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MECHANICS

Nonlocal Modulation Equations for Viscous-Fluid Flows in Layers and Spatially Localized Perturbations

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INTRODUCTION

For a variety of hydrodynamic problems, it is physically reasonable to analyze them in unbounded two-dimensional (2D) and three-dimensional (3D) cylindrical domains [1–7]. However, the standard setting of the initial and boundary value problems in the form of Navier–Stokes equations is insufficiently determinate in this case. As additional conditions at infinity, the mean flux $\mathcal{F}$ or the mean gradient $\mathcal{P}$ of pressure in the directions of the cylinder generatrix can be proposed. As an example, we consider the 3D Poiseuille problem of a viscous incompressible flow between parallel plates. In solving this problem, the Davey–Hocking–Stewartson (DHS) modulation system [8] turns out to be the generalization of the one-dimensional complex Ginzburg–Landau equation that describes 2D problems of the Poiseuille flow near the stability threshold. We complement the DHS system by different sets of nonlocal conditions. Thereby, we are able to find correct constrains to the DHS system for problems with one unbounded variable. It turns out to be possible to set a problem for which there exist solutions localized over the variable transverse to the mainstream direction. These solutions are the principal terms in the expansions for the corresponding exact solutions to the system of Navier–Stokes equations.

1. MULTIPARAMETER EXPANSIONS

Starting from the 1960s, formal multiparameter asymptotic expansions were used by F. Busse, A. Newell, T. Stuart, W. Eckhaus, and others to derive modulation equations of the Ginzburg–Landau type. In [8], this approach was extended to problems of layered flows of viscous fluids, e.g., for the Poiseuille flow $V_{\mathrm{Pois}}(t, x, y) = (1 - y^2, 0, 0)^t$ between two parallel walls. In this case, the motion of a fluid occurs in the domain $\Omega = \mathbb{R}^2 \times (-1, 1)$ with the coordinates x_1, x_2, y, [where $(x_1, x_2) \in \mathbb{R}^2$], whereas the pressure gradient drives the entire system; i.e., $p(x, y) = -\frac{2}{\mathrm{R}}x_1$. Here, we use dimensionless variables. The velocity $\mathbf{v}(t, x, y) \in \mathbb{R}^3$ has components v_1, v_2 and v_3 along the directions x_1, x_2 and y, which satisfy the problem

$$\left.\begin{aligned} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p - \frac{1}{\mathrm{R}}\Delta \mathbf{v} &= 0 \\ \operatorname{div}\mathbf{v} &= 0 \end{aligned}\right\} \text{ in } \Omega \tag{1}$$

$$\text{and} \quad \mathbf{v} = 0 \quad \text{on} \quad \partial\Omega,$$

where p is pressure and R is the Reynolds number corresponding to a certain typical velocity. We now define

$$[\![f]\!] = \lim_{L\to\infty}\frac{1}{8L^2}\int_{-L}^{L}\int_{-L}^{L}\int_{-1}^{1} f(x_1, x_2, y)dx_1dx_2dy.$$

Then the basic steady-state one-dimensional flow is uniquely determined if, for example, we suppose that $[\![\partial_{x_1} p]\!] = -\frac{2}{\mathrm{R}}$ and $[\![v_2]\!] = 0$ (see [2]). These conditions correspond to the constant mean pressure gradient and the zero mean flux (in streamwise and spanwise directions with respect to the basic flow, respectively). We term this Poiseuille problem as $\mathcal{PF}$. The next problem with $[\![v_1]\!] = \frac{4}{3}$ and $[\![v_2]\!] = 0$ deals with a fixed mean flux along each direction, and we call this problem $\mathcal{FF}$. Below, for brevity, we use the notation $(1)_{\mathcal{PF}}$ and $(1)_{\mathcal{FF}}$. Our next goal is deriving adequate amplitude equations for these two different cases. Of course, two other cases with constant mean pressure gradients in both directions, i.e., $[\![\partial_{x_1} p]\!] = -\frac{2}{\mathrm{R}}$ and $[\![\partial_{x_2} p]\!] = 0$ or $[\![v_1]\!] = \frac{4}{3}$ and $[\![\partial_{x_2} p]\!] = 0$, can be analyzed in a similar way. We denote the last two problems as $\mathcal{PP}$ and $\mathcal{FP}$, respectively.

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We are interested in small perturbations of the basic Poiseuille flow, i.e., $u = v - V_{\text{Pois}}$. Since V_{Pois} is independent of $x \in \mathbb{R}^2$, the classical representation is

$$u(t, x, y) = e^{\lambda t + i\langle k, x\rangle}\Phi_k(y),$$

where $k = (\alpha, \beta)$ can be employed for linearizing problem (1) in the case of the Poiseuille flow. Using numerical information together with the Squire transformation (see, e.g., [2, 8]) shows that for $\alpha \in (\alpha_4, \alpha_1)$, where $\alpha_4 \approx 0.98787$ and $\alpha_1 \approx 1.0973$, the instability threshold $\mathrm{R} = \mathrm{R}_{\text{cr}}(\alpha)$ occurs for the wave vector $k = (\alpha, 0)$, which is parallel to the basic flow. The minimal critical Reynolds number is $\mathrm{R}_0 \approx 5772.222$, and the corresponding wave vector is $k_0 = (\alpha_2, 0)^t$, where $\alpha_2 \approx 1.02055$. The expansion coefficients

$$\lambda(\mathrm{R} - \mathrm{R}_0, k - k_0) = i\alpha_2 c_0 + \lambda_{0,1}(\mathrm{R} - \mathrm{R}_0) \\ - i\langle c_{\text{gr}}, k - k_0\rangle - \langle\Lambda(k - k_0), k - k_0\rangle + \text{h. o. t.}$$

can be found numerically: $c_0 \approx 0.2640$, $\lambda_{0,1} \approx (0.1682 + i0.8113) \times 10^{-5}$, $c_{\text{gr}} \approx (0.3831, 0)^t$, $\Lambda = \operatorname{diag}(\Lambda_{11}, \Lambda_{22}) \approx \operatorname{diag}(0.187 + i0.0275,\ 0.004663 + i0.08083)$. These values refine the results of [8]. The corresponding eigenfunction can be expressed in terms of the eigenfunction $\varphi\colon (-1, 1) \longrightarrow \mathbb{C}$ of the Orr–Sommerfeld equation as $\Phi_{k_0}(y) = \left(\frac{d}{dy}\varphi(y), 0, -i\alpha_2\varphi(y)\right)^t$ (see [2]).

In order to preserve the normalization suggested in [8], we assume that $\mathrm{R} = \mathrm{R}_0 + \rho\varepsilon^2$ with $\rho = \dfrac{1}{\operatorname{Re}(\lambda_{0,1})}$ and denote $(\xi_1, \xi_2) = \varepsilon(x - c_{\text{gr}}t)$. Then the multiscale expansion near the stability threshold for the solutions to problem (1) takes the form

$$u(t, x, y) = \varepsilon A(\varepsilon^2 t, \xi)E(t, x)\Phi_{k_0}(y) + \text{c. c} + \text{h. o. t.},$$

$$p(t, x, y) = p_0(t) - \frac{2x_1}{\mathrm{R}} + \varepsilon P(\varepsilon^2 t, \xi) + \text{h. o. t.},$$

where $E(t, x) = e^{i(\omega t + \langle k_0, x\rangle)}$. Equating to zero the corresponding coefficients ahead of $\varepsilon^j E^m$, we arrive at the DHS system

$$\begin{gathered} \partial_r A - \Lambda_{11}\partial^2_{\xi_1}A - \Lambda_{22}\partial^2_{\xi_2}A - \mu A - c_1|A|^2A - c_2A\partial_{\xi_1}P = 0, \\ \operatorname{div}_\xi(\nabla_\xi P + \gamma(|A|^2, 0)^t) = 0, \\ \xi \in \mathbb{R}^2, \end{gathered} \tag{2}$$

where $\mu = \dfrac{\lambda_{0,1}}{\operatorname{Re}\lambda_{0,1}}$, $c_1 \approx 29.69 - i143.7$, $c_2 \approx -28.03 + i642.4$, and $\gamma \approx 0.04525$. The second equation relates to the lowest-order term in the equation $\operatorname{div} v = 0$. In this case, the mean flux is given by the expression

$$[\![(v_1, v_2)^t]\!] = \left(\frac{4}{3}, 0\right)^t - \frac{2}{3}\varepsilon^2\mathrm{R}_0[\![\nabla_\xi P + (\gamma|A|^2, 0)^t]\!] + \text{h. o. t.}$$

Hence, it follows that for modeling problems $(1)_{\mathcal{PF}}$ and $(1)_{\mathcal{FF}}$ we should complement system (2) by nonlocal conditions

$$\begin{aligned} &\mathcal{PF}\colon\ [\![P_{\xi_1}]\!] = 0 \ \text{ and } \ [\![P_{\xi_2}]\!] = 0; \\ &\mathcal{FF}\colon\ [\![|A|^2 + \gamma^{-1}P_{\xi_1}]\!] = 0 \ \text{ and } \ [\![P_{\xi_2}]\!] = 0. \end{aligned} \tag{3}$$

Analysis of problems $(1)_{\mathcal{PP}}$, $(1)_{\mathcal{FP}}$ leads to similar constraints.

In the absence of conditions (3), the system of modulation equations (2), which was first derived in [8] (see Eqs. 2.27, 2.28, and 2.34), cannot properly describe the mass flux and the pressure gradient of the fluid flow under consideration (cf. [2, Sect. 5]). The necessity of additional constraints was pointed out in [7]; however, there are some inconsistencies in the coefficients introduced there {see Eq. (6.35) of [7]}.

The appearance of the second equation for pressure (which has no time derivative) is typical of all flows of viscous incompressible fluids in problems with two unbounded directions. However, we now can return to the classical complex Ginzburg–Landau equation and reduce it to the case of a single unbounded variable, which corresponds to certain constraints for an appropriate subspace of solutions.

We consider problem $(1)_{\mathcal{PF}}$ and, first, search for the solutions independent of x_2, hence, for the solutions of the type of $(2)_{\mathcal{PF}}$, which are independent of ξ_2. Then, the second equation can be integrated so that $\partial_{\xi_1}P = -\gamma|A|^2 + \delta(t)$. From $(3)_{\mathcal{PF}}$, it follows that $\delta = \gamma[\![|A|^2]\!]$, and hence, the system of equations (2) is reduced to

$$\begin{gathered} \mathcal{PF}_{\xi_2-\text{indep}}\ \ \partial_\tau A - \Lambda_{11}\partial^2_{\xi_1}A - \mu A \\ - (c_1 - \gamma c_2)|A|^2A - \gamma c_2[\![|A|^2]\!]A = 0 \end{gathered} \tag{4}$$

with a nonlocal term (cf. [12]). Second, following [2] and [5], we may consider solutions to the set of equations (1), which are periodic in the downstream direction x_1 with a period close to E. This corresponds to searching for solutions to Eqs. (2) in the form $A(\tau, \xi) = e^{i\beta\xi_1}\tilde{A}(\tau, \xi_2)$ and $P(\tau, \xi) = \tilde{P}(\tau, \xi_2)$. Thus, we find

$$\begin{gathered} \mathcal{PF}_{\xi_1-\text{per}}\ \ \partial_\tau\tilde{A} - \Lambda_{22}\partial^2_{\xi_2}\tilde{A} \\ - [\mu - \Lambda_{11}\beta^2]\tilde{A} - c_1|\tilde{A}|^2\tilde{A} = 0. \end{gathered} \tag{5}$$

For problem $(2)_{\mathcal{FF}}$, we may seek solutions independent of x_2, and hence, solutions to the system of equations (2), which are independent of ξ_2. Then, integrating the second equation yields $\partial_{\xi_1} P = -\gamma|A|^2 + \delta(t)$. It follows from Eqs. (3) that $\delta = 0$, and hence, system (2) is reduced to the equations

$$\begin{aligned}\mathcal{FF}_{\xi_2-\text{indep}}\quad &\partial_\tau A - \Lambda_{11}\partial^2_{\xi_1}A - \mu A\\ &-(c_1-\gamma c_2)|A|^2A = 0.\end{aligned}\tag{6}$$

We now consider the solutions to problem $(1)_{\mathcal{FF}}$, which are periodic in the downstream direction x_1 with a period close to E. This corresponds to the search for solutions of the form $A(\tau, \xi) = e^{i\beta\xi_1}\tilde{A}(\tau, \xi_2)$, $P(\tau, \xi) = \tilde{P}(\tau, \xi_2) + \delta(t)\xi_1$ to Eqs. (2) in which we have to admit variations of the downstream pressure gradient. In this case, we arrive at

$$\begin{aligned}\mathcal{FF}_{\xi_1-\text{per}}\quad &\partial_\tau\tilde{A} - \Lambda_{22}\partial^2_{\xi_2}\tilde{A} - [\mu - \Lambda_{11}\beta^2]\tilde{A}\\ &-c_1|\tilde{A}|^2\tilde{A} + \gamma c_2[\![\,|\tilde{A}|^2]\!]\tilde{A} = 0.\end{aligned}\tag{7}$$

2. SINGLE-PULSE AND MULTIPULSE SOLUTIONS TO THE NAVIER–STOKES EQUATIONS IN THREE-DIMENSIONAL LAYERS

For problems (4)–(7) there exist pulse solutions of the form

$$A(\tau, \xi_j) = d_1 e^{i\nu\tau}[\cosh(\eta\xi_j)]^{d_2}$$

with $d_1, d_2 \in \mathbb{C}$, $\nu \in \mathbb{R}$, $\eta > 0$ and $\mathrm{Re}\,d_2 < 0$. We note that the nonlocal term with

$$[\![\,|A|^2]\!] = \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}|A(\xi_j)|^2 d\xi_j$$

vanishes for solutions decreasing at infinity.

There is a key difference concerning the interrelation between these modulation problems and the original Poiseuille problem. To prove the existence of such pulse solutions for the original problem, we need to use the reflection symmetry $\xi_j \mapsto -\xi_j$ that is inherent in problems (4) to (7). At the same time, problems $(1)_{\mathcal{PF}}$ and $(1)_{\mathcal{FF}}$ are invariant with respect to the reflection $x \mapsto (x_1, -x_2)$, but not with respect to the transformation $x \mapsto (-x_1, x_2)$. Therefore, we are only able to prove the existence of a pulse-type solution symmetric with respect to the reflection but not for the case of Eqs. (4) and (6) (cf. [5]). In order to transform the pulse solution to its stationary form, we introduce new parameters and write out the Ginzburg–Landau equation in the form

$$\begin{aligned}\partial_t A &= (1+i\kappa)\\ \times[\partial_x^2 A &- (1+i\theta)^2 A + (1+i\Omega)(2+i\Omega)|A|^2A].\end{aligned}\tag{8}$$

If θ and Ω are small, then this equation can be considered as a perturbation of either the real Ginzburg–Landau equation (if $\kappa \approx 0$) or the nonlinear Schrödinger equation (if $|\kappa| \gg 1$). As was first observed in [9] for $\theta = \Omega$, this equation has the explicit steady-state solution $H_\theta(x) = e^{i\gamma}[\cosh x]^{-(1+i\theta)}$. There are also more general steady-state pulse solutions to Eq. (8). In particular, in applications of the Ginzburg–Landau equation to laser optics, we are interested in the multipulse solutions, i.e., solutions in which $|A(\cdot)|^2$ has several well-distinguished maxima. Such solutions were found in [10] for the case $0 < |\Omega| \ll 1$ and $|\theta - \Omega| \ll |\Omega|$. These results were generalized and refined in [3, 4]. It was shown in [5] that, as applied to the Poiseuille problem, Ω lies between ~3.4 and ~22.74 for $\alpha \in (\alpha_4, \alpha_1)$. It is worth noting that, in problem (8), Ω is determined from the relation $\Omega^2 + 3M\Omega = 2$ with $M = \dfrac{\mathrm{Re}\left(\dfrac{c_1}{\Lambda_{22}}\right)}{\mathrm{Im}\left(\dfrac{c_1}{\Lambda_{22}}\right)}$.

We have no proof for the existence of solutions to the Navier–Stokes equations (1), which correspond to the Hocking–Stewartson pulses and are based on employing the DHS system and the multiparameter expansion. Instead of this, we use the Kirchgässner reduction (see [2, 5]). We fix the downstream periodicity $\dfrac{2\pi}{\alpha}$ and look for the solutions in the form of traveling waves

$$\begin{aligned}v(t, x_1, x_2, y) &= \tilde{v}(x_1 - ct, x_2, y)\\ &= \tilde{v}\left(x_1 - ct + \frac{2\pi}{\alpha}, x_2, y\right).\end{aligned}\tag{9}$$

The arising elliptic problem is reduced to four-dimensional reversible ordinary differential equations. In this case, we restrict our analysis to the wave-number interval $\alpha \in (\alpha_4, \alpha_1)$, since only the fact that the most unstable modes are caused by large-period spanwise perturbations is used in the reduction. The suitable replacement of the variables (cf. [2, 4]) leads to the equation

$$\begin{aligned}&\frac{d^2a}{d\zeta^2} - (1+i\omega)^2 a + (1+i\Omega)(2+i\Omega)|a|^2a\\ &\quad+\eta^2 G_0\left(\omega, \Omega, \eta, a, \bar{a}, \frac{da}{d\zeta}, \frac{d\bar{a}}{d\zeta}\right) = 0,\end{aligned}$$

where ω, Ω, and η are functions of the physical parameters R, c, and α. In particular, $\eta^2 = \mathbb{O}(|\mathrm{R} - \mathrm{R}_{\mathrm{crit}}(\alpha)| + |c - c_{\mathrm{crit}}(\alpha)|)$, whereas η is considered to be a small parameter. For $\eta = 0$ and $\omega = \Omega$, this equation has the single-pulse solution $H_\Omega(\zeta)$. Using methods of the theory of dynamical systems, we can prove the persistence of this single-pulse solution under variations of η for almost all ω. The exceptional values of ω give rise to an exceptional set $\mathcal{A} = \{a_1, a_2, \ldots\} \subset (\alpha_4, \alpha_1)$ for the Poiseuille problem, with $a_1 \approx 1.044$, $a_2 \approx 1.032$. Finally, we arrive at the conclusion on the existence of single-pulse solutions to problems $(1)_{\mathcal{PF}}$ and $(1)_{\mathcal{FF}}$.

Statement 1. *For* $\alpha_* \in (\alpha_4, \alpha_1)\backslash\mathcal{A}$, *there exist the functions* $c^{(1)}(\cdot, \alpha_*)$: $(\mathrm{R}_{\mathrm{crit}}(\alpha_*) - \varepsilon, \mathrm{R}_{\mathrm{crit}}(\alpha_*)] \longrightarrow \mathbb{R}$ *and* $\varepsilon > 0$ *such that for an arbitrary* $\mathrm{R}_* \in (\mathrm{R}_{\mathrm{crit}}(\alpha_*) - \varepsilon, \mathrm{R}_{\mathrm{crit}}(\alpha_*))$ *and* $c_* = c^{(1)}(\mathrm{R}_*, \alpha_*)$, *problem* (1), (9) *with parameters* R, c, α) = $(\mathrm{R}_*, c_*, \alpha_*)$ *has the single-pulse solution* $\tilde{v}^{(1)} = V_{\mathrm{Pois}} + \tilde{u}^{(1)}$, *where*

$$\tilde{u}^{(1)}(\xi, x_2, y) = \frac{\eta}{\rho}\mathrm{Re}([\cosh(\eta x_2)]^{-(1+i\Omega)}\Psi_0(\xi, y)) + \mathbb{O}(\eta^2 e^{-|\eta z|}),$$

with $\Psi_0(\xi, y) = e^{i\alpha\xi}(i\varphi'(y), 0, \alpha\varphi(y))^t$; $\xi = x - c_* t$, *and* $\eta = (\mu(\mathrm{R}_{\mathrm{crit}}(\alpha_*) - \mathrm{R}_*))^{1/2}$. *Here,* μ, ρ, *and* Ω *are positive constants depending only on* α_*.

In the vicinity of the single-pulse solution, there are sequences of multipulse solutions.

Statement 2. α_*, R_*, $c_* = c^{(1)}(\mathrm{R}_*, \alpha_*)$ *and* $\tilde{u}^{(1)}$ *be as in the preceding theorem. Then, for an arbitrary integer* $n \geq 2$ *and an arbitrarily small* $r > 0$, *there exists a parameter triplet* $(\mathrm{R}^{(n)}, c^{(n)}, \alpha^{(n)})$ *with* $|(\mathrm{R}_*, c_*, \alpha_*) - (\mathrm{R}^{(n)}, c^{(n)}, \alpha^{(n)})| \leq r$ *such that, for* $(\mathrm{R}, c, \alpha) = (\mathrm{R}^{(n)}, c^{(n)}, \alpha^{(n)})$, *problem* (1), (9) *has the n-pulse solution* $\tilde{v}^{(n)} = V_{\mathrm{Pois}} + \tilde{u}^{(n)}$. *This implies that, for* $k = 1, 2, \ldots, n$, *there exist* $z_k^{(n)} \in \mathbb{R}$ *and* $\beta_k^{(n)} \in S_\alpha = \mathbb{R}/(2\pi/\alpha)\mathbb{Z}$ *such that*

$$\sup\left\{\left\|\tilde{u}^{(n)}(\cdot x_2, \cdot) - \sum_{k=1}^{n}\tilde{u}^{(1)}(\cdot - \beta_k^{(n)}, x_2 - z_k^{(n)}, \cdot)\right\|_{H^2(S_\alpha \times [-1, 1])} : x_2 \in \mathbb{R}\right\} \leq r.$$

The additive representation of the n-pulse solution is meaningful, since the shifts $\left|z_k^{(n)} - z_{k+1}^{(n)}\right|$ between the single pulse solutions tend to infinity as $r \longrightarrow 0$. Thereby, the n-pulse solution $\tilde{u}^{(n)}$ looks like n copies of the single-pulse solution $\tilde{u}^{(1)}$, which are shifted along the x_2 direction.

The stability of the pulse solutions to the Poiseuille problem is determined by the stability of the solution H_θ to the Ginzburg–Landau equation written out in the form of Eq. (8). As is clear, it can be stable only provided that the asymptotic state $A \equiv 0$ is stable as well. From the analysis of the continuous spectrum, we obtain the necessary stability condition, $-1 + 2\kappa\theta + \theta^2 \leq 0$ [11]. In addition, there also exist discrete eigenvalues. In particular, due to the translational and rotational invariance, we always obtain the double eigenvalue $\lambda = 0$.

In the general case, the discrete spectral component can be determined only numerically (see [1, 6].) The calculations performed in [l] confirm the fact that the single-pulse solutions to the Poiseuille problem are unstable for $\alpha \in (\alpha_4, \alpha_1)$ due to the existence of a real positive eigenvalue. We have no data on the behavior of the single-pulse solution branch for small R. However, if there exists a turning point for a certain R, then pulses with a finite amplitude can be stable. In this case, they may play an important role in the space–time chaotization of the 3D Poiseuille problem much lower than the classical instability threshold. From the results of [3, 4], it is possible to prove the existence of spatially chaotic (with respect to x_2) solutions (9) to problem (1). It would be interesting to study properties of attractors in problems (2), (3), in particular, to estimate the value of the ε-entropy per unit volume as was done in the case of Ginzburg–Landau equation (see, e.g., [11]).

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