is the following theorem.

Theorem 3.6. *Let the conditions of Theorem* 1.2 *hold. Then*

$$\mathbf{H}_\varepsilon\big(\mathcal{A}^{\text{traj}}|_{(0,T)}, W^2_{\text{bd}}((0,T))\big) \leqslant C\left[T + \log_+ \frac{R_0}{\varepsilon}\right] \log_+ \frac{R_0}{\varepsilon} \tag{3.5}$$

in which $\log_+ \gamma := \max\{0, \log\gamma\}$ *and the constants* C *and* R_0 *do not depend on* T *and* ε.

The proof of this theorem is based on a series of intermediate results and will be completed after Lemma 3.9.

Proposition 3.7. *Let* $\mathcal{K} \subset W^2_{\text{bd}}(\mathbb{R})$ *be the essential set of* (0.4) *defined in Theorem* 2.4. *For any* $\alpha > 0$ *there is a constant* $C = C(\alpha)$ *such that the estimate*

$$\|u_1 - u_2\|_{W^2(T)} \leqslant C \sup_{t\in\mathbb{R}} \big\{ e^{-\alpha|T-t|} \|u_1 - u_2\|_{W^0(t)} \big\} \tag{3.6}$$

holds for any two solutions $u_1, u_2 \in \mathcal{K}$ *and any* $T \in \mathbb{R}$.

Proof. Indeed, let us introduce the function $v(t) = u_2(t) - u_1(t)$. Then v satisfies the equation (2.5). Moreover, it follows from the abstract interior estimate applied to this equation that

$$\|v\|_{W^2(T)} \leqslant C\|h_v\|_{W^0(T-1,T+2)}, \quad \text{where} \quad h_v := L_1(t)v + L_2(t)\dot{v}, \tag{3.7}$$

and C does not depend on $T \in \mathbb{R}$. Multiplying (3.7) by $e^{-\alpha|T-M|}$ and taking the supremum over $T \in \mathbb{R}$, we derive the following analogue of the regularity theorem in the weight spaces $W^2_{\mathrm{bd},e^{-\alpha|T-M|}}$:

$$\sup_{T\in\mathbb{R}}\bigl\{e^{-\alpha|T-M|}\|v\|_{W^2(T)}\bigr\} \leqslant C_1 \sup_{T\in\mathbb{R}}\bigl\{e^{-\alpha|T-M|}\|h_v\|_{W^0(T)}\bigr\}, \tag{3.8}$$

where the constant C_1 does not depend on M. Thus, it remains to estimate the right-hand side of (3.8). To this end, we note that the condition (0.5)(a) and the boundedness of $\mathcal{K}$ in $W^2_{\mathrm{bd}}(\mathbb{R})$ imply the estimate

$$\|L_1(t)\|_{H^{3/2}\to H} + \|L_2(t)\|_{H^{1/2}\to H} \leqslant R_*, \tag{3.9}$$

where R_* does not depend on t and $u_1, u_2 \in \mathcal{K}$. Hence, by (3.9) and the interpolation inequalities (see [35]) we have

$$\|h_v(t)\|_{W^0(T)} \leqslant C_2 R_* \|v\|_{W^{3/2}(T)} \leqslant \mu\|v\|_{W^2(T)} + C_\mu \|v\|_{W^0(T)}, \tag{3.10}$$

where $\mu > 0$ is arbitrary and $C_\mu > 0$ is a constant depending on μ and independent of T and $u_1, u_2 \in \mathcal{K}$. Choosing the value $\mu = 1/(2C_1)$ in the inequality (3.10) and substituting it into (3.8), we see that

$$\sup_{T\in\mathbb{R}}\bigl\{e^{-\alpha|T-M|}\|v\|_{W^2(T)}\bigr\} \leqslant 2C_1C_\mu \sup_{T\in\mathbb{R}}\bigl\{e^{-\alpha|T-M|}\|v\|_{W^0(T)}\bigr\}. \tag{3.11}$$

The obvious inequality

$$\|v\|_{W^2(M)} \leqslant e^{-\alpha} \sup_{T\in\mathbb{R}}\Bigl\{e^{-\alpha|T-M|}\|v\|_{W^2(T)}\Bigr\} \tag{3.12}$$

completes the proof of Proposition 3.7.

Let us now consider the family of weight functions

$$\phi_R(t) = \begin{cases} 1 & \text{for } |t| < R, \\ e^{R-|t|} & \text{for } |t| \geqslant R, \end{cases} \tag{3.13}$$

where $R \in \mathbb{R}^+$. These functions certainly satisfy the inequality (3.4) with the constants $\mu = 1$ and $C_\phi = 1$, which are thus independent of R.

Corollary 3.8. *Let $\mathcal{K} \subset W^2_{\mathrm{bd}}(\mathbb{R})$ be the essential set of* (0.4). *Then there is a constant $C_{\mathcal{K}} > 0$ such that the estimate*

$$\|u_1 - u_2\|_{W^2_{\mathrm{bd},\phi_R}(\mathbb{R})} \leqslant C_{\mathcal{K}} \|u_1 - u_2\|_{L^2_{\mathrm{bd},\phi_R}(\mathbb{R})} \tag{3.14}$$

holds for any $u_1, u_2 \in K$ and $R > 0$.

Indeed, setting $\alpha = 2$ in the inequality (3.6), multiplying it by $\phi_R(T)$, and passing to the supremum $\sup_{T\in\mathbb{R}}$, we obtain the estimate (3.14) after simple manipulations (see [42]). Moreover, since the functions ϕ_R satisfy the condition (3.4) uniformly with respect to $R > 0$, it follows that the constant $C_{\mathcal{K}}$ in (3.14) is also independent of R.

We note that the weight functions ϕ_R are chosen in such a way that

$$\|v|_{(0,R)}\|_{W^2_{\rm bd}((0,R))} \leqslant \|v\|_{W^2_{{\rm bd},\phi_R}(\mathbb{R})} \tag{3.15}$$

for any $v \in W^2_{\rm bd}(\mathbb{R})$, and hence

$$\mathbf{H}_\varepsilon\big(\mathcal{A}^{\rm traj}|_{(0,R)}, W^2_{\rm bd}((0,R))\big) \leqslant \mathbf{H}_\varepsilon(\mathcal{K}, W^2_{{\rm bd},\phi_R}(\mathbb{R})). \tag{3.16}$$

Thus, following [42], to estimate the entropy of the set $\mathcal{A}\big|_{(0,R)}$, we shall find an upper bound for the entropy of the essential set $\mathcal{K}$ of (0.4) in the weighted spaces $W^2_{{\rm bd},\phi_R}(\mathbb{R})$. To this end, we need the following lemma, whose proof is based on the estimate (3.14) and the fact that $W^2_{\rm bd}(\mathbb{R})$ is compactly embedded in $L^2_{{\rm bd},\phi_R}(\mathbb{R})$.

Lemma 3.9. *The entropy of the essential set $\mathcal{K}$ of* (0.4) *admits the recursion estimate*

$$\mathbf{H}_{\varepsilon/2}(\mathcal{K}, W^2_{{\rm bd},\phi_R}(\mathbb{R})) \leqslant L\left[R + 1 + \log_+ \frac{R_0'}{\varepsilon}\right] + \mathbf{H}_\varepsilon(\mathcal{K}, W^2_{{\rm bd},\phi_R}(\mathbb{R})), \tag{3.17}$$

in which the constants L and R_0' do not depend on $R > 0$ and $\varepsilon > 0$.

Proof. Let $\{B(u^i, \varepsilon, W^2_{{\rm bd},\phi_R}) : i = 1, \dots, N_\varepsilon\}$ be an ε-covering of the set $\mathcal{K}$ (here and henceforth, $B(v, \mu, X)$ stands for the ball of radius μ centred at a point v with respect to the metric of the space X). We note that $\mathcal{K} \cap B(u^i, C\varepsilon, W^2_{{\rm bd},\phi_R}(\mathbb{R}))$ is a compact set in $L^2_{{\rm bd},\phi_R}(\mathbb{R})$, and hence each of these sets can be covered by finitely many $\varepsilon/(2C_{\mathcal{K}})$-balls $\{B(u^{i,j}, \varepsilon/(2C_{\mathcal{K}}), L^2_{{\rm bd},\phi_R}(\mathbb{R})),\ j = 1, \dots, \mathcal{M}_i(\varepsilon)\}$, where $C_{\mathcal{K}}$ is the same as in (3.14) and

$$\mathcal{M}_i(\varepsilon) := N_{\varepsilon/(2C_{\mathcal{K}})}\big(\mathcal{K} \cap B(u^i, \varepsilon, W^2_{{\rm bd},\phi_R}(\mathbb{R})), L^2_{{\rm bd},\phi_R}(\mathbb{R})\big). \tag{3.18}$$

It now follows from the estimate (3.14) that the system $\{B(u^{i,j}, \varepsilon/2, W^2_{{\rm bd},\phi_R}(\mathbb{R})) :$ $i = 1, \dots, N_\varepsilon,\ j = 1, \dots, \mathcal{M}_i(\varepsilon)\}$ of balls is an $\varepsilon/2$-covering of $\mathcal{K}$, which leads to the recursion estimate

$$\mathbf{H}_{\varepsilon/2}\big(\mathcal{K}, W^2_{{\rm bd},\phi_R}(\mathbb{R})\big) \leqslant \max_{i=1,\dots,N_\varepsilon} \log \mathcal{M}_i(\varepsilon) + \mathbf{H}_\varepsilon\big(\mathcal{K}, W^2_{{\rm bd},\phi_R}(\mathbb{R})\big). \tag{3.19}$$

Thus, it remains to estimate the quantities $\mathcal{M}_i(\varepsilon)$ introduced in (3.18). To this end, we note that $\|u\|_{W^2_{\rm bd}(\mathbb{R})} \leqslant C_*$ for any $u \in \mathcal{K}$ by (1.3), and hence

$$\|u\|_{W^2_{\phi_R}(T)} \leqslant \frac{\varepsilon}{4C_{\mathcal{K}}} \quad \text{for} \quad |T| \geqslant T_\varepsilon \equiv R + \log_+ \frac{4C_* C_{\mathcal{K}}}{\varepsilon}\,. \tag{3.20}$$

Therefore,

$$\begin{aligned} \mathcal{M}_i(\varepsilon) &\leqslant N_{\varepsilon/(4C_{\mathcal{K}})}\big(\mathcal{K} \cap B(u^i, \varepsilon, W^2_{{\rm bd},\phi_R}(\mathbb{R})), L^2_{{\rm bd},\phi_R}((-T_\varepsilon, T_\varepsilon))\big) \\ &\leqslant N_{\varepsilon/(4C_{\mathcal{K}})}\big(B(u^i, \varepsilon, W^2_{{\rm bd},\phi_R}((-T_\varepsilon, T_\varepsilon))), L^2_{{\rm bd},\phi_R}((-T_\varepsilon, T_\varepsilon))\big) \\ &\leqslant N_{1/(4C_{\mathcal{K}})}\big(B(0, 1, W^2_{{\rm bd},\phi_R}((-T_\varepsilon, T_\varepsilon))), L^2_{{\rm bd},\phi_R}((-T_\varepsilon, T_\varepsilon))\big). \end{aligned} \tag{3.21}$$

In the first inequality we used the estimate of the 'tails' in (3.20), in the second inequality we deleted the symbol $\mathcal{K}\cap$, thus increasing the value of $N_{\varepsilon/(4C_{\mathcal{K}})}(\dots)$, and in the third inequality we used the invariance of the family of balls in a Banach space under homotheties and translations.

Thus, it remains to estimate the entropy of the embedding operator

$$W^2_{\mathrm{bd},\phi_R}((-T,T)) \subset L^2_{\mathrm{bd},\phi_R}((-T,T)). \tag{3.22}$$

To this end, we introduce a smooth analogue $\psi_R \in C^\infty(\mathbb{R})$ of the weight function ϕ_R such that

$$\max\{|\dot\psi_R(t)|, |\ddot\psi_R(t)|\} \leqslant \psi_R(t) \quad \text{and} \quad C'\phi_R(t) \leqslant \psi_R(t) \leqslant C''\phi_R(t),$$

where the constants C' and C'' do not depend on R. In this case it can readily be seen that the map $\mathbb{F}_R\colon u \to \psi_R^{1/2}u$ realizes a linear isomorphism of the Banach pairs

$$\mathbb{B}(T) = (W^2_{\mathrm{bd}}((-T,T)), L^2_{\mathrm{bd}}((-T,T))),$$
$$\mathbb{B}_R(T) = (W^2_{\mathrm{bd},\phi_R}((-T,T)), L^2_{\mathrm{bd},\phi_R}((-T,T))).$$

Moreover,

$$\|\mathbb{F}_R\|_{\mathbb{B}(T)\to\mathbb{B}_R(T)} + \|\mathbb{F}_R^{-1}\|_{\mathbb{B}_R(T)\to\mathbb{B}(T)} \leqslant C_2, \tag{3.23}$$

where C_2 does not depend on T and R (see [42]). Thus,

$$\begin{aligned}\log \mathcal{M}_i(\varepsilon) &\leqslant \mathbf{H}_{1/(4C_{\mathcal{K}})}\bigl(B(0,1,W^2_{\mathrm{bd},\phi_R}((-T_\varepsilon,T_\varepsilon))), L^2_{\mathrm{bd},\phi_R}((-T_\varepsilon,T_\varepsilon))\bigr)\\ &\leqslant \mathbf{H}_{1/(4C_{\mathcal{K}}C_2^2)}\bigl(B(0,1,W^2_{\mathrm{bd}}((-T_\varepsilon,T_\varepsilon))), L^2_{\mathrm{bd}}((-T_\varepsilon,T_\varepsilon))\bigr).\end{aligned} \tag{3.24}$$

The obvious estimate

$$\begin{aligned}&\mathbf{H}_\mu\bigl(B(0,1,W^2_{\mathrm{bd}}((-T,T))), L^2_{\mathrm{bd}}((-T,T))\bigr)\\ &\qquad\leqslant (2T+1)\mathbf{H}_{\mu/2}\bigl(B(0,1,W^2_{\mathrm{bd}}((-1,1))), L^2_{\mathrm{bd}}((-1,1))\bigr)\end{aligned}$$

now completes the proof of Lemma 3.9.

Completion of the proof of Theorem 3.6. By (1.3), $\mathbf{H}_{C_*}(\mathcal{K}, W^2_{\mathrm{bd},\phi_R}(\mathbb{R})) = 0$ for any $R \geqslant 0$. Iterating the estimate (3.17) k times, we get that

$$\mathbf{H}_{2^{-k}C_*}\bigl(\mathcal{K}, W^2_{\mathrm{bd},\phi_R}(\mathbb{R})\bigr) \leqslant L\biggl(R + 1 + \log_+ \frac{R_0' 2^{k-1}}{C_*}\biggr)k \quad \text{for any} \quad k \in \mathbb{N}. \tag{3.25}$$

For a given $\varepsilon > 0$ let k be such that $2^{-k}C_* \leqslant \varepsilon < 2^{-k+1}C_*$. Substituting this k in (3.25), we obtain the estimate (3.5). This completes the proof of Theorem 3.6.

We conclude this section by applying Theorem 3.6 to the study of the elliptic boundary-value problem (0.6) in a cylindrical domain $\Omega_+ = \mathbb{R}^+ \times \omega$, where $\omega \Subset \mathbb{R}^n$ is a smooth bounded domain. For simplicity, we formulate the conditions on the non-linear function $f(u,\dot u)$ only in the case $n \leqslant 3$. These conditions are of the following form:

$$\begin{cases} (1)\ \ f \in C^1(\mathbb{R}^k \times \mathbb{R}^k, \mathbb{R}^k), \\ (2)\ \ f(u,v)\cdot u \geqslant -C, \ \ \mathrm{D}_u f(u,v) \geqslant -C, \\ (3)\ \ |f(u,v)| + |\mathrm{D}_u f(u,v)| \leqslant C(1+|u|^{k_1})(1+|v|^{k_2}), \\ (4)\ \ |\mathrm{D}_v f(u,v)| \leqslant C(1+|u|^{k_1}), \end{cases} \tag{3.26}$$

where $0 \leqslant k_2 < 1$ and $0 \leqslant k_1 < \dfrac{n+3}{n-1}(1-k_2)$. We note that if f does not depend explicitly on $\partial_t u$ (and hence $k_2 = 0$), then we obtain the standard conditions on the growth of the non-linear function $f(u)$ that are stated in [2].

Corollary 3.10. *Let* $\gamma = \gamma^* \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$, *let* $g \in L^2(\omega)$, *let* $n \leqslant 3$, *and let the non-linear function* f *satisfy the conditions* (3.26). *Then the equation* (0.6) *satisfies all the conditions of Theorem* 3.6, *and hence has a trajectory attractor* $\mathcal{A}^{\mathrm{traj}}$ *whose entropy admits an upper estimate of the form* (3.5).

Proof. Let us rewrite the equation (0.6) in the abstract form of (0.4). Indeed, in this case we have $H = L^2(\omega)$, $A = -\Delta_x$ (with Dirichlet boundary conditions), and $F(u,v) := f(u,v) + g$. Hence, we must prove that the operator $F(u,v)$ thus defined satisfies the conditions (0.5). To this end, we recall that the space $H^s := D((-\Delta_x)^{-s/2})$ satisfies the embedding $H^s \subset H^s(\omega)$ for $s \geqslant 0$, where the $H^s(\omega)$ are the classical Sobolev spaces (see, for instance, [35]). We also note that for $n \leqslant 3$ we have the embedding $H^{3/2} \subset L^p$ for any $p < \infty$, and hence for any fixed $p < \infty$ there is a $\delta = \delta(p) > 0$ such that $H^{3/2-\delta} \subset L^p$.

Let us verify the condition (0.5)(a). Indeed, it follows from (3.26)(3) and the Hölder inequality that

$$\begin{aligned} \|\mathrm{D}_u F(u,v)\theta\|_{L^2}^2 &\leqslant C\|(1+|u|^{2k_1})(1+|v|^{2k_2})|\theta|^2\|_{L^1} \\ &\leqslant (1+\|v\|_{L^2}^{2k_2})(1+\|u\|_{L^{4k_1/(1-k_2)}}^{2k_1})\|\theta\|_{L^{4/(1-k_2)}}^2 \\ &\leqslant Q(\|u\|_{H^{3/2-\delta}} + \|v\|_{L^2})\|\theta\|_{H^{3/2-\delta}}^2 \end{aligned}$$

if $\delta > 0$ is sufficiently small. The estimate

$$\|\mathrm{D}_v F(u,v)\theta\| \leqslant Q(\|u\|_{H^{3/2-\delta}} + \|v\|_{H^{1/2-\delta}})\|\theta\|_{H^{1/2-\delta}}$$

can be proved similarly. The operators F, $\mathrm{D}_u F$, and $\mathrm{D}_v F$ are continuous, as can be proved in the standard way from the condition (3.26)(1). Thus, the condition (0.5)(a) is verified. The conditions (0.5)(b) and (0.5)(c) are immediate consequences of the assumption (3.26)(2). Therefore, it remains to derive the condition (0.5)(d) from (3.26)(3). Indeed, by the Hölder inequality and the Sobolev embedding theorem we have

$$\|F(u,v)\|_{L^2}^2 \leqslant C(1+\|v\|_{L^2}^2) + \|u\|_{L^p}^p \leqslant C(1+\|v\|_{L^2}^2) + C_1\|u\|_{H^s(\omega)}^p, \tag{3.27}$$

where $p := 2k_1/(1-k_2)$ and $1/p = 1/2 - s/n$. If $s \leqslant 1/2$, then the condition (0.5)(d) follows immediately from (3.27). Suppose that $s \geqslant 1/2$. Then

$$\|u\|_{H^s(\omega)} \leqslant C_2\|u\|_{H^2(\omega)}^{(2s-1)/3}\|u\|_{H^{1/2}(\omega)}^{(4-2s)/3} \tag{3.28}$$

by the interpolation inequality. We note that the condition (0.5)(d) follows from (3.27) and (3.28) if $p(2s-1)/3<2$. Therefore, it suffices to verify the last inequality. We note that $p<2(n+3)/(n-1)$ by our assumptions, and hence $s<2n/(n+3)$, which implies that $p(2s-1)/3<[2(n+3)/3(n-1)]\cdot[3(n-1)/(n+3)]=2$. Thus, the condition (0.5)(d) is also verified. This completes the proof of Corollary 3.10.

§ 4. Entropy of an attractor: an example of a sharp lower bound

In this section we construct an example of an equation (0.4) for which the entropy estimate (3.5) is sharp in a sense. Moreover, the fractal dimension of the corresponding attractor $\mathbb{A}$ turns out to be infinite. This example makes heavy use of the counterexample (constructed in [27], [12]) to the analogue of the Floquet theory for abstract elliptic boundary-value problems in cylindrical domains.

Let us consider the following linear elliptic equation in the strip $\mathbb{R}\times\omega\equiv\mathbb{R}\times(0,\pi)$:

$$\ddot{u}+\partial_x^2u=L_1(t)u+L_2(t)\dot{u},\qquad u\big|_{\mathbb{R}\times\partial\omega}=0. \tag{4.1}$$

Here $L_1(t)$ and $L_2(t)$ stand for linear operators that are T-periodic with respect to t. The main idea of the above counterexample is to construct operators L_1 and L_2 in such a way that the equation (4.1) admits a solution u that decreases more rapidly than any exponential function as $t\to\pm\infty$. This fact certainly contradicts the Floquet theory; if this theory were valid, then any solution would be a linear combination of products of exponential functions and periodic functions. More precisely, the following result was proved in the monograph [12] (see Appendix A, pp. 261–262).

Theorem 4.1. *There are T-periodic operators $L_j(t)$ satisfying the conditions*

$$L_1\in C^\infty_{\mathrm{bd}}(\mathbb{R},\mathcal{L}(H^{s+1},H^s))\ \ \textit{and}\ \ L_2\in C^\infty_{\mathrm{bd}}(\mathbb{R},\mathcal{L}(H^{s},H^s))\ \ \textit{for any}\ \ s\in\mathbb{R} \tag{4.2}$$

and the condition $L_j(t+T)=L_j(t)$ for any $t\in\mathbb{R}$ and such that the equation (4.1) *has a solution $u\in W^2(\mathbb{R})$ which admits the estimate*

$$C_1e^{-\beta t^2}\leqslant\|u(t)\|_{L^2(\omega)}\leqslant C_2e^{-t^2}\quad \textit{for}\quad t\in\mathbb{R} \tag{4.3}$$

for some positive constants β, C_1, and C_2. Moreover, the Fourier coefficients $u_n(t)=\int_0^\pi u(t,x)\sin(nx)\,dx$ of this solution are compactly supported in $\mathbb{R}$. (*Here the symbol H^s stands for the scale of spaces generated by the Laplacian in ω with Dirichlet boundary conditions.*)

We note that not only the function u but also the functions obtained from u by the shifts by kT, $k\in\mathbb{Z}$ (that is, $\mathcal{T}_{kT}u\colon t\mapsto u(t+kT)$) are linearly independent solutions of the equation (4.1), and hence the kernel of the linear elliptic operator defined in $L^2(\mathbb{R}\times\omega)$ by (4.1) is infinite-dimensional. We introduce the set

$$\mathbb{L}\equiv\biggl\{v=\sum_{k\in\mathbb{Z}}a_k\mathcal{T}_{kT}u:a_k\in\mathbb{R},\ \sup_{k\in\mathbb{Z}}|a_k|<\infty\biggr\}. \tag{4.4}$$

The following assertion holds.

Lemma 4.2. *Let u be the solution constructed in Theorem* 4.1 *and let* $\mathbb{L}$ *be defined by the formula* (4.4). *Assume also that the support of the first Fourier coefficient* $u_1\colon t\mapsto\int_0^\pi u(t,x)\sin x\,dx$ *of the function u satisfies the condition* $\varnothing\neq\operatorname{supp}u_1\subset[r,r+N]$ *for some* $r\in\mathbb{R}$ *and* $N\in(0,T)$. *Then*

$$C_1\sup_{k\in\mathbb{Z}}|a_k|\leqslant\|v\|_{W^2_{\mathrm{bd}}(\mathbb{R})}\leqslant C_2\sup_{k\in\mathbb{Z}}|a_k| \tag{4.5}$$

for any $v\in\mathbb{L}$.

Proof. Indeed, the left-hand inequality in (4.5) follows from the fact that the sets $\operatorname{supp}\mathcal{T}_{kT}u_1$ are disjoint for distinct values of k.

To prove the right-hand inequality, we note that the estimate (4.3) and the regularity of solutions of elliptic equations imply the estimate

$$\|u\|_{W^2(\tau)}\leqslant C_2e^{-\tau^2/2} \tag{4.6}$$

for the solution u. Thus,

$$\|v\|_{W^2(\tau)}\leqslant\sum_{k\in\mathbb{Z}}|a_k|\|u\|_{W^2(\tau-kT)}\leqslant\sum_{k\in\mathbb{Z}}|a_k|C_2e^{-(\tau-kT)^2/2}\leqslant C\sup_{k\in\mathbb{Z}}|a_k|. \tag{4.7}$$

This proves Lemma 4.2.

Replacing the period T by lT ($l\in\mathbb{N}$) if necessary, we can assume that the condition of Lemma 4.2 concerning the support $\operatorname{supp}u_1$ holds for the function u introduced in Theorem 4.1.

Lemma 4.3. *Let the conditions of Lemma* 4.2 *be satisfied and let*

$$\mathbb{L}_R=\{v\in\mathbb{L}:\|v\|_{W^2_{\mathrm{bd}}(\mathbb{R})}\leqslant R\}. \tag{4.8}$$

Then there are positive constants $C(R)$, M_0, *and* ε_0 *such that the lower estimate*

$$\mathbf{H}_\varepsilon\big(\mathbb{L}_R|_{(0,M)},W^2_{\mathrm{bd}}((0,M))\big)\geqslant C(R)M\log\frac{1}{\varepsilon} \tag{4.9}$$

holds for $M\geqslant M_0$ *and* $\varepsilon\in(0,\varepsilon_0)$.

Proof. Without loss of generality we can assume that $\operatorname{supp}u_1\subset[0,T]$. Let $\mathbb{L}_R^k=\{v\in\mathbb{L}_R:v(t)=\sum_{i=1}^k a_ku(t-iT)\}$. Then the estimate

$$\|v^1-v^2\|_{W^2_b((0,(k+1)T))}\geqslant\|v^1-v^2\|_{C([0,(k+1)T],H)}\geqslant K\sup_{i=1,\dots,k}|a_i^1-a_i^2| \tag{4.10}$$

holds for any $v^1,v^2\in\mathbb{L}_R^k$, where $K=\|u_1\|_{C([0,T])}>0$.

For any sufficiently large M we choose $k=k_M$ as the solution of the inequality $(k+1)T>M\geqslant kT$. In this case the right-hand estimate in (4.5) ensures that the function $v\in\mathbb{L}$ belongs to $\mathbb{L}_R$ if all the coefficients satisfy the inequality

$|a_k| \leqslant R/C_2$. Moreover, by (4.10), two function v_1 and v_2 in $\mathbb{L}_R^k$ are ε-distinguishable if $|a_i^1 - a_i^2| \geqslant \varepsilon/K$ for at least one index $i \in \{0, \dots, k\}$. Hence,

$$N_{\varepsilon/2}\big(\mathbb{L}_R^k, W_{\mathrm{bd}}^2([0, M])\big) \geqslant \left(2\left[\frac{KR}{C_2\varepsilon}\right] + 1\right)^k. \tag{4.11}$$

Since $k \sim M$, the inequality (4.11) proves Lemma 4.3.

We can now construct an equation (of the form (0.4)) satisfying the conditions of Theorem 3.6 and such that its attractor $\mathcal{A}$ contains the set $\Pi_+\mathbb{L}_R$ for sufficiently small R. Then Lemma 4.3 gives the needed lower bound for the ε-entropy of the attractor.

Let the operators L_1 and L_2 be the same as in Theorem 4.1. Since these operators are T-periodic (for simplicity, we assume below that $T = 2\pi$), it follows that there are smooth families of operators $\widehat{L}_1 \in C^\infty(\mathbb{R}^2, \mathcal{L}(H^{s+1}, H^s))$ and $\widehat{L}_2 \in C^\infty(\mathbb{R}^2, \mathcal{L}(H^s, H^s))$ such that

$$\widehat{L}_1(w_1, w_2) = \widehat{L}_2(w_1, w_2) = 0 \quad \text{for} \quad |w_1|^2 + |w_2|^2 \geqslant 2 \tag{4.12}$$

and

$$L_1(t) = \widehat{L}_1(\cos t, \sin t), \qquad L_2(t) = \widehat{L}_2(\cos t, \sin t). \tag{4.13}$$

Moreover, the pair $w(t) = (\cos t, \sin t)$ can be obtained as a solution of the system of ordinary differential equations

$$\ddot{w} - w = \frac{2(|w|^2 w - 3w)}{1 + |w|^4}\,. \tag{4.14}$$

Let $\phi_R(z)\colon \mathbb{R} \to \mathbb{R}$ be a truncating function such that $\phi_R = 1$ for $|z| \leqslant R^2$ and $\phi_R = 0$ for $|z| \geqslant 2R^2$. We consider the system

$$\begin{cases} \ddot{w} - w = 2(|w|^2 w - 3w)/(1 + |w|^4), \\ \ddot{u} + \partial_x^2 u = \phi_R\big(\|u\|^2_{H^1(\omega)} + \|\dot{u}\|^2_{L^2(\omega)}\big)\big[\widehat{L}_1(w_1, w_2)u + \widehat{L}_2(w_1, w_2)\dot{u}\big]. \end{cases} \tag{4.15}$$

Then, on the one hand, the embedding $W_{\mathrm{bd}}^2(\mathbb{R}) \subset C_{\mathrm{bd}}(\mathbb{R}, H^1) \cap C_{\mathrm{bd}}^1(\mathbb{R}, H)$ implies the inequality

$$\|u(t)\|^2_{H^1} + \|\dot{u}(t)\|^2_{L^2} \leqslant P^2 \|u\|^2_{W^2_{\mathrm{bd}}(\mathbb{R})} \tag{4.16}$$

for some positive constants P, and hence the essential set $\mathcal{K}$ of this system contains the product $\{(\cos t, \sin t)\} \times \mathbb{L}_{R/P}$ as a subset. On the other hand, this equation satisfies (0.5) and (0.7) with $H = \mathbb{R}^2 \times L^2(\omega)$, $A = \mathrm{diag}\{1, 1, -\partial_x^2\}$, $\gamma = 0$, and

$$F = \begin{pmatrix} 2(|w|^2 w - 3w)/(1 + |w|^4) \\ \phi_R\big(\|u\|^2_{H^1(\omega)} + \|\dot{u}\|^2_{L^2(\omega)}\big)\big[\widehat{L}_1(w_1, w_2)u + \widehat{L}_2(w_1, w_2)\dot{u}\big] \end{pmatrix}.$$

Indeed, it follows from the definition of the operators $\widehat{L}_i$, $i = 1, 2$, that

$$\|\widehat{L}_1(w_1, w_2)u\|_{L^2} + \|\widehat{L}_2(w_1, w_2)\dot{u}\|_{L^2} \leqslant C(\|u\|_{H^1} + \|\dot{u}\|_{L^2}),$$

where the constant C does not depend on w, and hence $\|F(w,u,\dot u)\|_{L^2}\leqslant C'_R$, where C'_R does not depend on w, u, and $\dot u$. Thus, the conditions (0.5)(c) and (0.5)(d) are verified. Similarly, it follows from the smoothness of $\widehat L_i$ with respect to w and the choice of the truncating function ϕ_R that

$$\|\,\mathrm{D}_w\,F(w,u,\dot u)\|_{\mathbb{R}^2\to H}+\|\,\mathrm{D}_u\,F(w,u,\dot u)\|_{H^1\to H}+\|\,\mathrm{D}_{\dot u}\,F(w,u,\dot u)\|_{H^0\to H}\leqslant C''_R,$$

where C''_R does not depend on w, u, and $\dot u$. Therefore, the conditions (0.5)(a) and (0.7) are also satisfied. It remains to note that the condition (0.5)(b) follows immediately from the last estimate.

Thus, we have proved that the non-linear function $F(w,u,\dot u)$ indeed satisfies (0.5) and (0.7), and hence the equation (4.15) satisfies the assumptions of Theorems 2.7 and 3.6. In particular, combining Theorem 3.6 and Lemma 4.3, we obtain the following two-sided estimate for the ε-entropy of the attractor.

Theorem 4.4. *The equation* (4.15) *has the trajectory attractor* $\mathcal{A}^{\mathrm{traj}}$, *and there are positive constants* T_0, ε_0, C_1, *and* C_2 *such that the* ε*-entropy of* $\mathcal{A}^{\mathrm{traj}}$ *satisfies*

$$C_1T\log\frac{1}{\varepsilon}\leqslant \mathbf{H}_\varepsilon\big(\mathcal{A}^{\mathrm{traj}}|_{(0,T)},W^2_{\mathrm{bd}}((0,T))\big)\leqslant C_2\bigg(T+\log\frac{1}{\varepsilon}\bigg)\log\frac{1}{\varepsilon}\tag{4.17}$$

for $T\geqslant T_0$ *and* $\varepsilon\in(0,\varepsilon_0)$.

We note that the left-hand inequality in (4.17) is sharp for $T\geqslant\log(1/\varepsilon)$ but is far from being optimal for $T\ll\log(1/\varepsilon)$. In particular, (4.17) gives no information concerning the entropy of the global attractor $\mathbb{A}$ on the section. The next theorem gives a lower bound for the entropy of the global attractor provided that the trajectory attractor satisfies the estimate (4.17).

Theorem 4.5. *Let the conditions of Theorem* 2.7 *hold and let the* ε*-entropy of the attractor* $\mathcal{A}^{\mathrm{traj}}$ *of* (0.4) *satisfy* (4.17). *Then there are positive constants* C *and* ε_0 *such that the entropy of the global attractor* $\mathbb{A}$ *constructed in Theorem* 2.7 *admits the estimate*

$$\mathbf{H}_\varepsilon(\mathbb{A},\mathbb{H}^{3/2})\geqslant C\bigg(\log\frac{1}{\varepsilon}\bigg)^{3/2}\quad \textit{for}\quad \varepsilon\in(0,\varepsilon_0).\tag{4.18}$$

Proof. Indeed, since the solutions belonging to $\mathcal{A}^{\mathrm{traj}}$ are uniformly bounded, that is, $\|(u(t),\dot u(t))\|_{\mathbb{H}^1}\leqslant B_*$, it follows from the estimate (2.9) that

$$\begin{aligned}&\|(u_1-u_2,\dot u_1-\dot u_2)\|_{C([0,t],\mathbb{H}^1)}\\&\qquad\leqslant\|(u_1(0)-u_2(0),\dot u_1(0)-\dot u_2(0))\|_{\mathbb{H}^1}^{1-t/T}B_*^{t/T}e^{2M(M+4/T)t(T-t)}\end{aligned}$$

for any $t<T$. By setting $T=2t$ in this estimate and assuming (without loss of generality) that $t,M\geqslant 1$, we get that

$$\mathbf{H}_{\mu(\varepsilon)}\big(\mathcal{A}^{\mathrm{traj}},C([0,t],\mathbb{H}^1)\big)\leqslant\mathbf{H}_\varepsilon(\mathbb{A},\mathbb{H}^1),\quad\text{where}\quad \mu(\varepsilon)=\varepsilon^{1/2}B_*^{1/2}e^{20M^2t^2}.$$

After replacing t by T and applying the lower estimate (4.17) to the left-hand side of the resulting inequality, we obtain the inequality

$$\mathbf{H}_\varepsilon(\mathbb{A},\mathbb{H}^1)\geqslant C_1T\log\mu(\varepsilon)=C_1T\bigg(\frac12\log\frac1\varepsilon-\frac12\log B_*-20M^2T^2\bigg).$$

Maximizing the right-hand side of this inequality with respect to T (that is, setting $T:=(10M)^{-1}(\log(1/\varepsilon))^{1/2}$), we derive the estimate (4.18). This completes the proof of Theorem 4.5.

§ 5. Chaos in spatial dynamical systems

In this section we interpret the results of the previous sections in the spirit of the theory of dynamical systems. To this end, we must recall some quantitative characteristics of dynamical complexity used in the theory of dynamical systems. We begin with the classical notion of topological entropy (see, for instance, [19]).

Definition 5.1. Let (M, d) be a compact metric space and let $S_h\colon M \to M$, $h \in \mathbb{R}^+$, be a continuous semigroup acting on M. For any $R > 0$ we introduce a new metric d_R in M as follows:

$$d_R(m_1, m_2) := \sup_{h \leqslant R} d(S_h m_1, S_h m_2), \qquad m_1, m_2 \in M. \tag{5.1}$$

It is clear that (M, d_R) is also a compact metric space. By the *topological entropy* of the semigroup S_h in M we mean the number

$$h_{\text{top}}(S_h, M) := \lim_{\varepsilon \to 0} \limsup_{R \to \infty} \frac{1}{R} \mathbf{H}_\varepsilon(M, d_R), \tag{5.2}$$

where the symbol $\mathbf{H}_\varepsilon(M, d_R)$ stands for the ε-entropy of M with respect to the metric (5.1).

As is known [19], the topological entropy (5.2) does not depend on the choice of the metric d in M but depends only on the topology of the space M.

To apply the general Definition 5.1 to the study of the trajectory dynamical system (2.1), we must fix some metric in $\mathcal{A}^{\text{traj}}$. To this end, we need the following simple proposition.

Proposition 5.2. *Let a weight function $\phi \in C_{\text{bd}}(\mathbb{R})$ be such that $\lim_{|t| \to \infty} \phi(t) = 0$ and $\phi(t) > 0$. Then the topology induced in $\mathcal{A}^{\text{traj}}$ by the embedding $\mathcal{A}^{\text{traj}} \subset W^2_{\text{bd},\phi}(\mathbb{R}^+)$ coincides with the local topology induced by the embedding $\mathcal{A}^{\text{traj}} \subset W^2_{\text{loc}}(\mathbb{R}_+)$.*

Indeed, Proposition 5.2 is immediate, because $\mathcal{A}^{\text{traj}}$ is a bounded set in $W^2_{\text{bd}}(\mathbb{R}^+)$.

Let us now fix an arbitrary weight function ϕ satisfying the conditions of Proposition 5.2 and define a metric in $\mathcal{A}^{\text{traj}}$ by the formula

$$d_\phi(u_1, u_2) := \|u_1 - u_2\|_{W^2_{\text{bd},\phi}(\mathbb{R}^+)}, \qquad u_1, u_2 \in \mathcal{A}^{\text{traj}}. \tag{5.3}$$

Then it follows from Theorem 2.4 and Proposition 5.2 that $(\mathcal{A}^{\text{traj}}, d_\phi)$ is a compact metric space, and one can define the topological entropy $h_{\text{top}}(\mathcal{T}_h, \mathcal{A}^{\text{traj}})$ of the trajectory dynamical system (2.1) by the formula (5.2). The next proposition gives a more convenient formula for computing the entropy.

Proposition 5.3. *The topological entropy $h_{\text{top}}(\mathcal{T}_h, \mathcal{A}^{\text{traj}})$ does not depend on the choice of the weight function ϕ and can be computed by the formula*

$$h_{\text{top}}(\mathcal{T}_h, \mathcal{A}^{\text{traj}}) = \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \mathbf{H}_\varepsilon\big(\mathcal{A}^{\text{traj}}|_{(0,T)}, W^2_{\text{bd}}(0, T)\big). \tag{5.4}$$

The proof of the formula (5.4) is more or less obvious; it can be found, for example, in [39].

We now note that, in contrast to classical dynamical systems generated by ordinary differential equations and by most of the natural partial differential equations in bounded domains (see, for instance, [34]), the topological entropy can be infinite in our case. In particular, this is so for the system (4.15).

Corollary 5.4. *The topological entropy of the dynamical system* $(\mathcal{T}_h, \mathcal{A}^{\mathrm{traj}})$, *generated by the equation* (4.15) *is infinite*:

$$h_{\mathrm{top}}(\mathcal{T}_h, \mathcal{A}^{\mathrm{traj}}) = \infty. \tag{5.5}$$

Indeed, the formula (5.5) follows immediately from (5.4) and (4.17).

We recall that dynamical systems with infinite topological entropy arise naturally when studying the spatial and dynamical complexity of the attractors $\mathcal{A}^{\mathrm{glob}}$ of evolution partial differential equations in *unbounded* domains, in particular, for equations of the form (0.13) (see, for instance, [39], [40]). Thus, with regard to the embedding (0.14), it is natural to apply the methods developed in [39] and [40] to the study of the trajectory dynamical system (2.1). We begin with the formulation of one of the possible generalizations of the notion of topological entropy (see [22], [39]).

Definition 5.5. Let (M, d) be a compact metric space and let $S_h\colon M \to M$ be a semigroup on M. Then the *modified topological entropy* of the semigroup S_h is defined by

$$\widehat{h}_{\mathrm{top}}(S_h, (M, d)) := \limsup_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)^{-1} \limsup_{R\to\infty}\frac{1}{R}\mathbf{H}_\varepsilon(M, d_R), \tag{5.6}$$

where the metric d_R is defined by the formula (5.1). Moreover, following [22], we introduce the *mean topological dimension* $\dim_{\mathrm{top}}$ by the formula

$$\dim_{\mathrm{top}}(S_h, M) := \inf_{\widehat{d}} \widehat{h}_{\mathrm{top}}(S_h, (M, \widehat{d})), \tag{5.7}$$

where the infimum is taken over all metrics $\widehat{d}$ in M that generate the topology on M induced by d.

In contrast to the topological entropy, the expression (5.6) is preserved only under Lipschitz homeomorphisms (as is the fractal dimension). This is the reason for introducing the quantity (5.7), which is a topological invariant (as is the case for the classical topological entropy).

Thus, fixing a weight function ϕ satisfying the conditions of Proposition 5.2 and defining the metric d_ϕ on $\mathcal{A}^{\mathrm{traj}}$ by the formula (5.3), we can define the modified topological entropy $\widehat{h}_{\mathrm{top}}(\mathcal{T}_h, (\mathcal{A}^{\mathrm{traj}}, d_\phi))$ and the mean topological dimension $\dim_{\mathrm{top}}(\mathcal{T}_h, \mathcal{A}^{\mathrm{traj}})$ of the trajectory dynamical system (2.1) by the formulae (5.6) and (5.7). (We stress that the infimum in (5.7) is taken over *all* metrics on $\mathcal{A}^{\mathrm{traj}}$ defining the local topology rather than only over those of the form (5.3).)

The next assertion is an analogue of Proposition 5.3.

Proposition 5.6. *The modified topological entropy of the trajectory dynamical system* (2.1) *does not depend on the choice of the weight* ϕ *and can be computed by the formula*

$$\widehat{h}_{\mathrm{top}}(\mathcal{T}_h, (\mathcal{A}^{\mathrm{traj}}, d_\phi)) = \limsup_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)^{-1} \limsup_{T\to\infty}\frac{1}{T}\mathbf{H}_\varepsilon\big(\mathcal{A}^{\mathrm{traj}}|_{(0,T)}, W^2_{\mathrm{bd}}(0,T)\big). \tag{5.8}$$

The proof of (5.8) is similar to that of (5.4) (see [39]).

The expression on the right-hand side of (5.8) was interpreted in [10] as the (fractal) dimension of the attractor per unit volume (see (3.2)). In [10], [39], [41], [42], [14] it was proved that this quantity is finite and positive for the attractors $\mathcal{A}^{\mathrm{glob}}$ in a broad class of evolution partial differential equations in unbounded domains. The following result shows that this quantity can be strictly positive also for the attractors of equations of the form (0.4).

Corollary 5.7. *If the conditions of Theorem* 2.4 *are satisfied, then the modified topological entropy of the trajectory dynamical system* (2.1) *is finite*:

$$\widehat{h}_{\mathrm{top}}(\mathcal{T}_h, \mathcal{A}^{\mathrm{traj}}) \leqslant C < \infty. \tag{5.9}$$

Moreover, this quantity is strictly positive for the equation (4.15):

$$0 < C_1 \leqslant \widehat{h}_{\mathrm{top}}(\mathcal{T}_h, \mathcal{A}^{\mathrm{traj}}) \leqslant C < \infty. \tag{5.10}$$

Indeed, the estimate (5.9) follows immediately from (5.8) and (3.5), and the estimate (5.10) from (4.17).

We recall that in the classical theory of dynamical systems the topic of chaotic dynamics is usually studied with the help of homeomorphic embeddings of Bernoulli schemes in the dynamical system in question (see [19] and the references therein). However, it should be noted that the classical Bernoulli schemes with *finitely* many symbols have *finite* topological entropy, and hence cannot be regarded as an adequate model when studying the case of infinite entropy. In this case it is natural to use Bernoulli schemes with *infinitely* many symbols.

Definition 5.8. Let $\mathcal{M} := [-1,1]^{\mathbb{Z}}$ be the compact metric space with the product topology. We recall that $\mathcal{M}$ consists of the functions $v\colon \mathbb{Z} \to [-1,1]$, and the topology in $\mathcal{M}$ is generated by the standard metric

$$d(v_1, v_2) := \sum_{i=-\infty}^{\infty} 2^{-|i|} |v_1(i) - v_2(i)|. \tag{5.11}$$

Let us define the model dynamical system $(\mathcal{T}_l, \mathcal{M})$ of shifts in $\mathcal{M}$ by the formula

$$(\mathcal{T}_l v)(i) := v(i+l), \qquad i, l \in \mathbb{Z}, \quad v \in \mathcal{M}. \tag{5.12}$$

As is known, $h_{\mathrm{top}}(\mathcal{T}_l, \mathcal{M}) = \infty$ and $\widehat{h}_{\mathrm{top}}(\mathcal{T}_l, (\mathcal{M}, d)) = \dim_{\mathrm{top}}(\mathcal{T}_l, \mathcal{M}) = 1$.

The next theorem gives an embedding of the dynamical system $(\mathcal{T}_l, \mathcal{M})$ in the trajectory dynamical system generated by the equation (4.15).

Theorem 5.9. *Let $\mathcal{K}$ be the essential set of* (4.15). *Then there is a homeomorphic embedding $\kappa\colon \mathcal{M} \to \mathcal{K}$ such that*

$$\mathcal{T}_{Tl}\kappa(v) = \kappa(\mathcal{T}_l v), \qquad l \in \mathbb{Z}, \quad v \in \mathcal{M}, \tag{5.13}$$

where $T > 0$ is the period introduced in Theorem 4.1 (and fixed by the condition $T := 2\pi$ in (4.13)). Moreover, this embedding is Lipschitz in the following sense:

$$C_1 \sum_{i=-\infty}^{\infty} e^{-T|i|}|v_1(i) - v_2(i)| \leqslant \|\kappa(v_1) - \kappa(v_2)\|_{W^2_{\mathrm{bd},e^{-|t|}}(\mathbb{R})} \leqslant C_2 \sum_{i=-\infty}^{\infty} e^{-T|i|}|v_1(i) - v_2(i)|. \tag{5.14}$$

Proof. By the construction of the equation (4.15), the set $\mathbb{L}_R$ defined in (4.4) and (4.8) is contained in $\mathcal{K}$ for some $R > 0$. Hence, it suffices to construct an embedding $\kappa\colon \mathcal{M} \to \mathbb{L}_R$. We claim that such an embedding is given by the formula

$$\kappa(v)(t) := R \sum_{i=-\infty}^{\infty} v(i)u(t - iT), \qquad v \in \mathcal{M}, \tag{5.15}$$

where $u(t)$ is the solution constructed in Theorem 4.1. Indeed, the formula (5.13) follows immediately from (5.15). Thus, it remains to verify the estimates (5.14). The proof of the estimate

$$\|\kappa(v)\|_{W^2(\tau)} \leqslant C \sum_{i=-\infty}^{\infty} |v(i)|e^{-(\tau-iT)^2/2} \quad \text{for} \quad \tau \in \mathbb{R}$$

is similar to that of (4.7). The right-hand inequality in (5.14) follows immediately from this inequality. To verify the left-hand inequality in (5.14), we recall that the first Fourier coefficient $u_1(t)$ of the solution $u(t)$ has compact support $\operatorname{supp} u_1 \subset [0, T]$ by Theorem 4.1. Therefore,

$$\|\kappa(v)\|_{W^2(\tau)} \geqslant \|\langle\kappa(v), e_1\rangle\|_{W^2_2(\tau,\tau+1)} = |v([\tau/T])| \cdot \|u_1\|_{W^2_2(0,1)} \geqslant C|v([\tau/T])|.$$

This estimate readily implies the left-hand inequality in (5.14), which completes the proof of Theorem 5.9.

We recall that the equation (4.15) satisfies the conditions of Theorem 2.7, and therefore the trajectory dynamical system (2.1) is topologically conjugate on the section to the dynamical system $(\mathbb{S}_h, \mathbb{K})$ defined by the formula (2.16). The next result is an analogue of Theorem 5.9 for this dynamical system.

Corollary 5.10. *Let $\mathbb{A} = \Pi_0\mathcal{A}^{\mathrm{traj}}$ be the global attractor of (4.15). Then there is a homeomorphic embedding $\widehat{\kappa}\colon \mathcal{M} \to \mathbb{A}$ such that*

$$\mathbb{S}_{Tl}\widehat{\kappa}(v) = \widehat{\kappa}(\mathcal{T}_l v) \quad \textit{for any} \quad l \in \mathbb{Z} \quad \textit{and} \quad v \in \mathcal{M}. \tag{5.16}$$

Moreover, this homeomorphism preserves the quantities (5.6) and (5.7), that is,

$$\begin{aligned} \widehat{h}_{\mathrm{top}}(\mathbb{S}_h, \widehat{\kappa}(\mathcal{M})) &= T^{-1}\widehat{h}_{\mathrm{top}}(\mathcal{T}_l, \mathcal{M}), \\ \dim_{\mathrm{top}}(\mathbb{S}_h, \widehat{\kappa}(\mathcal{M})) &= T^{-1}\dim_{\mathrm{top}}(\mathcal{T}_l, \mathcal{M}). \end{aligned} \tag{5.17}$$

Proof. By Corollary 2.6, the trace operator $\Pi_0\colon \mathcal{K}\to\mathbb{A}$ defines a Hölder-continuous homeomorphism whose Hölder exponent is as close to 1 as desired for an appropriate choice of the weight function in $\mathcal{K}$ (see (2.21) and (2.22)). Let us introduce the homeomorphism $\widehat{\kappa}:=\Pi_0\circ\kappa$. Then (5.16) follows immediately from (5.13). Since the mean topological dimension is a topological invariant, it follows that the second equality in (5.17) is obvious. The factor T^{-1} appears because of the 'time' rescaling (the semigroup $(\mathcal{T}_l,\mathcal{M})$ is conjugate to $(\mathcal{T}_{Tl},\mathcal{K})$ by means of κ). To verify the first equality of (5.17), we recall that κ is Lipschitz in the sense of (5.14). Therefore, since the modified topological entropy is invariant under Lipschitz homeomorphisms and the formula (5.6) does not depend on the choice of the weight function ϕ satisfying the conditions of Proposition 5.2 with respect to the metric (5.3), it follows that

$$\widehat{h}_{\rm top}(\mathcal{T}_h,\kappa(\mathcal{M}))=T^{-1}\widehat{h}_{\rm top}(\mathcal{T}_l,\mathcal{M}). \tag{5.18}$$

Similarly, since Π_0 is a Hölder-continuous homeomorphism whose Hölder exponent is as close to 1 as desired, we have $\widehat{h}_{\rm top}(\mathcal{T}_h,\kappa(\mathcal{M}))=\widehat{h}_{\rm top}(\mathbb{S}_h,\widehat{\kappa}(\mathcal{M}))$. This proves Corollary 5.10.

Corollary 5.11. *The mean topological dimension of the dynamical system generated by the equation* (4.15) *is strictly positive*:

$$\dim_{\rm top}(\mathcal{T}_h,\mathcal{A}^{\rm traj})=\dim_{\rm top}(\mathbb{S}_h,\mathbb{A})\geqslant T^{-1}>0. \tag{5.19}$$

Indeed, as is known [22], we have $\dim_{\rm top}(\mathcal{T}_l,\mathcal{M})=1$. Therefore, the estimate (5.19) follows from (5.17).

The next result shows that any finite-dimensional dynamics can be realized, up to a homeomorphism, by restricting $\mathbb{S}_h$ to the corresponding invariant set of the attractor $\mathbb{A}$ of the equation (4.15).

Corollary 5.12. *Let $\mathbb{A}$ be the global attractor of* (4.15), *let $K\subset\mathbb{R}^N$ be an arbitrary compactum, and let $F\colon K\to K$ be an arbitrary homeomorphism. There is a homeomorphic embedding $\tau\colon K\to\mathbb{A}$ such that*

$$\mathbb{S}_{NT}\circ\tau(k)=\tau(Fk)\quad \textit{for}\quad \textit{any}\quad k\in K. \tag{5.20}$$

Proof. Indeed, by Corollary 5.10, it suffices to embed the dynamical system (F,K) in $(\mathcal{T}_l,\mathcal{M})$. Moreover, without loss of generality we can assume that $K\subset[-1,1]^N$. In this case the desired embedding is given by the formula

$$\widetilde{\tau}(k)(i):=(F^{(m)}(k))_j,\qquad i=mN+j,\quad m,j\in\mathbb{Z},\quad 0\leqslant j\leqslant N-1, \tag{5.21}$$

where $F^{(m)}$ stands for the mth iteration of the map F and $(k)_j$ for the jth coordinate of a point $k\in[-1,1]^N$. Then the map $\tau:=\widehat{\kappa}\circ\widetilde{\tau}$ obviously satisfies all conditions of the corollary.

Remark 5.13. We recall that an abstract elliptic equation (0.4) can be formally interpreted as a second-order evolution equation. Moreover, this interpretation is partially justified (under the assumptions of Theorem 2.7) by constructing the continuous dynamical system $(\mathbb{S}_h,\mathbb{K})$ from the equation (0.4). However, in contrast to the case of evolution equations, the dynamical systems generated by elliptic equations *are not Lipschitz* in general (which means that we cannot set $\alpha = 0$ in (2.18)) but only Hölder continuous, with Hölder exponent as close to 1 as desired (as is the case of the equation (4.15)). Since these systems can be non-Lipschitz, they can have infinite-dimensional attractors with infinite topological entropy. Moreover, one can show by standard arguments that, if some equation satisfying the conditions of Theorem 2.7 also admits the Lipschitz property

$$\|\mathbb{S}_h(z_1) - \mathbb{S}_h(z_2)\|_{\mathbb{H}^1} \leqslant Q_h(\|z_1\|_{\mathbb{H}^{3/2}} + \|z_2\|_{\mathbb{H}^{3/2}})\|z_1 - z_2\|_{\mathbb{H}^1}, \qquad z_i \in \mathbb{K},$$

then the global attractor of this equation has finite fractal dimension, and hence its topological entropy is also finite (as is the case for evolution equations).

Remark 5.14. We note that, as in the case of classical dynamical systems, our embedding of a Bernoulli scheme in the trajectory dynamical system generated by the equation (4.15) is based on finding an appropriate homoclinic orbit. Indeed, the solution $t \mapsto (\sin t, \cos t, u(t))$, where $u(t)$ is the function constructed in Theorem 4.1, is a homoclinic orbit with respect to the 2π-periodic solution $t \mapsto (\sin t, \cos t, 0)$. However, in contrast to the classical case, in our situation we can sum spatial shifts of the original homoclinic orbit with coefficients not only in $\{0, 1\}$ but also in the interval $[-1, 1]$ (see (5.15)). Moreover, Theorem 5.9 shows that the periodic orbit $(\sin t, \cos t, 0)$ is 'infinitely degenerate'. Indeed, in a neighbourhood of this orbit we have a one-parameter family of 2π-periodic orbits that is parametrized by the constants $v_\varepsilon \in \mathcal{M}$, $\varepsilon \in [-1, 1]$ (that is, $v_\varepsilon(i) \equiv \varepsilon$ for $i \in \mathbb{Z}$). Similarly, there is a two-parameter family of 4π-periodic solutions, a three-parameter family of 6π-periodic solutions, and so on. The closure of this vast family of periodic orbits defines the embedding (constructed in Theorem 5.9) of the Bernoulli scheme $(\mathcal{T}_l, \mathcal{M})$.

Remark 5.15. Homeomorphic embeddings of the model dynamical system $(\mathcal{T}_l, \mathcal{M})$ in the spatial dynamical system $(\mathcal{T}_h, \mathcal{A}^{\mathrm{glob}})$ acting on the attractor were constructed in [39], [40] for a rather broad class of systems of reaction-diffusion equations of the form (0.13). Moreover, it follows from the construction suggested there that the image of $\mathcal{M}$ belongs to the strongly unstable manifold (with respect to the variable of evolution) of some spatially homogeneous equilibrium state of the evolution equation in question. Theorem 5.9 shows that (for some evolution equations in unbounded domains) the spatially chaotic behaviour of this type can also happen on the set $\mathcal{K}$ of equilibrium states, which is *a priori* much smaller than $\mathcal{A}^{\mathrm{glob}}$.

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