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# *Bifurcations of Poiseuille Flow between Parallel Plates: Three-Dimensional Solutions with Large Spanwise Wavelength*

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**Abstract**

The "spatial dynamics" approach is applied to the analysis of bifurcations of the three-dimensional Poiseuille flow between parallel plates. In contrast to the classical studies, we impose time periodicity as well as spatial periodicity with period $2\pi/\alpha$ in the streamwise direction. However, we make no assumptions on the behavior in the spanwise direction, except the uniform closeness of the bifurcating solution to the basic flow. In an abstract setting it is shown how the dimension of the critical eigenspace of the spatial dynamics analysis can be uniquely determined from the classical linear stability problem. For the three-dimensional Poiseuille problem we are able to find all relevant coefficients from the analysis of the purely two-dimensional problem. Moreover, we are able to analyze precisely the influence of a spanwise pressure gradient and the associated spanwise mass flux. The study of the reduced problem shows that there are two different kinds of solutions (spirals and ribbons) which are $2\pi/\beta$ periodic in the spanwise direction, as in the Couette-Taylor problem, and both of them bifurcate in the same direction.

## 1. Introduction

In the theory of hydrodynamic stability, the Poiseuille flow, as well as the Taylor-Couette problem, plays the role of a benchmark for analytical and numerical approaches. In both cases the neutral stability curves can only be obtained numerically, in contrast to the Bénard problem. However, by using analytical tools of bifurcation theory, such as center-manifold techniques, many otherwise difficult accessible solution types can be derived together with their stability properties. To that end, the relevant coefficients of the reduced problem on the center manifold are again calculated numerically.

These hydrodynamic problems are posed on an infinite cylinder $Q = \Omega \times \mathbb{R}^1$, where $\Omega$ is a bounded cross-section in one or two dimensions. The unbounded axial coordinate is denoted by $z$. The general problem is to describe the behavior of perturbations of the "fully" symmetric steady flow near the instability threshold. The classical approach to the problem is to impose spatial periodicity along the unbounded coordinate with the period $2\pi/\beta$. Then the spectrum of the linear problem of stability appears to be discrete, and the description of the local behavior of the evolution problem reduces to an ordinary differential equation via the center-manifold argument (see for instance IOOSS [1984]). This approach is natural in the case when the minimal critical Reynolds number $R_0(\beta)$ corresponds to $\beta_0 \neq 0$. If $\beta_0 = 0$, the idea of fixing rather small $\beta$, in order to obtain a relevant description of the flows in the extended domain, does not seem to be very fruitful.

When the assumption of spatial periodicity is dropped, the linear operator has a continuous spectrum, and instability appears at a whole band of wave lengths. For this situation the Ginzburg-Landau formalism produces a reduced problem: the so-called Ginzburg-Landau equation, which describes the amplitude and phases of the patterns. The mathematical justification of this formalism in the sense of rigorous error estimates recently became possible for model problems in COLLET & ECKMANN [1990], VAN HARTEN [1991]; also see SCHNEIDER [1994A] and the references therein. First results for hydrodynamics are given in SCHNEIDER [1994B].

Here we use an approach which also allows arbitrary $z$-dependence of the perturbations. Yet we restrict the temporal behavior to be either stationary or periodic. Then, time plays the role of a compact cross-sectional variable while the unbounded axial variable $z$ takes over the role as the evolutionary variable, leading to the so-called *spatial dynamics* formulation. In KIRCHGÄSSNER [1982] it was first observed that a spatial center manifold can be constructed such that it contains all small bounded perturbations. Moreover, the lowest-order terms of the ordinary differential equation on the spatial center manifold can be identified with the steady or time-periodic part of the Ginzburg-Landau equation. These ideas were already applied to hydrodynamic stability problems in IOOSS, MIELKE, & DEMAY [1989], IOOSS & MIELKE [1991, 1992]. The first paper treated the case when the critical mode is steady. In the other two papers oscillating critical modes were investigated. In the second one it was supposed that the critical spatial wave number $\beta_0$ is different from 0. This is the case of two-dimensional Poiseuille flow (when $z$ denotes the downstream variable) and the Couette-Taylor problem with counter-rotating cylinders. However, in these cases we do not find the classical complex Ginzburg-Landau equation. The third paper treats the case $\beta_0 = 0$ leading to bifurcations of patterns with large wave length which can be described by a complex Ginzburg-Landau equation.

The purpose of the present work is threefold. First, we develop a general result concerning the relation of the classical linear stability approach and the spatial dynamics theory. We give a new result on the spectral properties of the linear spatial operator. Second, we include the spanwise flux as an additional parameter. Previous work, such as DAVEY, HOCKING, & STEWARTSON [1974], does not account properly for non-zero spanwise mass flux, and in BRIDGES [1989] the spanwise mass flux is zero. Third, we give a detailed study of the (in)stability of the Poiseuille flow with respect to perturbations with fixed downstream periodicity and with large spanwise wavelength.

To be more precise, consider the classical linear problem $u_t = L(\mu, \partial_z)u$ with the associated spectral problem $\lambda u = L(\mu, \partial_z)u$. Because of the translation invariance, it is sufficient to seek solutions in the form $u(z) = e^{i\beta z}v$, which leads to $\lambda v = L(\mu, ik)v$. The linear spatial problem reads $\frac{d}{dz}w = \mathscr{L}_{\mu,\partial_t}w = \mathscr{L}_{\mu,0}w + E\partial_t w$, where typically $w = (u, \frac{d}{dz}u)^T$ and $E\partial_t w = (0, \partial_t u)^T$. We assume time-periodicity with period $2\pi/\omega$; the spatial operator $\mathscr{L}_{\mu,\omega} = \mathscr{L}_{\mu,0} + i\omega E$ has the eigenvalue $i\beta$ if and only if $L(\mu, i\beta)$ has the eigenvalue $i\omega$. The question remaining open in IOOSS, MIELKE, & DEMAY [1989], IOOSS & MIELKE [1991, 1992] is the following: How large is the multiplicity of $i\beta$ as an eigenvalue of $\mathscr{L}_{\mu,\omega}$? We are interested in the case that $i\omega$ is a simple eigenvalue of $L(i\beta)$; then for all $k \approx \beta$ there is a simple eigenvalue $\lambda(k)$ of $L(\mu, ik)$ such that $\lambda(k)$ is smooth in $k$ and $\lambda(\beta) = i\omega$. Our result is that the algebraic multiplicity of the geometrically simple eigenvalue $i\beta$ of $\mathscr{L}_{\mu,\omega}$ is $d$ if and only if $\lambda(k) = i\omega + e_{0d}(k-\beta)^d + \mathscr{O}(|k-\beta|^{d+1})$ with $e_{0d} \neq 0$. Since $\lambda(k)$ is usually calculated to obtain the neutral curve, the determination of $d$ is immediate. Of course, certain symmetries influence the generic size of $d$.

The Poiseuille flow is posed in the domain $Q = \mathbb{R} \times (-1, 1) \times \mathbb{R}$ with the spatial variables $(x, y, z)$. At $y = \pm 1$ the flow field is zero. In the $x$-direction (downstream) we assume periodicity with period $2\pi/\alpha$. This problem possesses $SO(2) \times O(2)$ symmetry if we also prescribe spatial periodicity along the $z$-direction. Thus, the problem appears to be analogous to the classical Couette-Taylor problem (non-axisymmetric case), which was investigated independently in AFENDIKOV, BABENKO, & YURIEV [1982], CHOSSAT & IOOSS [1985] and GOLUBITSKY & STEWART [1986]. Two sorts of time-periodic solutions which appear in the bifurcation analysis are called "spirals" (travelling waves) and "ribbons" (standing waves) in CHOSSAT & IOOSS [1985]. In the three-dimensional Poiseuille problem there exist the same kind of solutions, as was mentioned in BRIDGES [1989] and even earlier, in DHANAK [1983], but there on the physical level of accuracy. In BRIDGES [1989] the spirals are called oblique travelling waves (OTWs) and the ribbons are called standing travelling waves (STWs). For completeness, we mention numerical calculations of the global branches of spirals and ribbons together with the discussion of the relevance of the results to the experiment in EHRENSTEIN & KOCH [1991].

The two-dimensional Poiseuille problem (in which there is no $z$-dependence and the $z$-component of the velocity vanishes) is treated in AFENDIKOV & VARIN [1991]. It is shown there that on the neutral curve in the $(R, \alpha)$-plane, there is a point $T_3 = (R_3, \alpha_3)$ such that Hopf bifurcation of $2\pi/\alpha$-periodic solutions that occurs is subcritical for $\alpha > \alpha_3$ and supercritical for $\alpha < \alpha_3$. It is well known by SQUIRE's [1933] theorem that the most dangerous modes for instability in Poiseuille flow are two-dimensional. However, this result has to be used with care. If we prescribe $\alpha$ in advance, this is no longer true. Analyzing this problem, we have found another point $T_4 = (R_4, \alpha_4)$ on the two-dimensional neutral curve such that $\alpha < \alpha_4$ implies that the most dangerous modes are three-dimensional with $z$-periodicity $2\pi/\beta$ where $\beta = \beta(\alpha) > 0$. Hence, the neutral curve of the three-dimensional problem coincides with the two-dimensional neutral curve only for $\alpha \geqq \alpha_4$ but lies to the left for $\alpha < \alpha_4$; see Fig. 0. Thus, *the study of perturbations with large*

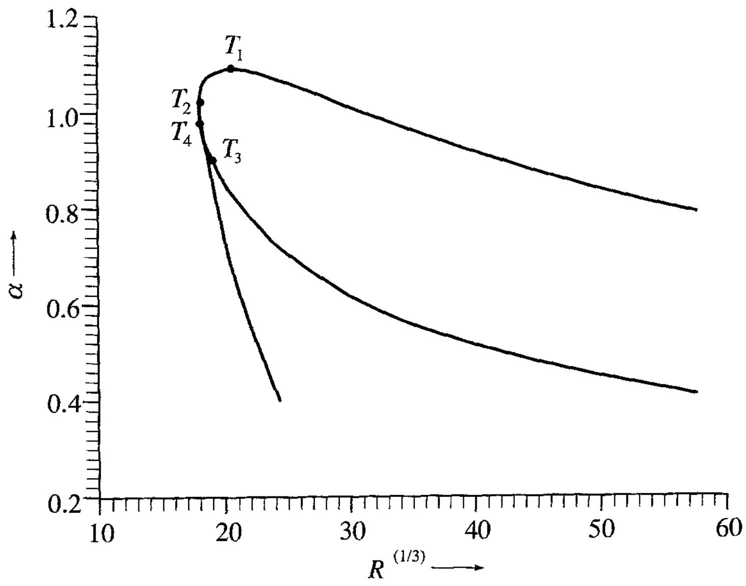


Fig. 0. Neutral curve for the two-dimensional and three-dimensional Poiseuille flow. $T_3 = (R_3, \alpha_3) \approx (6842.2, 0.90667)$, $T_4 = (R_4, \alpha_4) \approx (5874.7, 0.98787)$

*wavelengths is physically motivated only for* $\alpha \geqq \alpha_4$. For smaller $\alpha$ the relevant perturbations have shorter wavelengths and should be treated as in IOOSS & MIELKE [1991].

We show, for the Poiseuille problem, the existence of an associated spatial center manifold. Thus, we are able to classify all perturbations of the basic Poiseuille flow which are time-periodic and $x$-periodic with period $2\pi/\alpha$ but may have arbitrary $z$-dependence as long as they are uniformly close to the basic flow. In fact, all such solutions, which bifurcate from the basic flow at criticality, are downstream travelling waves, i.e., they only depend $2\pi$-periodically on the variable $\alpha(x - ct)$ rather than on $x$ and $t$ independently.

In particular, we are able to characterize all solutions having small spanwise mass flux. For instance, we find a three-parameter family of spirals, parametrized by the downstream and spanwise mass flux and the spanwise wavelength $\beta$. (Here $\alpha$ is fixed.) The downstream flux can only be defined for spanwise periodic solutions and is obtained as the mean over the period $2\pi/\beta$. Only a two-parameter subfamily of these spirals can be obtained by rotations of the two-dimensional travelling waves.

All these perturbations satisfy the reduced ordinary differential equation on the center manifold, which is a second-order equation for one complex variable as a function of $z$. In general, the computation of the coefficients of the lowest nonlinear terms is a troublesome task which can only be done numerically. From the computational viewpoint, the Poiseuille problem is more complicated than the Couette-Taylor problem. The interesting range of parameters is $R \in [5000, 7000]$.

This results in eigenfunctions with large gradients, and a very careful discretization of the spectral boundary-value problem is needed. In order to overcome this difficulty, we use in the discretization methods "without saturation" (see BABENKO [1986]). We show here that the knowledge of certain coefficients for the classical Liapunov-Schmidt method or "classical" center-manifold reduction can help in finding the necessary information for the ordinary differential equation on the "spatial center manifold" as studied here. For the problem treated here we are able to find the relevant coefficients from the purely two-dimensional case, as studied in CHEN & JOSEPH [1973] and AFFENDIKOV & VARIN [1991], by employing Squire's theorem. So, no additional numerical efforts are needed to find the reduced ordinary differential equation.

Moreover, we are now able to analyze the behavior of solutions which are also travelling waves in the $z$-direction (spirals) which are periodic with period $2\pi/\beta$. Also, the class of reflectionally symmetric $z$-periodic solution (ribbons) is studied. To this end we apply the Liapunov-Schmidt reduction to our spatial ordinary differential equation restricted to $2\pi/\beta$-periodic solutions. As a result, we find that spirals and ribbons always bifurcate into the same direction (super- or subcritical) for small enough $\beta$, a result which agrees with numerical results in BRIDGES [1994]. Our result is general and not restricted to hydrodynamics. It is interesting insofar as the general theory of Hopf bifurcation in the presence of $SO(2)\times O(2)$ symmetry tells us that spirals or ribbons can only be stable if both of them bifurcate supercritically (cf. GOLUBITSKY, STEWART, & SCHAEFFER [1988] and IOOSS & ADELMEYER [1992]). However, for the Poiseuille flow we do not find stable solutions since supercritical bifurcation with large wavelength occurs only for $\alpha < \alpha_3 < \alpha_4$. In this regime these solutions are already unstable to perturbations with much smaller spanwise wavelengths.

## 2. The linear spatial operator

Let us consider viscous incompressible fluid flow in the cylindrical domain $Q = \Omega \times \mathbb{R}$ with coordinates $(x, y, z)$, where $\Omega$ is a regular domain $\Omega \subset \mathbb{R}^2$ (or $\mathbb{R}^1$), $(x, y) \in \Omega$. The domain $\Omega$ can also be $\Omega = S^1 \times (-1, 1)$, which is the case, for instance, for the three-dimensional Poiseuille flow, if space periodicity along the $x$-axis is prescribed.

Let us suppose that in the domain $Q$ there exists a stationary $z$-independent solution $(V, P)$ of the Navier-Stokes system. The system for perturbations $(u, p)$ of the basic flow is

$$\begin{aligned} &\frac{\partial}{\partial t} u = \frac{1}{R}\Delta u - \operatorname{grad} p - (V\cdot\nabla)u - (u\cdot\nabla)V - (u\cdot\nabla)u, \\ &\operatorname{div}\ u = 0, \end{aligned} \tag{2.1}$$

and the boundary condition is

$$u|_{\partial\Omega\times\mathbb{R}} = 0. \tag{2.2}$$

Since $V$ is $z$-independent, the problem (2.1), (2.2) is invariant under translations $\tau_a$ in the $z$-direction $\tau_a: z \to z + a$. In a variety of cases, the problem is also invariant under the reflection

$$S\begin{pmatrix} u_1(t,x,y,z) \\ u_2(t,x,y,z) \\ u_3(t,x,y,z) \end{pmatrix} = \begin{pmatrix} u_1(t,x,y,-z) \\ u_2(t,x,y,-z) \\ -u_3(t,x,y,-z) \end{pmatrix}.$$

The usual approach of linear stability theory is to look for solutions of the linearization of problem (2.1), (2.2) in a form $u = v(x, y)\exp\{i\beta z + \lambda t\}$, where $v$ is a function of variables lying in $\Omega$. Exponentials in $z$ and $t$ appear because of the translation invariance. It is relevant to notice here that we also have to take $p = q(x, y)\exp\{i\beta z + \lambda t\}$, as the projection on the set of the divergence-free vector fields, which allows us to eliminate the pressure and requires periodicity conditions in the $z$-direction not only for the velocity field, but also for the pressure.

If we prescribe periodicity in $z$:

$$u(t,x,y,z) = u(t,x,y,z + 2\pi/\beta); \tag{2.3}$$

then in a suitable functional space $H$ (see for instance IOOSS [1984]) the problem (2.1), (2.2) can be written in the form of an evolution equation

$$\frac{du}{dt} = L_\mu u + B_\mu(u, u), \tag{2.4}$$

where $\mu$ represents the parameters of the problem.

The equivariance under the symmetries $\tau_a$ and $S$ means that

$$\begin{aligned} L_\mu Su &= SL_\mu u, \quad SB_\mu(u,u) = B_\mu(Su, Su), \\ \tau_a L_\mu u &= \mathscr{L}_\mu \tau_a u, \quad \tau_a B_\mu(u,u) = B_\mu(\tau_a u, \tau_a u). \end{aligned} \tag{2.5}$$

Note that $S$ acts as a reflection; hence, $S\tau_a = \tau_{-a}S$.

The operators $L_\mu$ and the symmetric bilinear map $B_\mu$ are well studied (see, e.g., LADYZHENSKAYA [1963] and IOOSS [1984]). In particular, $L_\mu$ has a compact resolvent. Define $L_{\mu,k}v = e^{-ikz}L_\mu(e^{ikz}v)$; the eigenvalue problem

$$(L_{\mu,k} - \lambda(\mu,k))v = 0 \tag{2.6}$$

with $k = \beta$ has a discrete set of eigenvalues depending continuously on $k$. In the classical approach one only considers real $k$; however, later on we also allow $k$ to vary in $\mathbb{C}$. This leads to no problems since $L_{\mu,k}$ depends analytically on $k$. (We distinguish between $k$ and $\beta$, since $k$ may be complex but $\beta$ is always assumed to be real.) The countable set of eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$ is assumed to be ordered in the following way

$$-\mathrm{Re}\lambda_0 \leqq -\mathrm{Re}\lambda_1 \leqq -\mathrm{Re}\lambda_2 \leqq \cdots.$$

The critical value of the parameter $\mu$ is defined through the condition $\mathrm{Re}\lambda_0(\mu,\beta) = 0$ where $\beta \in \mathbb{R}$. Notice that if $e^{i\beta z}v$ is the eigenvector of the problem

$(L_\mu - \lambda_0 E)e^{i\beta z}v = 0$, then from the condition (2.5) it follows that $e^{i\beta z}Sv$ is also an eigenvector belonging to the same eigenvalue $\lambda_0$. Therefore

$$\lambda_0(\mu, k) = \lambda_0(\mu, -k). \tag{2.7}$$

We are interested in the case that $\lambda_0(\mu,\beta) \notin \mathbb{R}$. Then, $\bar{v}$ is associated with the eigenvalue $\overline{\lambda_0(\mu,\beta)}$.

Suppose that $\mu(\beta)$ has the minimum $\mu = 0$ at $\beta = \beta_0$, which implies that

$$\operatorname{Re}\lambda_0(0,\beta_0) = 0, \quad \frac{\partial}{\partial k}\operatorname{Re}\lambda_0(0,\beta_0) = 0. \tag{2.8}$$

Let the parameter $\mu \in \mathbb{R}$ be chosen such that $\operatorname{Re}\frac{\partial}{\partial\mu}\lambda_0(0,\beta_0) > 0$. (In fact, further on we always use $R = R_0 + \mu$ where $\mu$ is the difference between the actual and the critical Reynolds number.) If the eigenvalue $\lambda_0(\mu,k)$ is simple, then $\lambda_0(\mu,k)$ is analytic, and according to (2.8),

$$\lambda_0(\mu,k) = i\omega_0 + \sum_{j,n=1}^{\infty} e_{jn}\mu^j(k-\beta_0)^n, \tag{2.9}$$

where $\operatorname{Re} e_{01} = 0$. For $\beta_0 = 0$ condition (2.7) yields $e_{j,2n-1} = 0$, $n = 1,2,3,\ldots$. If $\operatorname{Re} e_{02} \neq 0$, then the equation of the neutral curve near the instability threshold is $\mu(\beta) = \dfrac{\operatorname{Re} e_{02}}{\operatorname{Re} e_{10}}(\beta-\beta_0)^2 + O((\beta-\beta_0)^3)$.

Let us remark that the simplicity of the eigenvalue $\lambda_0(\mu,k)$ is sufficient but not necessary to give the expansion (2.9). For instance, in the Couette-Taylor problem for non-axisymmetric perturbations, the eigenvalues are twofold and semisimple, but the expansion still holds due to the symmetry group $SO(2)\times O(2)$.

Let us look now for solutions of (2.1) and (2.2) that are $2\pi/\omega$-periodic in time. The unbounded axial coordinate $z$ will play the role of the evolutionary variable. By setting $\theta = (u,w)^T$ ($T$ means transpose), where $w = R^{-1}\frac{\partial u}{\partial z} - pe_3$, $e_3 = (0,0,1)^T$, the equation (2.1) takes the form

$$\frac{d\theta}{dz} = \mathscr{L}_{\mu,\partial_t}\theta + \mathscr{B}_\mu(\theta,\theta). \tag{2.10}$$

Note that the construction of $w$ is such that there are no $z$-derivatives on the right-hand side. With the use of mass conservation we find that

$$w_3 = -R^{-1}\operatorname{div}_\Omega u - p, \quad \text{where } \operatorname{div}_\Omega u = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}.$$

The linear operator $\mathscr{L}_{\mu,\partial_t}\theta$ reads

$$\mathscr{L}_{\mu,\partial_t}\theta = \begin{pmatrix} w^* \\ \partial_t u - A_0 u - V_3 w^* - \nabla(w_3 + \operatorname{div}_\Omega u)\end{pmatrix},$$

$$w^* = (Rw_1, Rw_2, -\operatorname{div}_\Omega u)^T, \quad \nabla_\Omega = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)^T.$$

With the notations $\Delta_\Omega = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $u_\Omega = (u_1, u_2, 0)^T$, the operator $A_0$ is defined by $A_0 u = \frac{1}{R}\Delta_\Omega u_\Omega - (V \cdot \nabla)u - (u \cdot \nabla_\Omega)V$. For the bilinear operator $\mathscr{B}$ we have

$$\mathscr{B}(\theta, \theta) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_1 u_{1,x} + u_2 u_{1,y} + R u_3 w_1 \\ u_1 u_{2,x} + u_2 u_{2,y} + R u_3 w_2 \\ u_1 u_{3,x} + u_2 u_{3,y} - u_3 \operatorname{div}_\Omega u \end{pmatrix}. \tag{2.11}$$

Using the above notations, we can write the linearization of (2.1) in the form

$$\begin{aligned} u_t &= A_0 u + V_3 \frac{\partial u}{\partial z} + \frac{1}{R}\frac{\partial^2 u}{\partial z^2} - \nabla_\Omega p - \partial_z p e_3, \\ 0 &= \operatorname{div}_\Omega u + \frac{\partial u_3}{\partial z}, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{2.12}$$

The usual approach to investigate stability of the stationary solution $V$ is to look for solutions of (2.12) in the form

$$u(t, x, y, z) = e^{\lambda t + ikz} v(x, y), \quad p(t, x, y, z) = e^{\lambda t + ikz} P(x, y).$$

With $\Phi = (v, P)$ these representations yield the eigenvalue problem

$$E_k^\lambda \Phi = 0, \quad \text{with } E_k^\lambda = \begin{pmatrix} \lambda + k^2 - A_0 - ikV_3 & \nabla_\Omega + ike_3 \\ \operatorname{div}_\Omega + ike_3 & 0 \end{pmatrix}. \tag{2.13}$$

Equivalently, we can look for solutions of the linearized problem (2.10) in the form $u = e^{\lambda t + ikz} v(x, y)$ and $w = e^{\lambda t + ikz} \tilde{w}(x, y)$. This approach results in

$$(\tilde{\mathscr{L}}_{\mu,\lambda} - ik)\begin{pmatrix} v \\ \tilde{w} \end{pmatrix} = 0, \tag{2.14}$$

where $\tilde{\mathscr{L}}_{\mu,\lambda}$ is obtained from $\mathscr{L}_{\mu,\partial_t}$ by replacing the term $\partial_t u$ with $\lambda u$.

The key relation between the operator $\tilde{\mathscr{L}}_{\mu,\lambda}$ and the $L_{\mu,\lambda}$ is that if $L_{\mu,k}$ has the eigenvalue $\lambda = \lambda(\mu, k)$, then $\tilde{\mathscr{L}}_{\mu,\lambda}$ has the eigenvalue $k$, and vice versa. This is true not only for $k \in \mathbb{R}$, but also for general $k \in \mathbb{C}$. Since our point of view is restricted to time-periodic solutions we choose to confine $\lambda$ to lie in $i\mathbb{R}$.

We consider $\tilde{\mathscr{L}}_{\mu,\lambda}$ as a differential operator with domain

$$\mathscr{D}(\tilde{\mathscr{L}}_{\mu,\lambda}) = \{(u, w)^T \in [H^2(\Omega)]^3 \times [H^1(\Omega)]^3 : u = \operatorname{div}_\Omega u = w_1 = w_2 = 0 \text{ on } \partial\Omega\},$$

and we can investigate the spectral problem (2.14) for the closure of $\tilde{\mathscr{L}}_{\mu,\lambda}$ in $[L_2(\Omega)]^6$.

For $\lambda = 0$ we always find $k = 0$ as an eigenvalue in (2.14) with the eigenvector $\Phi_0 = (0, 0, 0, 0, 0, 1)^T$. This corresponds to the variations of the pressure. Moreover, there is a generalized eigenvector $\Phi_1 = (0, 0, \frac{1}{2}(1 - y^2), 0, 0, 0)^T$ satisfying the equation $\tilde{\mathscr{L}}_{\mu, 0}\Phi_1 = \Phi_0$. This vector measures the mean flux in the $z$-direction.

To rule out this trivial part of the kernel of $\tilde{\mathscr{L}}_{\mu, \lambda}$, we restrict the linear analysis to solutions with zero mean flux in the $z$-direction. The assumption of zero mass flux is no restriction, since the base flow $V$ may carry the spanwise mass flux. This viewpoint will be useful in the next sections. For the nonlinear problem in Sections 4 and 5 we recover full generality in allowing for any small spanwise flux. (Note that $\alpha(z, t) = \int_\Omega u_3 \, dx \, dy$ satisfies $\frac{\partial}{\partial z}\alpha = -\int_\Omega \operatorname{div}_\Omega u \, dx \, dy = \int_{\partial\Omega} u \cdot \eta \, d\sigma = 0$. Hence, we could describe any time-periodic function $\alpha$. But this would destroy the reflection symmetry $z \to -z$.)

In addition to the assumption about the mean flux, we project out the mean value of $w_3$ to suppress the indeterminacy of the pressure. Thus, we impose $\int_\Omega w_3 \, dx \, dy = -\int_\Omega p dx \, dy = 0$ for all $z$, $t$.

**Theorem 2.1.** *Suppose that $\lambda_0(\mu, k)$ is a simple eigenvalue of* (2.13) *which depends analytically on $(\mu, k)$ in some neighborhood of* $(0, k_0)$ *such that expansion* (2.9) *holds.*

*Then $ik_0$ is a geometrically simple eigenvalue of $\tilde{\mathscr{L}}_{0, \lambda_0}$. Its algebraic multiplicity is $d$ if and only if*

$$\lambda_0(0, k) = i\omega_0 + e_{0d}(k - ik_0)^d + O(|k - k_0|^{d+1}) \quad \textit{with } e_{0d} \neq 0. \tag{2.15}$$

**Proof.** With $E_k^\lambda$ from (2.13) let us define the operators

$$M_0 = E_{k_0}^{\lambda_0}, \quad M_1 = \frac{\partial E_k^\lambda}{\partial k}(k_0, \lambda_0), \quad M_2 = \frac{\partial E_k^\lambda}{\partial \lambda}(k_0, \lambda_0).$$

Note that $\dfrac{\partial^2 E_k^\lambda}{\partial k^2}(\lambda_0, k_0) = 2M_2$ and that $\dfrac{\partial^p E}{\partial k^p}(\lambda_0, k_0) = 0$ for $p \geqq 3$. If we let $\lambda_n = \dfrac{\partial^n \lambda_0}{\partial k^n}(0, k_0)$ and $\Phi_n = \dfrac{\partial^n \Phi}{\partial k^n}(k_0)$, then the differentiation of $E_k^{\lambda_0(0,k)}\Phi_0(k) = 0$ $n$ times with respect to $k$ yields

$$M_0 \Phi_n = F_n := -\sum_{m=1}^{n} \binom{n}{m} \lambda_m M_2 \Phi_{n-m} + n M_1 \Phi_{n-1} + n(n-1) M_2 \Phi_{n-2}. \tag{2.16}$$

$M_0$ is a Fredholm operator (Ladyzhenskaya [1963], Iooss [1984]; hence the adjoint operator $M_0^*$ has an eigenvector $\Phi_0^*$ such that $(M_2\Phi_0, \Phi_0^*)_4 = 1$, where $(\cdot, \cdot)_k$ denotes the standard scalar product in $[L_2(\Omega)]^k$. Now equation $M_0\Phi = g$ is solvable if and only if $(g, \Phi_0^*)_4 = 0$. Therefore we can calculate all $\lambda_j$ and $\Phi_j$ successively from (2.16). Under the assumption of the theorem, $\lambda_1 = \cdots = \lambda_{d-1} = 0$; hence,

$$-\frac{1}{(M_2\Phi_0, \Phi_0^*)_4}(dM_1\Phi_{d-1} + d(d-1)M_2\Phi_{d-2}, \Phi_0^*)_4 = \lambda_d = e_{0d} d! \neq 0.$$

This implies that

$$M_0\Phi_n + nM_1\Phi_{n-1} + n(n-1)M_2\Phi_{n-2} = 0, \quad \text{for } n = 0,1,2,\ldots,d-1, \tag{2.17}$$

whereas

$$M_0\Phi_d + d(d-1)M_1\Phi_{d-1} + d(d-1)M_2\Phi_{d-2} = -\lambda_d M_2\Phi_0. \tag{2.18}$$

Let us now compare this approach with the spatial formulation. A Jordan chain takes the form

$$(\tilde{\mathscr{L}}_{0,\lambda_0} - ik_0)\theta_n = \theta_{n-1}, \quad n = 0,1,2,\ldots,d-1, \tag{2.19}$$

with $\theta_n = (\hat{v}^n, \hat{w}^n)^T$, $\theta_0 \neq 0$, and $\theta_{-1} = 0$. Denote by $\theta_0^*$ the eigenvector of the formal adjoint problem $(\tilde{\mathscr{L}}_{0,\lambda}^* + ik_0)\theta_0^* = 0$. Then the condition $(\theta_{d-1}, \theta_0^*)_6 \neq 0$ implies that the length of the Jordan chain is exactly $d$. Of course, we have $(\theta_n, \theta_0^*)_6 = 0$ for $n = 0, \ldots, d-2$.

If we define $\hat{P}_n = -\operatorname{div}_\Omega \hat{u}^n - w_3^n e_3$, then the first three components of (2.19) just imply that $\hat{w}^n = ik\hat{u}^n - \hat{u}^{n-1} - \tilde{P}_n e_3$. Hence $\hat{w}^n$ and $\hat{w}^{n-1}$ can be eliminated from (2.19). Introducing $\hat{\Phi}^n = (\hat{v}^n, \hat{P}^n)^T$ we find that $M_0\hat{\Phi}_n + iM_1\hat{\Phi}_{n-1} - M_2\hat{\Phi}_{n-2} = 0$. Thus, if (2.17) holds, we can recover a Jordan chain of order $d$ via $(\hat{u}^n, \hat{P}^n)^T = \hat{\Phi}_n = \frac{(-i)^n}{n!}\Phi_n$.

The opposite conclusion, starting from the Jordan chain $(\hat{v}^n, \hat{w}^m)$, also holds. One finds $\Phi_n$ such that (2.17) holds, and hence (2.16) with $\lambda_1 = \cdots = \lambda_{d-1} = 0$ also holds. The conclusion $\lambda_d \neq 0$ is valid, since the length of the chain is exactly $d$. QED

*Remark 2.2.* If the eigenvalue $\lambda(\mu, k)$ is not simple, then the correspondence between the length of the Jordan chain (or chains) of the problem (2.14) and properties of the spectrum of (2.13) is not so straightforward. Yet, let us suppose that $\lambda(0, k_0)$ is semisimple of order $n_0$. In general the eigenvalue $\lambda$ splits with varying $k$ into $n_0$ different eigenvalues which can be represented by a Puiseux expansion; see KATO [1966]. If for any reason each branch $\lambda_j(\mu, k)$ is an analytic function with expansion (2.9), then in problem (2.14) one recovers $n_0$ Jordan chains of the corresponding lengths $d_j$.

*Remark 2.3.* In one-parameter families the condition $e_{02} \neq 0$ is generically satisfied, since $L_{\mu,k}$ is not selfadjoint and $\lambda_0(0,k) \in \mathbb{C}$. Yet, there are examples where $\lambda_0(0,k) \in \mathbb{R}$ and the principle of the exchange of stability is fulfilled. Then the length of the Jordan block is at least 4. The latter case (with $k_0 = 0$) appears in the Kolmogorov problem (flow on a two-dimensional rectangle with periodic boundary conditions); see AFENDIKOV & BABENKO [1986].

For the case of critical modes with $k_0 \neq 0$, the simplicity of the eigenvalue $\lambda(0, k_0) = i\omega_0$ leads to the following consequences.

**Corollary 2.4.** *Assume that all the conditions of Theorem 2.1 hold. Moreover, let (2.5) be satisfied.*

1. *If* $\omega_0 \neq 0$ *and* $k_0 \neq 0$, *then the dimension* $n_0$ *of the critical eigenspace is* $n_0 = 4d$.
2. *If* $\omega_0 = 0$, $k_0 \neq 0$, *and* $v \neq \pm \overline{Sv}$, *then the dimension* $n_0$ *of the critical eigenspace is* $n_0 = 4d$.
3. *If* $\omega_0 = 0$, $k_0 \neq 0$, *and* $v = \pm \overline{Sv}$, *then the dimension* $n_0$ *of the critical eigenspace is* $n_0 = 2d$.

Let us suppose now that $\omega_0 \neq 0$ and $k_0 = 0$. Moreover, assume that the shift symmetry $\tau_a$ as well as the reflection symmetry $S$ is present. Since $Sv(k)e^{-ikz}$ is an eigenvector of problem (2.6) belonging to the same eigenvalue $\lambda(\mu)$ as $v(-k)e^{-ikz}$, we find that $\lambda(\mu, -k) = \lambda(\mu, k)$. If we can suppose that the eigenvalue $i\omega_0$ of (2.6) is simple, then we can use Theorem 2.1 to find the length of the Jordan chain in (2.14). Since $S$ is an isometrical operator in $[L_2(\Omega)]^3$, we have $v(-k) = \pm Sv(k)$. This gives two different possibilities for the symmetries in the reduced problem.

## 3. Stability problem of the three-dimensional Poiseuille flow

We now specialize the analysis to the case of the three-dimensional Poiseuille flow. Hence, the spatial domain is $Q = \Omega \times \mathbb{R}$ with $\Omega = S^1 \times (-1, 1)$. Here we assume periodicity in the $x$-direction with period $2\pi/\alpha$. The trivial basic flow is the Poiseuille flow, which reads in nondimensional form

$$V_\delta(y) = (U(y), 0, \delta U(y))^T \quad \text{with } U(y) = 1 - y^2.$$

The parameter $\delta$ is used as an additional parameter. If $v = V_\delta + u$ is the total velocity, then $u$ satisfies problem (2.1) with the boundary conditions written out explicitly as

$$u(t, x, y, z) = u(t, x + 2\pi/\alpha, y, z), \quad u|_{y=\pm 1} = 0. \tag{3.1}$$

By mass conservation ($\operatorname{div} u = 0$) we find that

$$\frac{d}{dz}\int_\Omega u_3 \, dx\, dy = -\int_\Omega \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right) dx\, dy = 0.$$

Hence, the additional spanwise mass flux is a free parameter: $Q_{\text{add}}(t) = \int_\Omega u_3 \, dx\, dy$. Henceforth, we restrict the analysis to a time-independent flux $Q$. Without loss of generality we may assume that $Q_{\text{add}} \equiv 0$. Then the total spanwise flux of $v = V_\delta + u$ equals $4\delta/3$.

The problem admits the following symmetries: translation invariance along the $x$- and $z$-directions and the reflection symmetry $z \to -z$. This means that all mappings in the left-hand side of (2.1) commute with

$$\rho_b\colon\ (\delta, u(x,y,z)) \to (\delta, u(x+b,y,z)),$$

$$\tau_a\colon\ (\delta, u(x,y,z+a)) \to (\delta, u(x,y,z+a)),$$

$$S\colon\ (\delta, u(x,y,z)) \to (-\delta, \hat{D}u(\hat{D}(x,y,z)^T)),$$

where $\hat{D} = \operatorname{diag}(1, 1, -1)$ (cf. (2.5)). Obviously, $S^2 = 1$, $S\tau_a = \tau_{-a}S$, $\rho_b S = S\rho_b$, and $\rho_b\tau_a = \tau_a\rho_b$. If the flow is periodic in the spanwise direction, we take $b \in S^1$, and the symmetry group of the problem is $SO(2) \times O(2)$. If we consider $\delta$ as a fixed parameter, then the reflection symmetry $z \to -z$ is only valid for $\delta = 0$.

Under the assumption that $u(t,x,y,z) = \chi(y)\exp(\lambda t + i(\alpha x + \beta z))$, the linearization of (2.1) and (2.2) yields the spectral problem of stability in the form

$$\left.\begin{aligned} \lambda\chi_1 + i(\alpha + \delta\beta)U\chi_1 + \chi_2 U' + i\alpha p &= R^{-1} l_\kappa \chi_1, \\ \lambda\chi_2 + i(\alpha + \delta\beta)U\chi_2 + p' &= R^{-1} l_\kappa \chi_2, \\ \lambda\chi_3 + i(\alpha + \delta\beta)U\chi_3 + \delta\chi_2 U' + i\beta p &= R^{-1} l_\kappa \chi_3, \\ i\alpha\chi_1 + i\beta\chi_3 + \chi_2' &= 0 \end{aligned}\right\} \quad \text{for } y \in (-1, 1), \tag{3.2}$$

$$\chi = 0 \quad \text{for } y = \pm 1,$$

where $\chi = (\chi_1, \chi_2, \chi_3)^T$, $\kappa = (\alpha^2 + \beta^2)^{1/2}$, and $l_\kappa = \dfrac{d^2}{dy^2} - \kappa^2$. Note that in the spatial dynamics approach, the variable $\beta$ will play the role of the spectral parameter $k$ used in the previous section. With the use of SQUIRE's [1933] transformation, this eigenvalue problem can be reduced to the two-dimensional one by introducing

$$w = (\alpha\chi_1 + \beta\chi_3)/\kappa, \quad \tilde{w} = \beta\chi_1 - \alpha\chi_3.$$

We arrive at the equivalent problem

$$\left.\begin{aligned} \lambda\frac{\kappa}{\alpha + \delta\beta}\chi_2 + i\kappa U\chi_2 + \frac{\kappa}{\alpha + \delta\beta}p' &= \frac{\kappa}{R(\alpha + \delta\beta)} l_\kappa \chi_2, \\ \lambda\frac{\kappa}{\alpha + \delta\beta}w + i\kappa U w + \chi_2 U' + i\kappa\frac{\kappa}{\alpha + \delta\beta}p &= \frac{\kappa}{R(\alpha + \delta\beta)} l_\kappa w, \\ i\kappa w + \chi_2' &= 0, \\ \lambda\frac{\kappa}{\alpha + \delta\beta}\tilde{w} + i\kappa U\tilde{w} + \beta\frac{\kappa(\beta - \delta\alpha)}{\alpha + \delta\beta}U'\chi_2 &= \frac{\kappa}{R(\alpha + \delta\beta)} l_\kappa \tilde{w} \end{aligned}\right\} \quad \text{for } y \in (-1, 1), \tag{3.3}$$

$$w = \chi_2 = \tilde{w} = 0 \quad \text{for } y = \pm 1.$$

Note that this problem is block-diagonal such that $(w, \chi_2, p)$ decouples from $\tilde{w}$. This constitutes a plane problem investigated below. The equation for $\tilde{w}$ with $\chi_2 = 0$ also generates a sequence of eigenvalues $\tilde{\lambda}_n$. However, multiplying $\tilde{\lambda}_n \frac{\kappa}{\alpha + \delta\beta} \tilde{w} + i\kappa U \tilde{w} = \frac{\kappa}{R(\alpha + \delta\beta)} l_\kappa \tilde{w}$ by $\bar{\tilde{w}}$ and integrating over $y$ easily gives $\mathrm{Re}\,\tilde{\lambda}_n < 0$.

Thus, the stability problem reduces to the same problem as in the two-dimensional case, if we define the two-dimensional Reynolds number $\tilde{R} = (\alpha + \delta\beta)R/\kappa$ and the two-dimensional eigenvalue parameter $\sigma = \frac{\kappa}{\alpha + \delta\beta}\lambda$. Hence, introducing the potential $\varphi$ with $\chi_2 = i\kappa\varphi$ and $w = -\varphi'$, we find that the spectral problem reduces to the Orr-Sommerfeld problem

$$i\kappa\tilde{R}((U - \sigma/i\kappa) l_\kappa \varphi - U''\varphi) = l_\kappa^2 \varphi \text{ for } y \in (-1, 1), \quad \varphi = \varphi' = 0 \text{ at } y = \pm 1. \quad (3.4)$$

Thus, the neutral curve of stability of the three-dimensional problem (3.2) is determined by the relation

$$\mathrm{Re}\,\sigma_0(\kappa, \tilde{R}) = \mathrm{Re}\,\sigma_0\left(\sqrt{\alpha^2 + \beta^2}, \frac{\alpha + \delta\beta}{\sqrt{\alpha^2 + \beta^2}} R\right) = 0, \quad (3.5)$$

where $\sigma_0$ is the eigenvalue of the Orr-Sommerfeld problem (which is two-dimensional) with minimal real part.

Choose any point $(\kappa_0, R_0)$ on the neutral curve of stability of the two-dimensional problem (3.4), with the eigenfunction $\varphi$. Moreover, choose the eigenfunction $\zeta(y)$ of the adjoint Orr-Sommerfeld problem such that $\langle l_{\kappa_0}\varphi, \zeta\rangle = 1$, where $\langle u, v\rangle = \int_{-1}^{1} u(y)\bar{v}(y)\,dy$. Then, as calculated in Afendikov & Varin [1991], we find the expansion

$$\sigma_0(\kappa, R) = -i\kappa_0 c_0 + E(R - R_0) + H(\kappa - \kappa_0) + \mathcal{O}(|\kappa - \kappa_0|^2 + |R - R_0|^2), \quad (3.6)$$

where

$$E = \frac{\partial\sigma_0(\kappa_0, R_0)}{\partial R} = -\frac{i\kappa_0}{R_0}\langle (U - c_0) l_{\kappa_0}\varphi - U''\varphi, \zeta\rangle,$$

$$H = \frac{\partial\sigma_0(\kappa_0, R_0)}{\partial\kappa} = -\frac{1}{R_0}\langle 4\kappa_0 l_{\kappa_0}\varphi + iR_0(U l_{\kappa_0}\varphi - U''\varphi) + 2i\kappa_0^2 R_0(c_0 - U)\chi_0, \zeta\rangle$$

with $c_0 = -\mathrm{Im}\,\sigma_0(\kappa_0, R_0)/\kappa_0$.

The relation between the three-dimensional and two-dimensional stability problem is given through $\lambda(\beta, R, \delta) = \frac{\alpha + \delta\beta}{\kappa}\sigma_0\left(\kappa, \frac{\alpha + \delta\beta}{\kappa} R\right)$ with $\kappa^2 = \alpha^2 + \beta^2$. In the case $\delta = 0$, we immediately see that the neutral stability curve $R = R(\beta, 0)$ always has an extremum at $\beta = 0$. It is a local minimum if $\mathrm{Re}\,e_{02} < 0$ and a local maximum if $\mathrm{Re}\,e_{02} > 0$ due to the expansion

$$\lambda(\beta, R, 0) = i\omega_0 + E(R - R_0) + e_{02}\beta^2 + \text{h.o.t.}, \quad \text{where } e_{02} = \frac{H}{2\alpha} + \frac{ic_0}{2\alpha} - \frac{R_0 E}{2\alpha^2}. \quad (3.7)$$

Here h.o.t. designates higher-order terms. Hence, both values can be found completely by the two-dimensional theory as developed in AFENDIKOV & VARIN [1991].

In particular, numerical calculations give $e_{02}(\alpha, R_0) \neq 0$ along the whole neutral curve in the $(\alpha, R)$ plane. Using Theorem 2.1 allows us to conclude that the

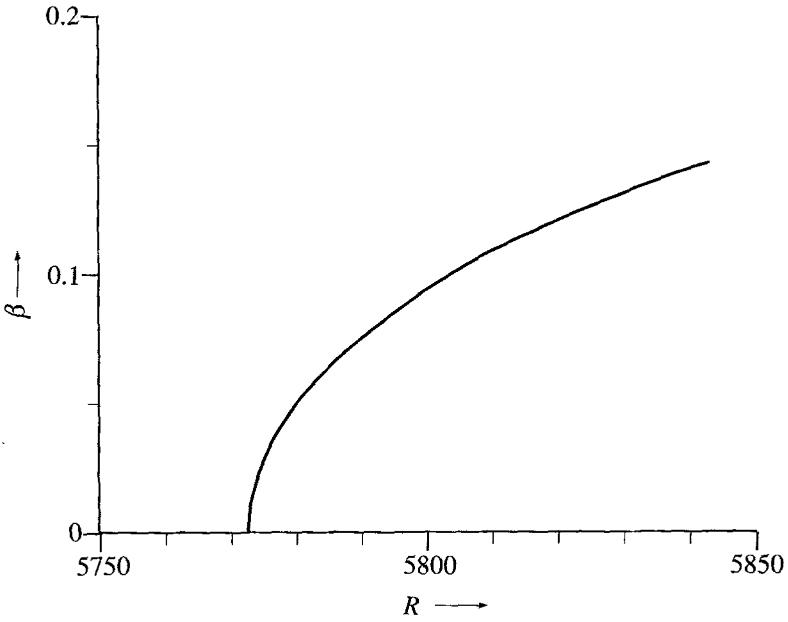


Fig. 1. $\alpha = 1.021$

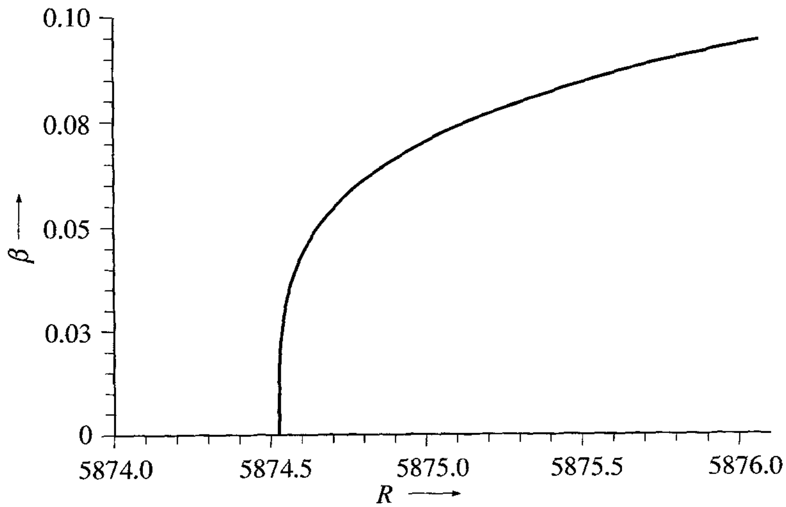


Fig. 2. $\alpha = 0.988$

length of the Jordan block is 2. Thus we are in the same situation as studied in Iooss & Mielke [1992] in a reaction-diffusion system (there the length of the block was not checked but was postulated to be 2). We shall find the same bifurcation equations for our three-dimensional Poiseuille flow problem if $\mathrm{Re}\, e_{02} \geqq 0$.

In Figures 1–4 neutral curves are displayed for different values of the parameter

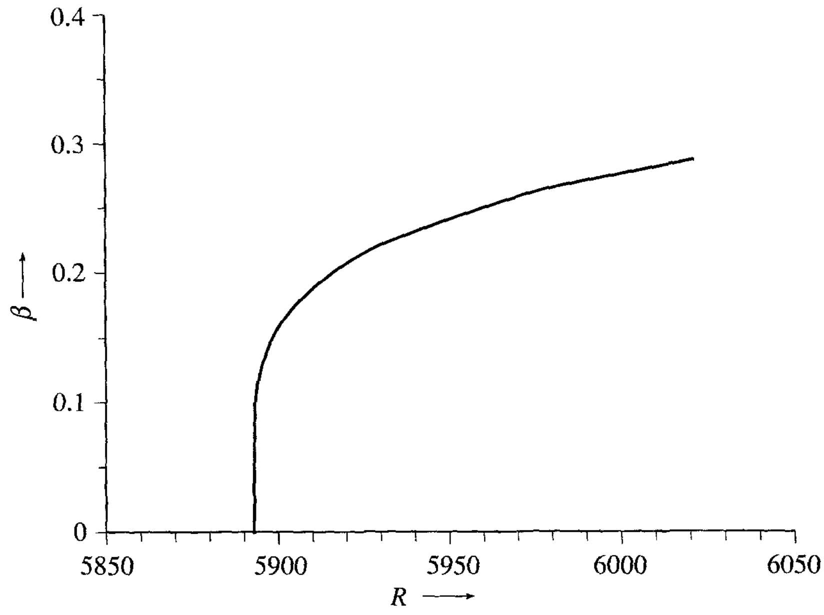


Fig. 3. $\alpha = 0.985$

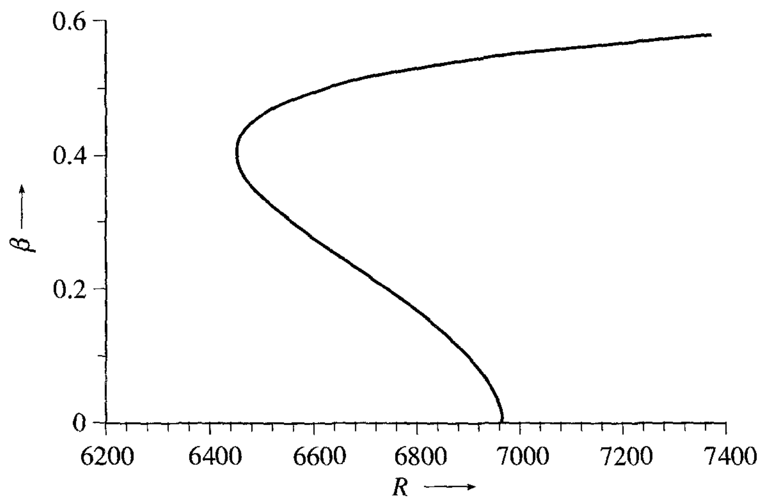


Fig. 4. $\alpha = 0.900$

$\alpha$. The direct computation of $e_{02}$ by the formula (3.7) gives for $\alpha = \alpha_4 \approx 0.9874$ that $\operatorname{Re} e_{02} = 0$, which, however, does not influence the length of the Jordan block.

Yet, for $\alpha < \alpha_4$ the neutral curve of instability no longer has its minimum at $\beta = 0$ but at the two values $\beta = \pm\beta_0$ with $\beta_0 > 0$. We do not study this case here. It would lead to an analysis similar to that in IOOSS & MIELKE [1991]. In order to simplify the problem, let us restrict ourselves to the study of travelling waves of the form

$$u(x - ct, y, z), \quad \text{with } c = c_0 + \varepsilon,\ c_0 = \omega_0/\alpha,\ \omega_0 = \operatorname{Im} \sigma_0(\alpha, R_0) \tag{3.8}$$

where $\varepsilon$ is a small parameter.

*Remark 3.1.* Using the results in IOOSS & MIELKE [1991], we could apply the center-manifold technique also for general time-periodic solutions with additional $x$ periodicity. However, analyzing the critical eigenfunctions and the action of the time shift and the shift in $x$-direction shows that they coincide on these eigenfunctions. By standard arguments in center-manifold theory (cf. IOOSS & ADELMEYER [1992]) this leads to the conclusion that *all* bifurcating solutions are in fact travelling waves. Thus, ansatz (3.8) is no restriction for the bifurcation problem.

We are looking for perturbations of the Poiseuille flow $V_\delta = (U, 0, \delta U)^T$ which are periodic in $x$ with the period $2\pi/\alpha$, vanish at the boundary and travel downstream with speed $c = c_0 + \varepsilon$. We assume that $(\alpha, R_0)$ is a point on the two-dimensional neutral curve and let $R = R_0 + \mu$. Note that the spanwise part $(0, 0, \delta U)^T$ can also be viewed as a small perturbation, since in the following definitions of function spaces we do not prescribe the spanwise flux. Let us introduce the operator $\mathscr{L}_{\mu,c,\delta}$: $D(\mathscr{L}_{\mu,c,\delta}) \to X$, with $X = [H_0^1(\Omega)]^3 \times [L_2(\Omega)]^3$ and

$$D(\mathscr{L}_{\mu,c,\delta}) = \{(u, w) \in [H_0^2(\Omega)]^3 \times [H^1(\Omega)]^3 : w_1, w_2, u_{1,x} + u_{2,y} \in H_0^1(\Omega)\}, \tag{3.9}$$

where $H_0^k(\Omega) = \{u \in H^k(\Omega) : u(x, y) = u(x + 2\pi/\alpha, y), u(x, \pm 1) = 0\}$ for $k = 1, 2$. The operator $\mathscr{L}_{\mu,c,\delta}$ is obtained from $\mathscr{L}_{\mu,\partial_t}$ defined in (2.10) by replacing $\frac{\partial}{\partial t} u$ by $-c\frac{\partial}{\partial x} u$.

We may write $\mathscr{L}_{\mu,c,\delta} = \mathscr{L}_{\mu,c,0} + \delta\mathscr{A}$ where $\delta\mathscr{A}$ is a small bounded perturbation. Hence, it will suffice to study $\mathscr{L}_{\mu,c,0}$. As in IOOSS, MIELKE, & DEMAY [1989], $\mathscr{L}_{\mu,c,0}$ satisfies the resolvent estimate

$$\|(\mathscr{L}_{\mu,c,0} - i\beta)^{-1}\|_{X \to X} = \mathscr{O}\left(\frac{1}{1 + |\beta|}\right) \quad \text{for } |\beta| \to \infty,\ \beta \in \mathbb{R}, \tag{3.10}$$

and has a compact resolvent. To verify this, it is enough to mention that the difference $\mathscr{L}_{\mu,c,0} - \mathscr{L}_{\mu,0,0}$ is an operator of lower order and can be treated in the same way as the "convective" terms in IOOSS, MIELKE, & DEMAY [1989]. From

(2.14) with $k = \beta \in \mathbb{R}$ after elimination of $w$, we obtain the problem

$$\left.\begin{aligned} -\frac{1}{R}(\Delta_\Omega - \beta^2)u + (U-c)\frac{\partial u}{\partial x} + U'u_2e_1 + \nabla_\Omega p + i\beta p e_3 = 0,\\ \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + i\beta u_3 = 0, \end{aligned}\right\} \quad \text{for } (x,y) \in \Omega, \tag{3.11}$$

$$u(x + 2\pi/\alpha, y) = u(x,y), \quad u = 0 \quad \text{for } y = \pm 1.$$

We may expand $u$ into a Fourier series $u = \sum_{-\infty}^{\infty} v_n(y) e^{i\alpha n x}$. Since $U$ is independent of $x$, the system decouples, and we can look for $u$ in the form $v_n(y)e^{i\alpha n x}$, $n \in \mathbb{Z}$. For fixed $n$ we write $v_n = (a_1, a_2, a_3)^T$ to simplify notations. For $n = 0$ we arrive at

$$\begin{aligned} -l_\beta a_1 + R[-i\beta a_1 + U'a_2] = -l_\beta a_2 + Rp' = 0,\\ -l_\beta a_3 + i\beta R p = a_2' + i\beta a_3 = 0. \end{aligned} \tag{3.12}$$

For the components $a_2$ and $a_3$ we have the identity

$$\int_{-1}^{1} (|a_2'|^2 + |a_3'|^2 + \beta^2(|a_2|^2 + |a_3|^2))\,dy = 0, \tag{3.13}$$

which is obtained by multiplying the second and the third equations in (3.12) by $\bar{a}_2$ and $\bar{a}_3$, respectively, and integrating by parts. For $\beta \neq 0$, (3.13) implies that $a_2 = a_3 = 0$ and hence that $a_1 = p = 0$. But if $\beta = 0$, then the vector $\Phi_0 = (0,0,0,0,0,1)^T$ corresponding to the variations of the pressure is in $\ker \mathscr{L}_{\mu,c,0}$ and, as was mentioned before, gives rise to the Jordan block of length 2.

All the cases $n \neq 0$ can be treated in the same way, since $\alpha$ can be replaced by $\tilde{\alpha} = n\alpha$. Hence, it is enough to deal with $n = 1$. There, we recover the spectral problem (3.2), which in turn is equivalent to (3.4). That is why for $R = R_0$ the linear operator $\mathscr{L}_{0,c_0,0}$ has no eigenvalues $i\beta$ for $\beta \neq 0$, but does have an additional eigenvector $\Psi_0$ corresponding to the eigenvalue $\beta = 0$,

$$\mathscr{L}_{0,c_0,0}\Phi_{2,3} = 0 \quad \text{where } \Phi_2 = \operatorname{Re}\Psi_0,\ \Phi_3 = \operatorname{Im}\Psi_0$$

with $\Psi_0(x,y) = e^{i\alpha x}(i\varphi'(y), \alpha\varphi(y), 0, 0, 0, R_0 p(y))^T$. The function $\varphi(y)$ is the eigenfunction of the Orr-Sommerfeld operator, and the "pressure" can be determined from (3.3). Since our numerical calculations give $e_{02}(\alpha, R_0) \neq 0$ along the neutral curve of the Orr-Sommerfeld problem, we know by Theorem 2.1 that the Jordan block has length 2. To calculate the generalized eigenvectors $\Phi_{4,5}$ satisfying $\mathscr{L}_{0,c_0,0}\Phi_{4,5} = \Phi_{2,3}$, let us denote by $K(t,y)$ Green's function for the problem

$$\psi'' - \alpha^2\psi = f \quad \text{for } y \in (-1,1),\ \psi(\pm 1) = 0.$$

Then $\Phi_4 = \operatorname{Re}\Psi_1$ and $\Phi_5 = \operatorname{Im}\Psi_1$ where

$$\Psi_1(x,y) = e^{i\alpha x}\left(0, 0, R_0 \int_{-1}^{1} K(t,y)p(t)\,dt, i\varphi'(y), \alpha\varphi(y), 0\right)^T.$$

## 4. Reduction of the nonlinear problem

The travelling waves under consideration satisfy the partial differential equation

$$\frac{d}{dz}\theta = \mathscr{L}_{\mu,c,\delta}\theta + \mathscr{B}(\theta,\theta), \tag{4.1}$$

which is obtained from (2.10) by replacing $\partial/\partial t$ with $-c\partial/\partial x$ when using $V_\delta = (U, 0, \delta U)^T$ as the basic flow. The functional-analytic setup for the linear operator $\mathscr{L}_{\mu,c,\delta}$ is defined as in (3.9).

It was demonstrated in the previous section that, for every fixed $(\alpha, R_0)$ on the two-dimensional neutral curve, $\mathscr{L}_{0,c_0,0}$ has the eigenvalue $\beta = 0$ with the generalized eigenspace

$$X_0 = \mathrm{Span}\{\Phi_0, \Phi_1, \mathrm{Re}\,\Psi_0, \mathrm{Im}\,\Psi_0, \mathrm{Re}\,\Psi_1, \mathrm{Im}\,\Psi_1\} \subset X.$$

Using the equivalence of the classical stability problem and the spatial dynamics description, we determine the unfolding of the fourfold eigenvalue 0 (neglecting the trivial zero eigenvalue due to the pressure indeterminacy). Recall the definitions of the small parameters $\mu = R - R_0$, $\varepsilon = c - c_0$, and of the spanwise mass flux $4\delta/3$ and recall that $k \in \mathbb{C}$ was used as a spectral parameter for spatial operators with $\beta = k$ in case of real $k$.

**Lemma 4.1.** *There exists a small neighborhood $\mathscr{U}$ of $0 \in \mathbb{R}^3$ such the critical eigenvalues of $\mathscr{L}_{\mu,c_0+\varepsilon,\delta}$ are given by 0 (double), $k_1, \bar{k}_1, k_2$, and $\bar{k}_2$, where $k_1$ and $k_2$ are the solutions of*

$$k^2 + ik\delta g_1(\mu,\varepsilon,\delta^2) + g_0(\mu,\varepsilon,\delta^2) = 0. \tag{4.2}$$

*The functions $g_1$ and $g_0$ are analytic in $(\mu,\varepsilon,\delta) \in \mathscr{U}$ and have the expansions*

$$\begin{aligned} e_{02}g_1(0,0,0) &= a := -c_0 + iR_0E/\alpha, \\ e_{02}g_0(\mu,\varepsilon,\delta^2) &= i\alpha\varepsilon - E\mu + \mathcal{O}(|(\mu,\varepsilon,\delta^2)|^2) \end{aligned} \tag{4.3}$$

*with $g_0(0,0,\delta^2) = 0$.*

**Proof.** By the equivalence of that classical and spatial descriptions, we see that $\mathscr{L}_{\mu,c_0+\varepsilon,\delta}$ has four nontrivial small eigenvalues $k \in \mathbb{C}$, which are solutions of the dispersion relation

$$0 = G(\mu,\varepsilon,\delta) := i\alpha(c_0+\varepsilon) + \frac{1+\delta k/\alpha}{\sqrt{1+k^2/\alpha^2}}\sigma_0\left(\sqrt{\alpha^2+k^2}, \frac{1+\delta k/\alpha}{\sqrt{1+k^2/\alpha^2}}(R_0+\mu)\right),$$

where $\sigma_0$ is the critical eigenvalue (see (3.5)). Since $G$ is analytic and $G(k,0,0,0) = e_{02}k^2 + \text{h.o.t.}$, the coefficient functions $g_j$ are obtained by rewriting $G$ with the aid of Weierstrass' Preparation Theorem in the form

$$G(k,\mu,\varepsilon,\delta) = T(\mu,\varepsilon,\delta^2)(k^2 + ik\delta g_1(\mu,\varepsilon,\delta^2) + g_0(\mu,\varepsilon,\delta^2))$$

with $T(0,0,0) = e_{02} \neq 0$.The evenness in $\delta$ follows from the reflection symmetry, which here means that $G(-k, \mu, \varepsilon, -\delta) = G(k, \mu, \varepsilon, \delta)$. Moreover, since by definition $G(0, \mu, \varepsilon, \delta)$ is independent of $\delta$, it follows that $g_0(0, 0, \delta^2) = 0$. If we employ the expansion (3.6) for $\sigma_0$, the desired result follows. QED

To apply the abstract center-manifold theorem in MIELKE [1988], we rewrite (4.1) in the form

$$\frac{d}{dz}\theta - \mathscr{L}_{0,c_0,0}\theta = \mathscr{N}(\mu, \varepsilon, \delta, \theta) := (\mathscr{L}_{\mu, c_0+\varepsilon, \delta} - \mathscr{L}_{0,c_0,0})\theta + \mathscr{B}(\theta, \theta).$$

The nonlinearity $\mathscr{N}$ is an analytical mapping from a neighborhood of zero in $\mathbb{R}^3 \times D(\mathscr{L}_{0,c_0,0})$ into $X$. To prove this, it is enough to note the $\mathscr{B}$ is bilinear and bounded. For the boundedness, use Sobolev's imbedding theorem $H^2(\Omega) \subset C^0(\Omega)$ (since $\Omega \subset \mathbb{R}^2$) and the fact that in every product (see the definition of $\mathscr{B}$ in (2.11)) one term is in $H^2(\Omega)$ and the other in $H^1(\Omega)$.

The unique spectral projection $P_0$: $X \to X$ (with $P_0 X = X_0$ and $\mathscr{L}_{0,c_0,0}P_0 = P_0\mathscr{L}_{0,c_0,0}$) defines the splitting $\theta = \theta_0 + \theta_1$ with $\theta_0 = P_0\theta \in X_0$ and $\theta_1 \in X_1 = (I - P_0)X$. For our case the center-manifold theorem in MIELKE [1988] reads as follows.

**Theorem 4.2.** *For each* $m \in \mathbb{N}$ *there exist neighborhoods of zero* $Y_0 \in X_0$, $Y_1 \subset X_1 \cap D(\mathscr{L}_{\mu,c_0})$, *and* $U \subset \mathbb{R}^3$ *and a function* $h(\mu, \varepsilon, \delta, \theta_0) \in C^m(U \times Y_0 \to Y_1)$ *satisfying*

1) $h(0,0,0) = D_{\theta_0}h(0,0,0) = 0$.
2) *For each* $(\mu, \varepsilon, \delta) \in U$ *the set* $\mathscr{M}_{\mu,\varepsilon,\delta} = \{\theta_0 + h(\mu, \varepsilon, \delta, \theta_0) \in D(\mathscr{L}_{0,c_0,0})\colon \theta_0 \in Y_0\}$ *is a local integral manifold of* (4.1).
3) *Every solution of* (4.1) *with* $\theta(z) \in Y_0 \times Y_1$ *for all* $z \in \mathbb{R}$ *belongs to* $\mathscr{M}_{\mu,\varepsilon,\delta}$.

This theorem allows us to reduce the study of all solutions of (4.1) which are *uniformly small* in $z \in \mathbb{R}$. Since these solutions lie in $\mathscr{M}_{\mu,\varepsilon,\delta}$, they have the form $\theta(z) = \theta_0(z) + h(\mu, \varepsilon, \delta, \theta_0(z))$ and satisfy the "reduced problem"

$$\frac{d}{dz}\theta_0 = \mathscr{L}_{0,c_0,0}\theta_0 + P_0\mathscr{N}(\mu, \varepsilon, \delta, \theta_0 + h(\mu, \varepsilon, \delta, \theta_0)).$$

If we let $\theta_0 = \gamma_1\Phi_0 + \gamma_2\Phi_1 + A\Psi_0 + \overline{A\Psi_0} + B\Psi_1 + \overline{B\Psi_1}$, the reduced problem can be written as the ordinary differential equation

$$\begin{aligned}
\frac{d\gamma_1}{dz} &= \gamma_2 + f_1(\mu, \varepsilon, \delta, \gamma_2, A, \bar{A}, B, \bar{B}),\\
\frac{d\gamma_2}{dz} &= f_2(\mu, \varepsilon, \delta, \gamma_2, A, \bar{A}, B, \bar{B}),\\
\frac{dA}{dz} &= B + f_3(\mu, \varepsilon, \delta, \gamma_2, A, \bar{A}, B, \bar{B}),\\
\frac{dB}{dz} &= f_4(\mu, \varepsilon, \delta, \gamma_2, A, \bar{A}, B, \bar{B}),
\end{aligned} \tag{4.4}$$

where $\gamma_1, \gamma_2, f_1 f_2 \in \mathbb{R}$, $A$, $B$, $f_3, f_4 \in \mathbb{C}$ with

$$f_j(\mu, \varepsilon, \gamma_2, A, \bar{A}, B, \bar{B}) = \mathcal{O}(|(\gamma_2, A, B)|^2 + |(\gamma_2, A, B)| \cdot |(\mu, \varepsilon, \delta)|).$$

Note that $\gamma_1$ does not appear in $f$ since the full problem (and hence the reduced one) is invariant under the addition of a constant to the pressure.

Because of the symmetries of problem (2.10), the reduced system (4.4) is reversible with respect to the involution $S = \mathrm{diag}(1, -1, 1, -1)$. This means that

$$f_j(\mu, \varepsilon, -\delta, -\gamma_2, A, \bar{A}, -B, -\bar{B}) = (-1)^j f_j(\mu, \varepsilon, \delta, \gamma_2, A, \bar{A}, B, \bar{B}) \quad \text{for } j = 1, 2, 3, 4.$$

The translations along the $x$-direction act, due to space periodicity in $x$, like the $SO(2)$-action

$$\tau_a : (\gamma_1, \gamma_2, A, B) \to (\gamma_1, \gamma_2, e^{ia}A, e^{ia}B). \tag{4.5}$$

Now we return to the question of the spanwise mass flux. On the one hand, we have fixed the base flow $V_\delta$ to have spanwise flux $4\delta/3$. On the other hand, the solutions on the center manifold may also have nonzero flux. We find for $(u, w) = \theta_0 + h(\mu, \varepsilon, \delta, \theta_0)$ the additional flux

$$Q_{\text{add}} = \int_\Omega u_3(x, y, z)\, dx\, dy = \tilde{Q}(\mu, \varepsilon, \delta, \gamma_2, A, B) = \frac{4\pi}{3\alpha}\gamma_2 + \text{h.o.t.}$$

The additional flux can be set equal to zero by solving for $\gamma_2$ as a function of $(\mu, \varepsilon, \delta, A, B)$.

After the elimination of $B$ from (4.4), we are led to a second-order equation for the complex amplitude $A = A(z)$:

$$\frac{d^2}{dz^2}A = g\left(\mu, \varepsilon, \delta, A, \bar{A}, \frac{d}{dz}A, \frac{d}{dz}\bar{A}\right). \tag{4.6}$$

Using the reversibility $(z, \delta, A) \to (-z, -\delta, A)$ and the $SO(2)$-action $\tau_a$, we find the expansion

$$\begin{aligned} g\left(\mu, \varepsilon, \delta, A, \bar{A}, \frac{d}{dz}A, \frac{d}{dz}\bar{A}\right) &= g_0(\mu, \varepsilon, \delta^2)A + \delta g_1(\mu, \varepsilon, \delta^2)\frac{d}{dz}A \\ &\quad + g_2(\mu, \varepsilon, \delta^2)|A|^2 A \\ &\quad + \mathcal{O}\left(\left|\frac{dA}{dz}\right|\left(\left|\frac{dA}{dz}\right|^2 + |A|^2\right) + |A|^5\right). \end{aligned} \tag{4.7}$$

It is clear that the coefficients of the linear terms can be calculated explicitly by the use of Lemma 4.1.

From the nonlinear terms only $b := e_{02} g_2(0, 0, 0)$ is relevant for the bifurcation analysis. To see this, note that $\nu^2 = |g_0(\mu, \varepsilon, \delta^2)|$ is small. If we assume that $b(0, 0) \neq 0$, then this implies that small solutions $A$ scale such that $\frac{d^j A}{dz^j} = \mathcal{O}(\nu^{j+1})$.

In fact, letting

$$\eta = \nu z, \quad A = \nu \hat{A}, \quad \mu = \nu^2 \hat{\mu}, \quad \varepsilon = \nu^2 \hat{\varepsilon}, \quad \delta = \nu \hat{\delta},$$

and using the expansion in Lemma 4.1, we arrive at the limit problem

$$-e_{02} \frac{d^2}{d\eta^2} \hat{A} = (i\alpha\hat{\varepsilon} - E\hat{\mu})\hat{A} + \alpha\hat{\delta}\frac{d}{d\eta}\hat{A} + b|\hat{A}|^2\hat{A}. \tag{4.8}$$

Let us now demonstrate that the coefficient $b(0,0)$ can be found from the analysis of the plane Poiseuille flow. In this way we avoid the troublesome calculations of the straightforward approach, which involves the construction of the biorthogonal basis of the generalized kernel of the adjoint $\mathscr{L}^*_{\mu,c,0}$ and the calculation of all quadratic terms in $h(0,0,\theta_0)$. The idea is to compare the predictions of our normal form (4.7) for the $z$-independent solutions with the predictions of the classical center-manifold theory for the two-dimensional Poiseuille flow as studied in AFENDIKOV & VARIN [1991].

For the two-dimensional Poiseuille flow we may also prescribe the downstream periodicity with fixed period $2\pi/\alpha$. We consider general time dependence; then the only remaining parameter is $\mu = R - R_0$. Classical center-manifold theory shows that a Hopf bifurcation occurs, which can be described by a reduced equation of the form

$$\frac{d}{dt}\tilde{A} = \sigma_0(R_0 + \mu)\tilde{A} + \tilde{b}(\mu)|\tilde{A}|^2\tilde{A} + \mathcal{O}(|\tilde{A}|^5).$$

Here $\sigma_0$ is again the critical eigenvalue from (3.5), yet $\tilde{A}$ and $\tilde{b}$ are not directly related to $A$ and $b$ in (4.7). We assume that the coefficient $b_0 = \tilde{b}(0)$ is known from the numerical calculations as explained in AFENDIKOV & VARIN [1991].

Using the expansion $\sigma_0(R_0 + \mu) = i\omega_0 + E\mu + \mathcal{O}(\mu^2)$ and assuming that $\operatorname{Re} b_0 \neq 0$, we find time-periodic solutions $\tilde{A}_\mu(t) = r(\mu)e^{i\omega(\mu)t}$ with

$$r^2(\mu) = -\frac{\operatorname{Re} E}{\operatorname{Re} b_0}\mu + \mathcal{O}(\mu^2) \quad \text{and} \quad \omega(\mu) = \omega_0 + \left(\operatorname{Im} E - \frac{\operatorname{Re} E}{\operatorname{Re} b_0}\operatorname{Im} b_0\right)\mu + \mathcal{O}(\mu^2).$$

The bifurcation is supercritical for $\operatorname{Re} E/\operatorname{Re} b_0 < 0$ and subcritical for $\operatorname{Re} E/\operatorname{Re} b_0 > 0$.

However, these solutions also appear as constant solutions on the "spatial center manifold". Thus, we have to find the $z$-independent solutions of (4.6) with travel speed $c = -\omega_0/\alpha + \varepsilon = -\omega/\alpha$ and $\delta = 0$. Looking for such solutions amounts to solving $g_0(\mu, \varepsilon, 0) + g_2(\mu, \varepsilon, 0)|A|^2 + \text{h.o.t.} = 0$. Using (4.3) we find $A(\mu)$ with

$$|A(\mu)|^2 = -\frac{\operatorname{Re} E}{\operatorname{Re} b(0,0)}\mu + \mathcal{O}(\mu^2),$$

$$\varepsilon(\mu) = \frac{1}{\alpha}\left(\operatorname{Im} E - \frac{\operatorname{Re} E}{\operatorname{Re} b(0,0)}\operatorname{Im} b(0,0)\right)\mu + \mathcal{O}(\mu^2).$$

Comparing the two expansions for the different description of the same two-dimensional flows, we conclude that $b(0,0) = \gamma\tilde{b}(0)$ for some $\gamma \in (0, \infty)$. The case

$\gamma < 0$ is excluded since the bifurcation direction has to be the same. The positive factor $\gamma$ can easily be compensated by the scaling $A \to A/\sqrt{\gamma}$, and hence it is not essential from the qualitative point of view.

Thus, all the relevant quantities in the limit problem (4.8) are obtained without carrying out the center-manifold reduction in the three-dimensional problem. The linear part was reduced to two dimensions by the Squire theory and the lowest nonlinear term is determined from the plane Poiseuille problem.

A general discussion of the reduced problem (4.6) or the scaled limit (4.8) would be of great interest. Below, we consider only periodic solutions. However, it seems very likely that for $\hat{\delta} \neq 0$ and suitable $(\hat{\varepsilon}, \hat{\mu})$ there exist other bounded solutions, such as heteroclinic solutions connecting the trivial state $\hat{A} = 0$ $(\eta \to -\infty)$ with spiral solutions of the form $\hat{A}(\eta) = re^{i(\omega\eta + \beta)}$ $(\eta \to +\infty)$. Such solutions would correspond to standing fronts in Poiseuille flow. The existence of such solutions is not yet known. Some preliminary results are given in IOOSS & MIELKE [1992], but they are not applicable here.

## 5. Discussion

### *5.1. Comparison to an amplitude equation*

It is interesting to compare our work on the three-dimensional Poiseuille problem to the results in DAVEY, HOCKING & STEWARTSON [1974]. There only the case $\alpha \approx \alpha_2 = 1.02055$ was considered, and the Ginzburg-Landau formalism of multiple scaling was used to derive amplitude equations for the three-dimensional disturbances of Poiseuille flow. By using the scaled variables $\tau = \mu t$, $\xi = \sqrt{\mu}(x + a_{1r}t)$, $\eta = \sqrt{\mu}z$, and $\tilde{A} = \sqrt{\mu}A$, the following limit problem was obtained:

$$\begin{aligned} \frac{\partial}{\partial \tau}\tilde{A} &= a_2 \frac{\partial^2}{\partial \xi^2}\tilde{A} + b_2 \frac{\partial^2}{\partial \eta^2}\tilde{A} + \frac{d_1}{d_{1r}}\tilde{A} + k|\tilde{A}|^2\tilde{A} + q\tilde{A}B, \\ 0 &= \frac{\partial^2}{\partial \xi^2}B + \frac{\partial^2}{\partial \eta^2}(B - |\tilde{A}|^2). \end{aligned} \tag{5.1}$$

Here the variable $B \in \mathbb{R}$ satisfies $|\tilde{A}|^2 - B = -\frac{R_0}{C_0}\frac{\partial}{\partial \xi}P_{01}$ where the pressure has the expansion $p = -2/R_0 x + \text{const.} + \sqrt{\mu}P_{01} + \text{h.o.t.}$ and $C_0 \approx 261.6$.

Reducing this system to our case of travelling waves which are periodic in the $x$-direction amounts to looking for solutions of the form $\tilde{A}(\tau, \xi, \eta) = e^{i(\delta\xi + \rho\tau)}\hat{A}(\eta) \in \mathbb{C}$ and $B(\tau, \xi, \eta) = \hat{B}(\eta) \in \mathbb{R}$, where $\delta$ and $\rho$ are related to the actual downstream wavelength $\alpha$ and the travel speed $c$. Thus, introducing the scaled downstream pressure gradient $\hat{P}_\xi = \hat{B} - |\hat{A}|^2$ we are led to the ordinary differential equation

$$-e_{02}\frac{d^2}{d\eta^2}\hat{A} = w(\alpha, c)\hat{A} + (k + q)|\hat{A}|^2\hat{A} + q\hat{P}_\xi\hat{A}, \quad 0 = \frac{d^2}{d\eta^2}\hat{P}_\xi. \tag{5.2}$$

This system has to be compared with our limit system (4.8) with the equations

$$d\hat{P}/d\eta = \hat{\delta}, \quad d\hat{\delta}/d\eta = 0$$

added. There are essential differences. In our system the parameter $\delta$ corresponds to a spanwise pressure gradient, and the reflection $\eta \to -\eta$ gives $\hat{\delta} \to -\hat{\delta}$ whereas in (5.2) the relation $\hat{P}_\xi \to \hat{P}_\xi$ holds. System (5.1) would match with (4.8) if the term $q\hat{A}\hat{P}$ were replaced by $q\frac{d\hat{A}}{d\eta}\frac{d\hat{P}}{d\eta}$. In conclusion, we see that the amplitude equation (5.1) cannot properly describe the spanwise mass flux and pressure gradients. The additional term $q\hat{A}\hat{P}_\xi$ relates to an extra downstream pressure gradient. Bounded solutions can only be obtained for constant $\hat{P}_\xi$, but this can be compensated for by adjusting Reynold's number, viz., by replacing $w(\alpha, c)$ with $w(\alpha, c) + q\hat{P}_\xi$.

### 5.2. *Spirals and ribbons with zero flux*

As one application of the spatial center-manifold reduction we study the bifurcation of solutions which are periodic in $z$-direction with a large period $2\pi/\beta$ in the case of zero spanwise mass flux, $\delta = 0$. (The case $\delta \neq 0$ will be commented on below.) Of course, this can be done by standard center-manifold theory as described in the beginning of Section 2 or by the Liapunov-Schmidt reduction applied to the full problem as done in Bridges [1989]. However, the limit $\beta \to 0$ is somewhat degenerate. Using the spatial center manifold constructed above, we have the advantage that our calculations involve only an ordinary differential equation, and the limit $\beta \to 0$ is easier to handle.

We are only looking for so-called *spiral solutions* and *ribbons*. The spirals are flow patterns depending only on $\alpha x + \beta z - ct$ and $y$. In the reduced form these solutions have the form $A(z) = e^{i\beta z} A_0$. The ribbons are flow patterns that are even and $2\pi/\beta$-periodic in $z$, and appear as standing time-periodic flow patterns. As we shall see, there are spiral solutions for every small spanwise flux, but the ribbons appear only for $\delta = 0$, as they are reflection symmetric. Both of these $z$-periodic solutions can be found by applying the classical Liapunov-Schmidt reduction to the reduced problem (4.6) when restricted to solutions which are periodic in $z$ with period $2\pi/\beta$. Here $\beta > 0$ is small, since the reduced system (4.6) is defined only for such solutions ($dA/dz$ has to be small).

The linearization at $A = 0$ in the space of $2\pi/\beta$-periodic functions has the spectrum $-m^2\beta^2 e_{02} + g_0(\mu, \varepsilon, 0)$, $m \in \mathbb{Z}$, with eigenvectors $e^{im\beta z}$. Because of (4.3) there is a smooth function $(\mu, \varepsilon) = n(\beta^2) = \mathcal{O}(\beta^2)$ such that $g_0(n(\beta^2), 0) = \beta^2 e_{02}$. Hence, at $(\mu_c, \varepsilon_c) = n(\beta^2)$ the linearization has the eigenvalue 0 with the complex two-dimensional kernel spanned by $e^{\pm i\beta z}$. Thus, the Liapunov-Schmidt reduction gives

$$A(z) = d_1 e^{i\beta z} + d_2 e^{-i\beta z} + r(\mu, \varepsilon, d, z), \quad \text{where } d = (d_1, d_2) \in \mathbb{C}^2, \tag{5.3}$$

with $|r| = \mathcal{O}(|d|(|(\mu,\varepsilon)-(\mu_c,\varepsilon_c)|+|d|^2))$ and $\int_0^{2\pi/\beta} e^{\pm i\beta z} r(\ldots,z)\,dz = 0$. Let us substitute (5.3) into (4.6) to obtain

$$\begin{aligned}0 &= g_0(\mu,\varepsilon,0)(d_1 e^{i\beta z}+d_2 e^{-i\beta z}) - \beta^2(d_1 e^{i\beta z}+d_2 e^{-i\beta z})\\ &\quad + g_2(\mu,\varepsilon,0)(d_1 e^{i\beta z}+d_2 e^{-i\beta z})(|d_1|^2+|d_2|^2+\bar{d}_1 d_2 e^{-2i\beta z}+\bar{d}_2 d_1 e^{2i\beta z})\\ &\quad + \mathcal{O}(\beta^2|d|^3+|d|^5).\end{aligned}$$

Projecting onto $e^{i\beta z}$ and $e^{-i\beta z}$ leads to the bifurcation equations

$$\begin{aligned}d_1(g_0-\beta^2+g_2(|d_1|^2+2|d_2|^2)+\mathcal{O}(|d|^4+\beta^2|d|^2)) &= 0,\\ d_2(g_0-\beta^2+g_2(2|d_1|^2+|d_2|^2)+\mathcal{O}(|d|^4+\beta^2|d|^2)) &= 0.\end{aligned}$$

The spirals are found by letting

$$d_2 = 0, \quad |d_1|^2 = \frac{\operatorname{Re}(\beta^2 - g_0(\mu,\varepsilon,0))}{\operatorname{Re} g_2(\mu,\varepsilon,0)} + \mathcal{O}(\mu^2+\beta^4)$$

or vice versa. The ribbons satisfy

$$d_1 = d_2, \quad |d_{1,2}|^2 = \frac{\operatorname{Re}(\beta^2 - g_0(\mu,\varepsilon,0))}{3\operatorname{Re} g_2(\mu,\varepsilon,0)} + \mathcal{O}(\mu^2+\beta^4).$$

Therefore, for $\beta$ small enough the direction of bifurcation is the same for both types of solutions. This agrees with the numerical results in BRIDGES [1994] where contradicting results from BRIDGES [1989] are revised.

As a consequence of the results in AFENDIKOV & VARIN [1991], we have that in the three-dimensional Poiseuille problem both "spirals" and "ribbons" are bifurcating for $\alpha \geqq \alpha_4$ in the subcritical direction and hence are unstable in the usual sense (see AFENDIKOV, BABENKO, & YURIEV [1982], CHOSSAT & IOOSS [1985] and GOLUBITSKY & STEWART [1986]). For $\alpha \leqq \alpha_4$ the most unstable perturbations are no longer two-dimensional (i.e., $\beta_0 > 0$), and at criticality one obtains a different kind of reduced bifurcation equation. Nevertheless, our analysis can be carried through for Reynolds numbers beyond the critical one. Then the bifurcation of spirals and ribbons with large wavelengths can even appear supercritical for $\alpha < \alpha_3 = 0.90667$ ($< \alpha_4$), see AFFENDIKOV & VARIN [1991]. Practically, these bifurcations are not relevant, since the system becomes unstable much earlier with patterns of smaller period.

The same arguments have to be applied to the stability results in BRIDGES [1989]. The center-manifold method employed there and the standard stability results from GOLUBITSKY, STEWART, & SCHAEFFER [1988] and IOOSS & ADELMEYER [1992] only prove stability for perturbations within the (four-dimensional) center manifold. However, this manifold may be unstable, even in the class of periodic functions with the same period $2\pi/\beta$ as the solution. In fact, for $\alpha - \alpha_4 > \rho(\beta) > 0$ there is instability with respect to the $z$-independent mode of the planar problem. For $\alpha - \alpha_4 < \hat{\rho}(\beta) < 0$ there is instability with respect to modes which are periodic in $z$ with period $2\pi/(m\beta)$ for some $m \geqq 2$. Our theory shows that $\rho(\beta)$, $\hat{\rho}(\beta) \to 0$ for $\beta \to 0$.

### 5.3. *Spirals with non-zero flux*

It should be noted that the spirals constructed in BRIDGES [1989] have zero mass flux. This is not stated explicitly there. However, it follows implicitly from the analysis. The assumption (on p. 356) that the operator $\omega_\beta I - L_0$ has a four-dimensional kernel excludes mass flux. Note, however, that the decomposition $(L_p(\Omega))^n = X_p \oplus G_p(\Omega)$ on p. 339 allows nonzero flux for $u \in X_p$. To make the results rigorous, a further projection onto the orthogonal complement of constant pressure gradients has to be included; see IUDOVICH [1966] and CHOSSAT & IOOSS [1994, Ch. II.2.1, 2.2] for a discussion of these decompositions in the case of periodicity.

Our reduction method allows us to construct spiral solutions for any small prescribed mass flux $4\delta/3$. Spirals are of the form $A(z) = de^{i\beta z}$. Substituting this into the reduced problem (4.6) we find, after division by $d$, the algebraic equation

$$0 = \beta^2 + g_0(\mu, \varepsilon, \delta^2) + i\beta\delta g_1(\mu, \varepsilon, \delta^2) + g_2(\mu, \varepsilon, \delta^2)|d|^2 + \mathcal{O}((|\beta| + |d|^2)|d|^2).$$

Thus, we can solve locally for $|d|^2$ and $\mu$ as functions of $(\varepsilon, \delta, \beta)$.

However, this family can be constructed from the family obtained for $\beta = 0$ as follows. Recall that the spirals correspond to a flow pattern of the form

$$v(t, x, y, z) = V_\delta + u_{\varepsilon,\delta,\beta}(\alpha x + \beta z - \alpha(c_0 + \varepsilon)t, y), \tag{5.4}$$

where $u_{\varepsilon,\delta,\beta}$ is $2\pi$-periodic in the first argument, is small, and has zero spanwise flux. Now we use the fact that the Poiseuille problem is invariant under rotations in the $(x, z)$-plane (see, e.g., OVSIANNIKOV [1982]). Rotating the solution (5.4) such that the wave vector $(\alpha, \beta)$ coincides with $(\tilde{\alpha}, \tilde{\beta}) = (\sqrt{\alpha^2 + \beta^2}, 0)$, we obtain a plane solution with no $z$ dependence. This new solution has the base flow

$$\tilde{V} = \left(\frac{\alpha + \delta\beta}{\tilde{\alpha}} U, 0, \frac{\alpha\delta - \beta}{\tilde{\alpha}} U\right)^T$$

and the associated spanwise flux is close to $4(\alpha\delta - \beta)/(3\sqrt{\alpha^2 + \beta^2})$.

This shows that it is easy to find all spiral solutions from the plane ones ($\beta = 0$) if the base flow carries nonzero spanwise flux ($\delta \neq 0$), whereas going from pure two-dimensional patterns with $\delta = 0$ and $\beta = 0$ to spirals with fixed $\beta \neq 0$ and nonzero mass flux is not possible.

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