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AN EVOLUTIONARY ELASTOPLASTIC PLATE MODEL DERIVED VIA Γ-CONVERGENCE

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This paper is devoted to dimension reduction for linearized elastoplasticity in the rate-independent case. The reference configuration of the three-dimensional elastoplastic body has a two-dimensional middle surface and a positive but small thickness. Under suitable scalings we derive a limiting model for the case in which the thickness of the plate tends to 0. This model contains membrane and plate deformations (linear Kirchhoff–Love plate), which are coupled via plastic strains. We establish strong convergence of the solutions in the natural energy space. The analysis uses an abstract Γ-convergence theory for rate-independent evolutionary systems that is based on the notion of energetic solutions. This concept is formulated via an energy-storage functional and a dissipation functional, such that energetic solutions are defined in terms of a stability condition and an energy balance. The Mosco convergence of the quadratic energy-storage functional follows the arguments of the elastic case. To handle the evolutionary situation the interplay with the dissipation functional is controlled by cancellation properties for Mosco-convergent quadratic energies.

Keywords: Linearized elastoplasticity; rate-independent system; Γ-convergence; Mosco convergence; hysteresis; generalized Prandtl–Ishlinskii operator.

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1. Introduction

The derivation of lower-dimensional theories for bodies such as rods, beams, membranes, plates and shells from a three-dimensional theory is of fundamental importance for engineering applications. A first rigorous justification of the plane membrane system and Kirchhoff's plate equation dates back to Morgenstern,[36,35] respectively. Here "justification" means that the convergence of the solutions of the full three-dimensional system toward the solutions of the limiting model is shown without any additional assumptions on the solutions. Later results for rods, linear and nonlinear plates, or shells can be found in Refs. 8, 24, 9, 4, 7, 13, 22, 10, 14 and the references therein. Most of the recent investigations use a variational approach that is based on the notion of Γ-convergence. This convergence assures, roughly speaking, that (almost) minimizers of the three-dimensional theory (subject to suitable boundary conditions and applied loads) converge to minimizers of the limiting lower-dimensional theory.

As the theory of Γ-convergence is purely static, there are only very few results concerning the justification of similar dimension reductions for evolutionary problems in nonlinear continuum mechanics, see Ref. 1 for a recent result. More often, lower-dimensional theories are derived by *ad hoc* assumptions via formal asymptotic expansions, see e.g. Refs. 33, 20 and 18.

In this paper, we want to give a rigorous justification of a new lower-dimensional elastoplastic plate model in the rate-independent case. It is the recently developed theory of Γ-convergence for rate-independent systems[29] that allows us to do a limit passage from linearized elastoplasticity in three dimensions to a model that combines two two-dimensional linear elastic models, namely the membrane model for in-plane displacements and Kirchhoff's plate equation for the out-of-plane displacement, with the plastic effects.

Rate-independent linearized elastoplasticity can be formulated in different equivalent forms, e.g. as variational inequality, as differential inclusion, or as energetic system. All three are formulated in terms of an energy functional

$$\mathcal{E}_\varepsilon(t, U, P) = \int_\Omega \mathbb{W}_\varepsilon(\mathbf{E}(U), P)\mathrm{d}x - \langle L(t), U\rangle,$$

defined as integral over the rescaled plate domain $\Omega := \omega \times \,]-1, 1[$. Here U and P are the rescaled displacements and plastic strains, respectively. The small parameter $\varepsilon > 0$ is proportional to the unscaled thickness of the plate and occurs in $\mathbb{W}_\varepsilon$ via the corresponding scalings of the strains.

Additionally we have a dissipation potential

$$\mathcal{R}(\dot{P}) = \int_\Omega R(\dot{P}(x))\mathrm{d}x,$$

which is assumed to be independent of ε after rescaling. Rate-independence manifests itself in the fact that $\mathcal{R}$ is positively homogeneous of degree 1, i.e. $\mathcal{R}(\lambda\dot{P}) = \lambda^1\mathcal{R}(\dot{P})$. A typical R has the form $R(\dot{P}) = \sigma_{\text{yield}}|\dot{P}|$, where σ_{yield} is the yield stress.

The solutions have to solve the differential inclusion

$$0 = \mathrm{D}_U\mathcal{E}_\varepsilon(t, U(t), P(t)), \quad 0 \in \partial_{\dot{P}}\mathcal{R}(\dot{P}(t)) + \mathrm{D}_P\mathcal{E}(t, U(t), P(t)), \tag{1.1}$$

where the first equation is the balance of forces and the second is the plastic flow rule.

Since for linearized elastoplasticity the energy $\mathcal{E}_\varepsilon(t, \cdot)$ is a convex quadratic functional, the differential inclusion is fully equivalent to the so-called energetic formulation, which is formulated in terms of an energetic stability condition and the total balance of energy, see (2.7). The advantage of the energetic formulation is that it is based on $\mathcal{E}_\varepsilon$ and $\mathcal{R}$ rather than on their derivatives. Thus, we can use convergence of functionals such as Γ-convergence, Mosco convergence, and continuous convergence.

The derivation of our limiting elastoplastic plate model will be described in Sec. 2 together with the underlying scalings. In Sec. 3 we provide an abstract Γ-convergence result for energetic rate-independent systems (RIS) $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, where $\mathbf{Q}$ is the underlying Hilbert space. In particular, we treat the more general case in which the dissipation functional is allowed to depend on ε as well. Our theory is a special case of the general theory,[29] as we assume that $\mathcal{E}_\varepsilon(t, \cdot)$ is a quadratic functional that converges to $\mathcal{E}_0(t, \cdot)$ in the sense of Mosco convergence, see (3.1). Moreover, $\mathcal{R}_\varepsilon$ is assumed to converge to $\mathcal{R}_0$ in the Mosco as well as in the sense of continuous convergence in the norm topology.

Under natural technical assumptions it is then shown in Theorem 3.1 that the solutions $q_\varepsilon : [0, T] \to \mathbf{Q}$ of the RIS $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, see (1.1) strongly converge to the solution q_0 of the RIS $(\mathbf{Q}, \mathcal{E}_0, \mathcal{R}_0)$. In particular, we emphasize that the exploitation of the quadratic structure of the energy and of the Mosco convergence allows us to avoid the commonly used continuous convergence of $\mathcal{R}_\varepsilon$ with respect to the weak topology, see, e.g. Refs. 21, 27 and 29. The abstract construction of the *mutual recovery sequences* is given in Proposition 3.1 and relies on refined estimates for Mosco-convergent functionals.

The main point here is that the limiting energy $\mathcal{E}_0$ is quite degenerate, as it is only finite if U lies in the space $\mathbf{U}_{\mathrm{KL}}$ of Kirchhoff–Love displacements (see (2.12))

$$U(y, x_3) = (\,V_1(y) - x_3\partial_{y_1}V_3(y), V_2(y) - x_3\partial_{y_2}V_3(y), V_3(y))^{\mathsf{T}}.$$

The proof of Mosco convergence of $\mathcal{E}_\varepsilon(t, \cdot)$ to $\mathcal{E}_0(t, \cdot)$ is given in Sec. 4 and is a generalization of the ideas in Ref. 4, as we have to take into account also the plastic variable.

In Sec. 5 we formulate the limit problem in terms of the in-plane displacements (V_1, V_2), the out-of-plane displacement V_3, and the plastic strain P, which is still defined on all of Ω. For an isotropic material, the limiting model takes the form

$$0 = -\mathrm{div}(\Sigma_0(2\mathbf{E}^{1,2}(V) - [P^{1,2}]_0)) - G_{\mathrm{memb}}(t, \cdot) \qquad \text{in } \omega, \tag{1.2a}$$

$$0 = \mathrm{div}\,\mathrm{div}\left(\Sigma_0\left(\frac{2}{3}\mathrm{D}^2 V_3 + [P^{1,2}]_1\right)\right) - g_{\mathrm{bend}}(t, \cdot) - \mathrm{div}\, G_{\mathrm{bend}}(t, \cdot) \qquad \text{in } \omega, \tag{1.2b}$$

$$0 \in \partial R(\dot{P}) + \mathrm{dev}([\![\Sigma_0(P^{1,2} - \mathbf{E}^{1,2}(V) + x_3\mathrm{D}^2 V_3) \,\|\, 0]\!]) + hP \qquad \text{in } \Omega, \tag{1.2c}$$

where $\Sigma_0(E) := \frac{2\lambda\mu}{\lambda+2\mu}\,\mathrm{tr}\, E I_2 + 2\mu E$. Here, $\mathbf{E}^{1,2}(V) \in \mathbb{R}^{2\times2}_{\mathrm{sym}}$ is the in-plane strain tensor and $\mathrm{D}^2 V_3 \in \mathbb{R}^{2\times2}_{\mathrm{sym}}$ the bending strain tensor.

Equation (1.2a) is the second-order membrane equation for (V_1, V_2), which is coupled to the plastic strain P via the integrals $[\,\cdot\,]_0$ over $x_3 \in\,]-1, 1[$. Equation (1.2b) is a generalization of Kirchhoff's plate equation (of order 4) for V_3. It is also coupled to the plastic strain P, but now with weighted averages $[\,\cdot\,]_1$. The flow rule (1.2c) exhibit the elastic strains as forcing in a very special manner concerning the dependence on x_3.

In the final Sec. 5.3 we show how the last equation can be eliminated using a vector-valued hysteresis operator of play-type. For a suitably defined generalized Prandtl–Ishlinskii operator $\mathfrak{P} = (\mathfrak{P}_E, \mathfrak{P}_H)$ we obtain the two-dimensional system

$$-\mathrm{div}(\Sigma_0(\mathbf{E}^{1,2}(V)) + \mathfrak{P}_E[\mathbf{E}^{1,2}(V), \mathrm{D}^2 V_3](t)) = G_{\mathrm{memb}}(t,\cdot) \quad \text{on } \omega, \tag{1.3a}$$

$$\mathrm{div}\,\mathrm{div}(\Sigma_0(\mathrm{D}^2 V_3) + \mathfrak{P}_H[\mathbf{E}^{1,2}(V), \mathrm{D}^2 V_3](t)) = g_{\mathrm{bend}}(t,\cdot) + \mathrm{div}\, G_{\mathrm{bend}}(t,\cdot) \quad \text{on } \omega. \tag{1.3b}$$

Of course, $\mathfrak{P} = (\mathfrak{P}_E, \mathfrak{P}_H)$ has a memory, which takes care of all the necessary information on previous plastic deformations.

Most of the results in this work were derived for the diploma thesis of the first author.[23]

2. Setup of the Elastoplastic Plate Model

We start from the classical elastoplastic models with hardening. We formulate it in terms of differential inclusions or equivalently as variational inequalities. In Sec. 2.2 we then focus on domains with a plate geometry, i.e. $\Omega_\varepsilon = \omega \times]-\varepsilon, \varepsilon[$, and discuss the suitable scalings to obtain a nontrivial limiting model. The final model will be presented in Sec. 2.3. The convergence proof is the content of Secs. 3 and 4, while Sec. 5 is devoted to a discussion of the derived model.

2.1. *Linearized elastoplasticity as a rate-independent system*

We consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, where $\Gamma_\mathrm{D} \subset \partial\Omega$ denotes the part of the boundary, where we have Dirichlet boundary conditions, i.e. the displacement is prescribed. We set

$$\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^d) := \{u \in \mathrm{H}^1(\Omega;\mathbb{R}^d)\,|\,u|_{\Gamma_\mathrm{D}} = 0\}.$$

We assume that the pair $(\Omega, \Gamma_\mathrm{D})$ satisfies a Korn inequality, i.e.

$$\begin{aligned} &\exists\, c_{\mathrm{Korn}} > 0\ \forall\ u \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^d): \quad \|\mathbf{e}(u)\|_{\mathrm{L}^2} \geq c_{\mathrm{Korn}}\|u\|_{\mathrm{H}^1},\\ &\text{where}\quad \mathbf{e}(u) = \frac{1}{2}(\nabla u + \nabla u^\mathsf{T}) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} := \{A \in \mathbb{R}^{d\times d}\,|\,A = A^\mathsf{T}\}. \end{aligned} \tag{2.1}$$

The elastoplastic properties of the body Ω are described in terms of the linearized strain tensor $\mathbf{e}$ and the plastic strain tensor

$$p \in \mathbb{R}^{d\times d}_{\mathrm{dev}} := \{A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}\,|\,\mathrm{tr}\,A = 0\}$$

via the stored energy density $\mathbb{W}: \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{dev}} \to \mathbb{R}$, which is assumed to be a quadratic function satisfying

$$\begin{aligned} \exists\, c, C > 0\ \forall\ (\mathbf{e}, p) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{dev}}:\ & c(|\mathbf{e}|^2 + |p|^2) \leq \mathbb{W}(\mathbf{e}, p)\\ & \leq C(|\mathbf{e}|^2 + |p|^2). \end{aligned} \tag{2.2}$$

Here "quadratic" means that $\mathbb{W}$ is given by a linear and symmetric operator B on $\mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{dev}}$, see, e.g. (2.5).

Moreover, the plastic flow rule of the material can be formulated in terms of a dissipation potential $R : \mathbb{R}^{d\times d}_{\mathrm{dev}} \to [0,\infty[$, which is assumed to satisfy the following conditions:

$$R \text{ is continuous, convex, and homogeneous of degree 1,} \tag{2.3}$$

where the latter condition means $R(\lambda\dot{p}) = \lambda R(\dot{p})$ for all $\lambda > 0$ and $\dot{p} \in \mathbb{R}^{d\times d}_{\mathrm{dev}}$. This property guarantees that the material response is rate-independent. The corresponding elastic domain $\mathbb{E} \subset \mathbb{R}^{d\times d}_{\mathrm{dev}}$ is defined via $\mathbb{E} := \partial R(0)$, which is the subdifferential of R at 0.

Given time-dependent volume and surface loadings $f_{\mathrm{vol}}(t,\cdot)$ and $f_{\mathrm{surf}}(t,\cdot)$, as well as time-dependent Dirichlet data $u_{\mathrm{D}}(t,\cdot)$, the full elastoplastic problem can be written in the form

$$\begin{aligned} &-\mathrm{div}(\partial_{\mathbf{e}}\mathbb{W}(\mathbf{e}(u),p)) = f_{\mathrm{vol}}(t,\cdot) && \text{in } \Omega,\\ &0 \in \partial R(\dot{p}) + \partial_p \mathbb{W}(\mathbf{e}(u),p) && \text{in } \Omega,\\ &u(t,\cdot) = u_{\mathrm{D}}(t,\cdot) && \text{on } \Gamma_{\mathrm{D}},\\ &\partial_{\mathbf{e}}\mathbb{W}(\mathbf{e}(u),p)\nu = f_{\mathrm{surf}}(t,\cdot) && \text{on } \partial\Omega\backslash\Gamma_{\mathrm{D}}, \end{aligned} \tag{2.4}$$

where ν denotes the outer normal vector on $\partial\Omega$. Here $\sigma = \partial_{\mathbf{e}}\mathbb{W} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ denotes the stress, while $\partial_p\mathbb{W} \in \mathbb{R}^{d\times d}_{\mathrm{dev}}$ contains the deviator of the stress as well as any plastic back stresses.

Example 2.1. Throughout we will use the isotropic stored energy density

$$\mathbb{W}(\mathbf{e},p) = \frac{\lambda}{2}(\mathrm{tr}\,\mathbf{e})^2 + \mu|\mathbf{e}-p|^2 + \frac{h}{2}|p|^2 \tag{2.5}$$

as an example. Here $\lambda, \mu > 0$ are the Lamé constants and $h > 0$ is a measure for kinematic hardening. For this example we have

$$\begin{aligned} \sigma = \partial_{\mathbf{e}}\mathbb{W} &= \lambda\,\mathrm{tr}\,\mathbf{e}\,I_d + \mu(\mathbf{e}-p) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \quad \text{and}\\ \partial_p\mathbb{W} &= -\mathrm{dev}\,\sigma + hp = \mu(p - \mathrm{dev}\,\mathbf{e}) + hp \in \mathbb{R}^{d\times d}_{\mathrm{dev}}, \end{aligned}$$

which shows that hp plays the role of the plastic back stress.

We reformulate the system (2.4) in abstract form for the pair $q = (u,p) \in \mathbf{Q}$ via the energy functional $\mathcal{E} : [0,T] \times \mathbf{Q} \to \mathbb{R}$ and the dissipation functional $\mathcal{R} : \mathbf{Q} \to [0,\infty[$ as follows:

$$\begin{aligned} \mathbf{Q} &:= \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d) \times \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}),\\ \mathcal{E}(t,q) &:= \int_\Omega \mathbb{W}(\mathbf{e}(u)(x),p(x))\mathrm{d}x - \langle \ell(t), q\rangle,\\ \mathcal{R}(\dot{q}) &:= \int_\Omega R(\dot{p}(x))\mathrm{d}x, \end{aligned}$$

where $\ell(t) \in \mathbf{Q}^*$ is defined via

$$\langle \ell(t), q\rangle := \int_\Omega f_{\rm vol}(t,x)\cdot u(x)\mathrm{d}x + \int_{\partial\Omega\setminus\Gamma_{\rm D}} f_{\rm surf}(t,x)\cdot u(x)\mathrm{d}a(x). \qquad (2.6)$$

Without loss of generality we set $u_{\rm D} \equiv 0$ from now on. Otherwise we could replace u by $u - u_{\rm D}$, which would just produce an additional term in $\ell(t)$. We call the triple $(\mathbf{Q}, \mathcal{E}, \mathcal{R})$ a *rate-independent system*.

By (2.1) and (2.2) the functional $\mathcal{E}(t,\cdot) : \mathbf{Q} \to \mathbb{R}$ is uniformly convex and can be written as $\mathcal{E}(t,q) = \mathcal{B}(q) - \langle \ell(t), q\rangle$ with a quadratic form $\mathcal{B}(q) = \frac{1}{2}\langle Aq, q\rangle$ (here obtained by the integral over $\mathbb{W}$). The operator $A : \mathbf{Q} \to \mathbf{Q}^*$ is a symmetric and positive definite isomorphism and we have $\mathrm{D}_q\mathcal{E}(t,q) = Aq - \ell(t)$.

We call a function $q = (u,p) : [0,T] \to \mathbf{Q}$ a solution to the RIS $(\mathbf{Q}, \mathcal{E}, \mathcal{R})$ (and hence to the above elastoplastic problem (2.4)), if it solves one of the following three equivalent problem formulations:

Differential inclusion:

$$0 \in \partial\mathcal{R}(\dot q(t)) + \mathrm{D}_q\mathcal{E}(t, q(t)); \qquad (2.7a)$$

Variational inequality:

$$\forall\, v \in \mathbf{Q}: \ \langle \mathrm{D}_q\mathcal{E}(t,q(t)), v - \dot q\rangle + \mathcal{R}(v) - \mathcal{R}(\dot q) \geq 0; \qquad (2.7b)$$

Energetic formulation:

$$\begin{aligned} &\text{(S) stability:} && \forall\, \tilde q \in \mathbf{Q}: \ \mathcal{E}(t,q(t)) \leq \mathcal{E}(t,\tilde q) + \mathcal{R}(\tilde q - q(t)), \\ &\text{(E) energy balance:} && \mathcal{E}(t,q(t)) + \int_0^t \mathcal{R}(\dot q)\mathrm{d}t = \mathcal{E}(0,q(0)) - \int_0^t \langle \dot\ell, q\rangle \mathrm{d}t. \end{aligned} \qquad (2.7c)$$

We refer to Sec. 2 in Ref. 25 for the equivalence between these three forms. It turns out that the energetic formulation will be especially useful for deriving our limiting model in the process of dimension reduction. Moreover, it is free of derivatives and thus applies also in the more general case, where $\mathcal{E}$ or $\mathcal{R}$ takes the value ∞.

The existence of solutions for (2.7) is classical, see, e.g. Refs. 34, 17, 2, 19, 31 and 25.

Theorem 2.1. *Assume that* $(\mathbf{Q}, \mathcal{E}, \mathcal{R})$ *is as above with* $\ell \in \mathrm{C}^{\rm Lip}([0,T], \mathbf{Q}^*)$ *and that* $q^0 \in \mathbf{Q}$ *is stable at* $t = 0$ *(i.e.* $0 \in \partial\mathcal{R}(0) + \mathrm{D}_q\mathcal{E}(0,q^0)$ *or equivalently* (S) *in* (2.7c) *holds with* $q(0)$ *replaced by* q^0*), then there is a unique solution* $q \in \mathrm{C}^{\rm Lip}([0,T], \mathbf{Q})$ *with* $q(0) = q^0$.

2.2. *Scalings for thin-plate domains*

We now specialize to the case that Ω is a thin plate, i.e. we assume Ω from above is replaced by

$$\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[, \quad \Gamma_{\rm D}^\varepsilon = \gamma_{\rm D} \times]-\varepsilon,\varepsilon[,$$

where $\omega \subset \mathbb{R}^2$ is a planar, bounded Lipschitz domain, the so-called mid-surface of the plate. The boundary part $\gamma_{\rm D} \subset \partial\omega \subset \mathbb{R}^2$ has a positive one-dimensional Hausdorff measure.

Throughout we keep the material laws given via $\mathbb{W}$ and R fixed and obtain an ε-dependent state space $\mathbf{Q}^\varepsilon$ and functionals $\mathcal{E}^\varepsilon$ and $\mathcal{R}^\varepsilon$ defined over Ω^ε. For each $\varepsilon > 0$ all the assumptions of the previous section are satisfied and Theorem 2.1 provides solutions $q^\varepsilon = (u^\varepsilon, p^\varepsilon) : [0, T] \to \mathbf{Q}^\varepsilon$ of the RIS $(\mathbf{Q}^\varepsilon, \mathcal{E}^\varepsilon, \mathcal{R}^\varepsilon)$. We want to study their behavior for $\varepsilon \to 0$. However, to obtain a nontrivial limit we have to do suitable scalings, which we explain now.

For linearized elasticity the scaling of the strains is arbitrary, because it is an infinitesimal theory by definition. In contrast, the theory of linearized elastoplasticity is no longer scaling invariant, because the boundary of the elastic domain $\mathbb{E} = \partial R(0)$ contains the given yield stresses of order 1, i.e. independent of ε. Thus, our theory needs a scaling where the plastic tensor p as well as most of the strains in $\mathbf{e}$ are of order 1.

It is already known from the theory of linearized elasticity[36,35,8] that the strain of in-plane displacements (membrane modes) are smaller than the out-of-plane modes (bending modes). Thus, we look for a scaling of the form

$$u^\varepsilon(x^\varepsilon) = \varepsilon^\alpha S_\varepsilon U(S_\varepsilon x^\varepsilon), \quad p^\varepsilon(x^\varepsilon) = \varepsilon^\beta P(S_\varepsilon x^\varepsilon), \quad \text{where } S_\varepsilon = \mathrm{diag}(1, 1, 1/\varepsilon). \tag{2.8}$$

To simplify the presentation we will choose $\alpha = \beta = 0$ at this stage and refer to Remark 2.1 for more general scalings.

Since $x^\varepsilon \in \Omega^\varepsilon$ is mapped to $x = S_\varepsilon x^\varepsilon \in \Omega_1$, the rescaled function $Q = (U, P)$ will be defined in

$$\mathbf{Q} := \mathrm{H}^1_{\Gamma_{\rm D}}(\Omega; \mathbb{R}^3) \times \mathrm{L}^2(\Omega; \mathbb{R}^{3\times3}_{\rm dev}), \quad \text{where } \Omega := \Omega_1 \quad \text{and} \quad \Gamma_{\rm D} := \gamma_{\rm D} \times]-1, 1[.$$

The scaling acts differently on the components of the strains in $\mathbf{e}(u^\varepsilon)$, as follows

$$\mathbf{e}(u^\varepsilon)(x^\varepsilon) = S_\varepsilon \mathbf{E}(U)(S_\varepsilon x^\varepsilon) S_\varepsilon = \begin{pmatrix} E_{11}(U) & E_{12}(U) & \frac{1}{\varepsilon} E_{12}(U) \\ E_{12}(U) & E_{22}(U) & \frac{1}{\varepsilon} E_{23}(U) \\ \frac{1}{\varepsilon} E_{13}(U) & \frac{1}{\varepsilon} E_{23}(U) & \frac{1}{\varepsilon^2} E_{33}(U) \end{pmatrix} \in \mathbb{R}^{3\times3}_{\rm sym},$$

where here $\mathbf{e}(u)$ is calculated via the gradient with respect to $x^\varepsilon \in \Omega^\varepsilon$, while $\mathbf{E}(U)$ is calculated via the gradient with respect to $x \in \Omega$. We continue to use capital letters for functions defined on $\Omega = \omega \times]-1, 1[$.

When substituting $q^\varepsilon = (u^\varepsilon, p^\varepsilon)$ into $\mathcal{E}^\varepsilon$ and $\mathcal{R}^\varepsilon$ we still have to take care of the change in the volume measure, namely $\mathrm{d}x^\varepsilon = \varepsilon \mathrm{d}x$. Hence we set

$$\mathcal{E}_\varepsilon(t, U, P) = \frac{1}{\varepsilon} \mathcal{E}^\varepsilon(t, u^\varepsilon, p^\varepsilon) \quad \text{and} \quad \mathcal{R}_\varepsilon(\dot{P}) = \frac{1}{\varepsilon} \mathcal{R}^\varepsilon(\dot{p}^\varepsilon).$$

To control the loading part of ℓ^ε defined in (2.6), we also have to assume a corresponding scaling of the loadings namely

$$f_{\rm vol}(t,x^\varepsilon) = \varepsilon^\alpha S_\varepsilon^{-1} F_{\rm vol}(S_\varepsilon x^\varepsilon) \quad \text{and} \quad f_{\rm surf}(t,x^\varepsilon) = \varepsilon^{\alpha+1} S_\varepsilon^{-1} F_{\rm surf}(S_\varepsilon x^\varepsilon)$$

for $x^\varepsilon \in \omega \times \{-\varepsilon, \varepsilon\}$ and $\alpha = 0$ as above. For simplicity, we assume that there are no surface loadings on $\partial\omega\backslash\gamma_{\rm D} \times]-\varepsilon,\varepsilon[$. They could be easily included, but need a different scaling. Then, $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ take the form

$$\mathcal{E}_\varepsilon(t,U,P) = \mathcal{B}_\varepsilon(U,P) - \langle L(t), U\rangle, \quad \mathcal{R}_\varepsilon(\dot{P}) = \int_\Omega R(\dot{P}(x))\mathrm{d}x, \tag{2.9a}$$

$$\mathcal{B}_\varepsilon(U,P) = \int_\Omega \mathbb{W}(S_\varepsilon \mathbf{E}(U)(x) S_\varepsilon, P(x))\mathrm{d}x, \tag{2.9b}$$

$$\langle L(t), U\rangle = \int_\Omega F_{\rm vol}(t,x)\cdot u(x)\mathrm{d}x + \int_{\omega\times\{-1,1\}} F_{\rm surf}(t,x)\cdot u(x)\mathrm{d}a(x). \tag{2.9c}$$

Thus, the only dependence in ε occurs through the scaling of the elastic strains.

By (2.1) and (2.2) and $\varepsilon \in]\,0,1]$ we have the uniform convexity

$$\begin{aligned}\mathcal{B}_\varepsilon(U,P) &\geq c(\|S_\varepsilon E(U) S_\varepsilon\|^2_{\mathrm{L}^2} + \|P\|^2_{\mathrm{L}^2}) \geq c(\|E(U)\|^2_{\mathrm{L}^2} + \|P\|^2_{\mathrm{L}^2}) \\ &\geq c_{\rm Korn} c\|U\|^2_{\mathrm{H}^1} + c\|P\|^2_{\mathrm{L}^2}\end{aligned} \tag{2.10}$$

independently of ε.

Remark 2.1. In principle we could use a different scaling for elastic and plastic strains in (2.8), i.e. $\alpha \neq \beta$. Moreover, for the dissipation potential one may consider the scaling $R^\varepsilon(p) = \varepsilon^\gamma R(p)$. Finally, the scaling of the total energy and total dissipation potential can be assumed in the form $\mathcal{E}_\varepsilon(t,U,P) = \frac{1}{\varepsilon^\delta}\mathcal{E}^\varepsilon(t,u^\varepsilon,p^\varepsilon)$ and $\mathcal{R}_\varepsilon(\dot{P}) = \frac{1}{\varepsilon^\delta}\mathcal{R}^\varepsilon(\dot{p}^\varepsilon)$. The scaling of both energetic terms must be the same to stay consistent with the energetic formulation (2.7c). One may now explore the space of all possible scalings and will find the same model as above, whenever we take $\alpha = \beta = \gamma$ and $\delta = 1 + 2\alpha$.

In the case $\alpha \neq \beta$ the proper scaling for the energy leads to $\delta = 1 + 2\min\{\alpha,\beta\}$. Applying this δ to the dissipation potential we see that $\mathcal{R}_\varepsilon$ tends to 0 if $\gamma > \delta - 1 - \beta$, which leads to the degenerate situation that plasticity does not dissipate energy. Hence it occurs immediately in such a way that all plastic stresses are 0. In the case $\gamma < \delta - 1 - \beta$ plastic changes would dissipate infinite energy and hence we find $\dot{P} \equiv 0$, which leads to pure elasticity. Only the case $\gamma = \delta - 1 - \beta$ produces classical plasticity.

Now returning to the choices of α and β: for $\alpha > \beta$, the hardening becomes infinite and we are led to pure elasticity with $P \equiv 0$, for $\alpha < \beta$ one obtains rigid plasticity where $u \equiv 0$.

2.3. *The limiting elastoplastic plate model*

Obviously, the energy $\mathcal{B}_\varepsilon$ blows up for $\varepsilon \to 0$ if the strains $E_{i3}(U)$ do not vanish. Thus, we expect the limit model to be defined on a reduced space, namely the so-called *Kirchhoff–Love displacements*

$$\mathbf{U}_{\mathrm{KL}} := \{U \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3) \,|\, \mathbf{E}_{13}(U) = \mathbf{E}_{23}(U) = \mathbf{E}_{33}(U) = 0\}. \tag{2.11}$$

The restrictions in $\mathbf{U}_{\mathrm{KL}}$ take the explicit form

$$\partial_{x_1} U_3 + \partial_{x_3} U_1 = \partial_{x_2} U_3 + \partial_{x_3} U_2 = \partial_{x_3} U_3 = 0 \quad \text{a.e. in } \Omega.$$

The last equation implies that U_3 is independent of x_3. Using this the first two equations imply that U_1 and U_2 are affine in x_3. Defining

$$\mathbf{V} := \{V = (V_1, V_2, V_3) \in \mathrm{H}^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^3) \,|\, V_3 \in \mathrm{H}^2(\omega), \nabla V_3 \cdot \nu = 0 \text{ on } \gamma_{\mathrm{D}}\}$$

the space of Kirchhoff–Love displacements can be characterized via

$$\mathbf{U}_{\mathrm{KL}} = \{U = \mathcal{K} V \,|\, V \in \mathbf{V}\} \quad \text{with}$$
$$\mathcal{K} V(x_1, x_2, x_3) = \begin{pmatrix} V_1(x_1, x_2) - x_3 \partial_{x_1} V_3(x_1, x_2) \\ V_2(x_1, x_2) - x_3 \partial_{x_2} V_3(x_1, x_2) \\ V_3(x_1, x_2) \end{pmatrix}, \tag{2.12}$$

see, e.g. Refs. 8 and 6. Note that the component U_3 in $\mathbf{U}_{\mathrm{KL}}$ has gained higher smoothness, namely $U_3 \in \mathrm{H}^2(\Omega)$.

The limit model will be defined in such a way that it is restricted to $\mathbf{U}_{\mathrm{KL}}$. The reduced energy is obtained by relaxing the strains $\mathbf{E}_{j3}$ in the following way. We decompose the six-dimensional space $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ into two three-dimensional components by setting

$$\mathbf{E}^{1,2} := \begin{pmatrix} \mathbf{E}_{12} & \mathbf{E}_{12} \\ \mathbf{E}_{12} & \mathbf{E}_{12} \end{pmatrix} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \quad \mathbf{E}^3 := (\mathbf{E}_{13}, \mathbf{E}_{23}, \mathbf{E}_{33}) \in \mathbb{R}^3. \tag{2.13a}$$

For $A \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ and $b \in \mathbb{R}^3$ we define $[\![A \,\|\, b]\!] \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ such that $\mathbf{E} = [\![\mathbf{E}^{1,2} \,\|\, \mathbf{E}^3]\!]$, i.e.

$$[\![A \,\|\, b]\!] = \begin{pmatrix} A_{11} & A_{12} & b_1 \\ A_{12} & A_{22} & b_2 \\ b_1 & b_2 & b_3 \end{pmatrix}. \tag{2.13b}$$

Now we define a relaxed energy density depending only on $\mathbf{E}^{1,2}$, namely

$$\overline{\mathbb{W}}(\mathbf{E}^{1,2}, P) := \min\{\mathbb{W}([\![\mathbf{E}^{1,2} \,\|\, b]\!], P) \,|\, b \in \mathbb{R}^3\}. \tag{2.14}$$

The minimization of the energy density $\mathbb{W}$ with respect to $b = \mathbf{E}^3$ is a common feature of linear and nonlinear plate theories.[4,6,22,14]

The definition of $\overline{\mathbb{W}}$ in (2.14) implies the important lower estimate

$$\mathbb{W}(S_\varepsilon \mathbf{E} S_\varepsilon, P) \geq \overline{\mathbb{W}}(\mathbf{E}^{1,2}, P) \quad \text{for all } \varepsilon \in [0,1],\ (\mathbf{E}, P) \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \times \mathbb{R}^{3\times 3}_{\mathrm{dev}}. \tag{2.15}$$

Moreover there is a linear mapping $N : \mathbb{R}^{2\times 2}_{\mathrm{sym}} \times \mathbb{R}^{3\times 3}_{\mathrm{dev}} \to \mathbb{R}^3$ such that

$$N(\mathbf{E}^{1,2}, P) = \operatorname{argmin}\{\mathbb{W}([\![\mathbf{E}^{1,2} \,\|\, b]\!], P) \,|\, b \in \mathbb{R}^3\}.$$

Some linear algebra shows that these definitions lead to the formula

$$\begin{aligned} \mathbb{W}([\![\mathbf{E}^{1,2}\,\|\,b]\!],P) &= \overline{\mathbb{W}}(\mathbf{E}^{1,2},P)+\mathbb{W}_2(b-N(\mathbf{E}^{1,2},P)) \\ \text{with } \mathbb{W}_2 &: \mathbb{R}^3\to[0,\infty[; \quad b\mapsto \mathbb{W}([\![0\,\|\,b]\!],0). \end{aligned} \tag{2.16}$$

Explicit formulas for a special case will be given in Example 2.2.

We now define the limit RIS $(\mathbf{Q},\mathcal{E}_0,\mathcal{R}_0)$ as follows:

$$\begin{aligned} &\mathcal{E}_0(t,U,P)=\mathcal{B}_0(U,P)-\langle L(t),U\rangle \quad\text{and}\quad \mathcal{R}_0(\dot P)=\int_\Omega R(\dot P(x))\mathrm{d}x, \quad\text{where} \\ &\mathcal{B}_0(U,P)=\begin{cases}\overline{\mathcal{B}}(U,P) & \text{if } U\in\mathbf{U}_{\mathrm{KL}},\\ \infty & \text{else,}\end{cases} \end{aligned} \tag{2.17}$$

with $\overline{\mathcal{B}}(U,P)=\int_\Omega \overline{\mathbb{W}}(\mathbf{E}^{1,2}(U)(x),P(x))\mathrm{d}x.$

The following convergence result, which is the central aim of this paper, shows that the solutions $Q_\varepsilon=(U_\varepsilon,P_\varepsilon)$ of the RIS $(\mathbf{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ converge, for $\varepsilon\to0$, to solutions $Q_0=(U_0,P_0)$ of the limiting RIS $(\mathbf{Q},\mathcal{E}_0,\mathcal{R}_0)$. The proof will be established in Sec. 4 on the basis of the abstract Γ-convergence theory for evolutionary problems developed in Sec. 3. The specific properties of the limit system are discussed in Sec. 5. There we will highlight in what sense the limit problem can be understood as an elastoplastic plate model.

Theorem 2.2. *Assume that the RIS $(\mathbf{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ are given as above for all $\varepsilon\in[0,1]$, where $L\in\mathrm{C}^1([0,T];\mathbf{Q}^*)$. Consider a family of solutions $Q_\varepsilon:[0,T]\to\mathbf{Q}$, as defined in (2.7c) for all $\varepsilon\in[0,1]$. If the initial conditions satisfy*

$$Q_\varepsilon(0)\to Q_0(0) \;\text{ and }\; \mathcal{E}_\varepsilon(0,Q_\varepsilon(0))\to\mathcal{E}_0(0,Q_0(0)) \quad \text{for } \varepsilon\to0,$$

then for all $t\in[0,T]$ we have the convergences

$$Q_\varepsilon(t)\to Q_0(t), \quad \mathcal{E}_\varepsilon(t,Q_\varepsilon(t))\to\mathcal{E}_0(t,Q_0(t)), \quad \int_0^t\mathcal{R}(\dot P_\varepsilon(s))\mathrm{d}s\to\int_0^t\mathcal{R}(\dot P_0(s))\mathrm{d}s.$$

We end this section by showing the result of the relaxed energy density $\mathbb{W}$ for the isotropic $\mathbb{W}$ considered in Example 2.1.

Example 2.2. We return to the isotropic $\mathbb{W}$ defined in (2.5), now for $d=3$. Using $\operatorname{tr}P=0$ we obtain the relaxed energy density

$$\overline{\mathbb{W}}(\mathbf{E}^{1,2},P)=\frac{\lambda\mu}{\lambda+2\mu}(\operatorname{tr}(\mathbf{E}^{1,2}-P^{1,2}))^2+\mu|\mathbf{E}^{1,2}-P^{1,2}|^2+\frac{h}{2}|P|^2$$

as well as the relations

$$N(\mathbf{E}^{1,2},P)=\left(P_{13},P_{23},-\frac{\lambda}{\lambda+2\mu}\operatorname{tr}\mathbf{E}^{1,2}+\frac{2\mu}{\lambda+2\mu}P_{33}\right)^{\mathsf{T}},$$

$$\mathbb{W}_2(b)=2\mu b_1^2+2\mu b_2^2+\frac{\lambda+2\mu}{2}b_3^2.$$

3. Γ-Convergence for RIS with Quadratic Energies

In this section we consider general families $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)_{\varepsilon\in]0,1]}$ of RIS and study the convergence of the associated solutions q_ε in the limit $\varepsilon \to 0$. The aim is to establish fairly general conditions on the convergences of $(\mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ to $(\mathcal{E}_0, \mathcal{R}_0)$ that guarantee that the solutions q_ε converge to the solution q of the limit system $(\mathbf{Q}, \mathcal{E}_0, \mathcal{R}_0)$, which we then call the Γ-limit of the above family.

For rate-independent systems a general strategy for Γ-convergence was developed in Ref. 29, which found numerous applications: e.g. fracture,[15] homogenization,[32] numerical approximation,[16,21,27] and delamination.[37,30] Here we specialize this theory to the case that $\mathcal{E}_\varepsilon(t,\cdot) : \mathbf{Q} \to \mathbb{R}_\infty$ is a quadratic functional, as it is the case in linearized elastoplasticity. Thus, the abstract theory is simplified in two respects. First, the systems under consideration have unique solutions and we do not need to consider subsequences. Second, the quadratic nature of the energy allows for a simpler construction of recovery sequences by using the quadratic trick introduced in Ref. 31, see Proposition 3.1.

3.1. *Abstract setup and* Γ*-convergence result*

Our result is formulated abstractly in terms of Γ-convergence of $\mathcal{E}_\varepsilon(t,\cdot)$ toward $\mathcal{E}_0(t,\cdot)$, where we use the weak and the strong topologies in the underlying separable Hilbert space $\mathbf{Q}$. We use the notions of *Mosco convergence*[3,11] and *continuous convergence* for functionals $\mathcal{I}_n$, denoted by $\mathcal{I}_n \xrightarrow{\mathrm{M}} \mathcal{I}$ and $\mathcal{I}_n \overset{\mathrm{c}}{\rightsquigarrow} \mathcal{I}$, respectively. The definitions are as follows:

$$\mathcal{I}_n \xrightarrow{\mathrm{M}} \mathcal{I} \Leftrightarrow \begin{cases} \text{(i)} & \text{Liminf estimate:} \\ & q_n \rightharpoonup q \Rightarrow \mathcal{I}(q) \le \liminf_{n\to\infty} \mathcal{I}_n(q_n), \\ \text{(ii)} & \text{Limsup estimate (existence of recovery sequences)} \\ & \forall \hat{q} \in \mathbf{Q}\, \exists (\hat{q}_n)_n : \ \hat{q}_n \to \hat{q} \quad \text{and} \quad \mathcal{I}(\hat{q}) \ge \limsup_{n\to\infty} \mathcal{I}_n(\hat{q}_n). \end{cases} \tag{3.1}$$

$$\mathcal{I}_n \overset{\mathrm{c}}{\rightsquigarrow} \mathcal{I} \Leftrightarrow (q_n \to q \Rightarrow \mathcal{I}_n(q_n) \to \mathcal{I}(q)). \tag{3.2}$$

Our precise assumptions on the family $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)_{\varepsilon\in[0,1]}$ are the following. Note that often the limit functionals $\mathcal{E}_0$ and $\mathcal{R}_0$ are included in the assumptions via $\varepsilon = 0$. The assumptions (3.3a)–(3.3c) provide some uniform *a priori* estimates, while (3.3d) and (3.3e) are the main convergence assumptions.

$$\mathcal{E}_\varepsilon(t,q) = \mathcal{B}_\varepsilon(q) - \langle \ell_\varepsilon(t), q\rangle \quad \text{with } \mathcal{B}_\varepsilon \text{ quadratic and weakly lower semicontinuous (wlsc) and } \ell_\varepsilon \in \mathrm{C}^1([0,T];\mathbf{Q}^*); \tag{3.3a}$$

$$\mathcal{R}_\varepsilon : \mathbf{Q} \to [0,\infty] \text{ is 1-homogeneous, wlsc, and convex;} \tag{3.3b}$$

$$\exists\, \beta, C > 0\ \forall\, (t,q) \in [0,T] \times \mathbf{Q}\ \forall\, \varepsilon \in [0,1] : \ \mathcal{B}_\varepsilon(q) \ge \frac{\beta}{2}\|q\|^2, \quad \|\ell_\varepsilon(t)\|_{\mathbf{Q}_*} \le C, \quad \|\dot{\ell}_\varepsilon(t)\|_{\mathbf{Q}_*} \le C; \tag{3.3c}$$

$$\mathcal{B}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{B}_0 \quad \text{and} \quad \forall\, t \in [0, T]:\ \ell_\varepsilon(t) \to \ell_0(t) \ \text{in } \mathbf{Q}^*; \tag{3.3d}$$

$$\mathcal{R}_\varepsilon \overset{\mathrm{c}}{\rightsquigarrow} \mathcal{R}_0 \quad \text{and} \quad \mathcal{R}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{R}_0. \tag{3.3e}$$

In the last condition "$\overset{\mathrm{c}}{\rightsquigarrow}$" means that every strongly converging sequence is a recovery sequence. The additional condition "$\xrightarrow{\mathrm{M}}$" is needed in order to guarantee $\mathcal{R}_0(q_0) \leq \liminf_{\varepsilon\to 0} \mathcal{R}_\varepsilon(q_\varepsilon)$ whenever $q_\varepsilon \rightharpoonup q_0$. Note that we only ask for continuous convergence in the norm topology, which is in contrast to Refs. 21, 29 and 27, where the more restrictive continuous convergence in the weak topology is used. Thus, we follow Ref. 32 and exploit the quadratic structure (3.3a) of $\mathcal{E}_\varepsilon$ for the construction of the *mutual recovery sequence*, see Proposition 3.1.

Our aim is to establish the following convergence result for the solutions q_ε of the RIS $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$. We obtain strong convergence of the solutions q_ε toward solutions q_0 of the limiting problem $(\mathbf{Q}, \mathcal{E}_0, \mathcal{R}_0)$. Moreover, the solutions $q_\varepsilon(t)$ are recovery sequences for $q_0(t)$, see (3.5b). The proof will be given in Sec. 3.3.

Theorem 3.1. *Let the assumptions* (3.3) *hold. Moreover, choose a family* $(q_\varepsilon^0)_{\varepsilon\in[0,1]}$ *of initial data such that the following conditions hold*:

$$\forall\, \varepsilon \in [0,1]\ \forall\, \tilde{q} \in \mathbf{Q}:\ \mathcal{E}_\varepsilon(0, q_\varepsilon^0) \leq \mathcal{E}_\varepsilon(0, \tilde{q}) + \mathcal{R}_\varepsilon(\tilde{q} - q_\varepsilon^0), \tag{3.4a}$$

$$q_\varepsilon^0 \rightharpoonup q_0^0 \quad \textit{and} \quad \mathcal{E}_\varepsilon(0, q_\varepsilon^0) \to \mathcal{E}_0(0, q_0^0). \tag{3.4b}$$

Then, the unique solutions $q_\varepsilon : [0, T] \to \mathbf{Q}$ *for the RIS* $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ *with* $q_\varepsilon(0) = q_\varepsilon^0$ *satisfy, for all* $t \in [0, T]$, *the convergences*

$$q_\varepsilon(t) \to q_0(t), \tag{3.5a}$$

$$\mathcal{E}_\varepsilon(t, q_\varepsilon(t)) \to \mathcal{E}_0(t, q_0(t)), \tag{3.5b}$$

$$\mathrm{Diss}_{\mathcal{R}_\varepsilon}(q_\varepsilon, [0, t]) \to \mathrm{Diss}_{\mathcal{R}_0}(q_0, [0, t]), \tag{3.5c}$$

$$\langle \dot{\ell}_\varepsilon(t), q_\varepsilon(t)\rangle \to \langle \dot{\ell}_0(t), q_0(t)\rangle. \tag{3.5d}$$

The assumption (3.4b) on the initial conditions should be seen as the counterpart to (3.5a) and (3.5b). Similarly, (3.4a) is necessary as energetic solutions are stable for all times, also $t = 0$.

Remark 3.1. One might ask the question, whether for a given limit solution $q_0 : [0, T] \to \mathbf{Q}$ there are suitable initial conditions such that (3.4), and hence (3.5) hold. The answer is yes, and they can be obtained by the construction

$$q_\varepsilon^0 = \mathrm{argmin}\{\mathcal{E}_\varepsilon(0, q) + \mathcal{R}_\varepsilon(q - q_0^0) \mid q \in \mathbf{Q}\}.$$

By standard arguments using the triangle inequality for $\mathcal{R}_\varepsilon$, see Theorem 3.2(i) in Ref. 25, such minimizers are stable, i.e. (3.4a) holds.

The convergences (3.3d) and (3.3e) imply that $\mathcal{J}_\varepsilon :\ q \mapsto \mathcal{E}_\varepsilon(0, q) + \mathcal{R}_\varepsilon(q - q_0^0)$ Mosco-converges to $\mathcal{J}_0$, which has the unique minimizer q_0^0 by (3.4a). By standard results from Γ-convergence we obtain $q_\varepsilon^0 \rightharpoonup q_0^0$ and $\mathcal{J}_\varepsilon(q_\varepsilon^0) \to \mathcal{J}_0(q_0^0) = \mathcal{E}_0(0, q_0^0)$. For the convergence of the energies $\mathcal{E}_\varepsilon(0, q_\varepsilon^0)$ we use the uniform convexity (3.3c) giving

$$\mathcal{J}_\varepsilon(q_\varepsilon^0) + \frac{\beta}{2}\|\tilde{q}_\varepsilon - q_\varepsilon^0\|^2 \leq \mathcal{J}_\varepsilon(\tilde{q}_\varepsilon) \to \mathcal{J}_0(q_0^0), \tag{3.6}$$

where $\tilde{q}_\varepsilon$ is chosen such that $\tilde{q}_\varepsilon \to q_0^0$ and $\mathcal{E}_\varepsilon(0, \tilde{q}_\varepsilon) \to \mathcal{E}_0(0, q_0^0)$. We find $\|\tilde{q}_\varepsilon - q_\varepsilon^0\| \to 0$ and conclude $q_\varepsilon^0 \to q_0^0$. Thus, $\mathcal{E}_\varepsilon(0, q_\varepsilon^0) = \mathcal{J}_\varepsilon(q_\varepsilon^0) - \mathcal{R}_\varepsilon(q_\varepsilon^0 - q_0^0) \to \mathcal{J}_0(q_0^0) = \mathcal{E}_0(0, q_0^0)$; and (3.4b) is established.

In (3.5c) and for any $q : [0, T] \to \mathbf{Q}$ and $0 \leq r < s \leq T$ we use the notation

$$\mathrm{Diss}_{\mathcal{R}}(q, [r, s]) := \sup \left\{ \sum_{j=1}^{N} \mathcal{R}(q(t_j) - q(t_{j-1})) \,\middle|\, N \in \mathbb{N},\ r < t_0 < t_1 < \cdots < t_N < s \right\},$$

which is defined for all functions (defined for all $t \in [0, T]$). For absolutely continuous functions we have

$$\mathrm{Diss}_{\mathcal{R}}(q, [r, s]) = \int_r^s \mathcal{R}(\dot{q}(t))\mathrm{d}t.$$

Using the liminf estimate from $\mathcal{R}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{R}_0$ it is standard to show that $\mathrm{Diss}_{\mathcal{R}_\varepsilon}$ is lower semicontinuous in the sense that

$$\begin{aligned} &(\forall\, t \in [0, T] :\ q_\varepsilon(t) \rightharpoonup q_0(t)) \\ &\quad \Rightarrow \mathrm{Diss}_{\mathcal{R}_0}(q_0, [0, T]) \leq \liminf_{\varepsilon \to 0} \mathrm{Diss}_{\mathcal{R}_\varepsilon}(q_\varepsilon, [0, T]). \end{aligned} \tag{3.7}$$

3.2. *Quadratic forms*

Here we discuss some abstract results on quadratic forms and their Γ-convergence. We refer to Chap. 13 in Ref. 11 for general background and more details.

A functional $\mathcal{B} : \mathbf{Q} \to \mathbb{R}_\infty$ is called a quadratic form, if it is homogeneous of degree 2 and satisfies the parallelogram identity, i.e.

$$\mathcal{B}(\lambda q) = \lambda^2 \mathcal{B}(q), \quad \mathcal{B}(q + \tilde{q}) + \mathcal{B}(q - \tilde{q}) = 2\mathcal{B}(q) + 2\mathcal{B}(\tilde{q})$$

for all $\lambda \in \mathbb{R}$ and $q, \tilde{q} \in \mathbf{Q}$. Note that $\mathcal{B}$ may take the value $+\infty$ here. Throughout we assume that our quadratic forms are *coercive* and *weakly lower semicontinuous*, i.e.

$$\exists\, \beta > 0\ \forall\, q \in \mathbf{Q} :\ \mathcal{B}(q) \geq \frac{\beta}{2}\|q\|^2, \quad q_n \rightharpoonup q \Rightarrow \mathcal{B}(q) \leq \liminf_{n \to \infty} \mathcal{B}(q_n).$$

We define the domain $\mathbf{V} = \mathrm{dom}\, \mathcal{B} = \{q \in \mathbf{Q} \,|\, \mathcal{B}(q) < \infty\}$, which is a Hilbert space, when equipped with the norm $\|q\|_{\mathcal{B}} = \mathcal{B}(q)^{1/2}$. Clearly, $\mathbf{V}$ is continuously embedded into $\mathbf{Q}$. We denote by $\mathbf{Q}_{\mathcal{B}}$ the closure of $\mathbf{V}$ in $\mathbf{Q}$.

There is a symmetric bounded linear operator $A : \mathbf{V} \to \mathbf{V}^*$ such that

$$\mathcal{B}(q) = \begin{cases} \frac{1}{2}\langle Aq, q\rangle & \text{for } q \in \mathbf{V}, \\ \infty & \text{for } q \in \mathbf{Q} \backslash \mathbf{V}. \end{cases}$$

Using the Gelfand triple $\mathbf{V} \overset{\text{dense}}{\subset} \mathbf{Q}_\mathcal{B} \cong \mathbf{Q}_\mathcal{B}^* \overset{\text{dense}}{\subset} \mathbf{V}^*$, we also define a unique self-adjoint operator $A^+ : \mathbf{D} \to \mathbf{Q}_\mathcal{B}^*$ with

$$\mathbf{D} := \{q \in \mathbf{V} \,|\, \exists\, C > 0\ \forall\, \tilde{q} \in \mathbf{V}:\ |\langle Aq, \tilde{q}\rangle| \le C\|\tilde{q}\|\}$$

and $\langle A^+ q, \tilde{q}\rangle = \langle A\tilde{q}, q\rangle$ for $q,\, \tilde{q} \in \mathbf{V}$. Clearly, we have

$$\mathbf{D} \overset{\text{dense}}{\subset} \mathbf{V} \overset{\text{dense}}{\subset} \mathbf{Q}_\mathcal{B} \tag{3.8}$$

with respect to the strong topology in $\mathbf{Q}$. We also introduce the $\mathbf{Q}$-orthogonal projection $P : \mathbf{Q} \to \mathbf{Q}_\mathcal{B}$ and observe that for each $\ell \in \mathbf{Q}^*$ the minimization problem

$$q \in \operatorname{argmin}\{\mathcal{B}(q) - \langle \ell, q\rangle \,|\, q \in \mathbf{Q}\}$$

has the unique minimizer $q = A^{-1}P^*\ell$.

We now consider a family $(\mathcal{B}_\varepsilon)_{\varepsilon\in[0,1]}$ of quadratic forms that are all weakly lower semicontinuous and uniformly coercive, i.e.

$$\exists\, \beta > 0\ \forall\, \varepsilon \in [0,1]\ \forall\, q \in \mathbf{Q}: \quad \mathcal{B}_\varepsilon(q) \ge \frac{\beta}{2}\|q\|^2. \tag{3.9}$$

We denote the corresponding subspaces by $\mathbf{D}_\varepsilon \subset \mathbf{V}_\varepsilon \subset \mathbf{Q}_\varepsilon = \overline{\mathbf{V}_\varepsilon} \subset \mathbf{Q}$ and the operators by $A_\varepsilon^+ : \mathbf{D}_\varepsilon \to \mathbf{Q}_\varepsilon^*$, $A_\varepsilon : \mathbf{V}_\varepsilon \to \mathbf{V}_\varepsilon^*$, and the projections $P_\varepsilon : \mathbf{Q} \to \mathbf{Q}_\varepsilon$.

We now study the situation that $\mathcal{B}_0$ is the Mosco limit of $\mathcal{B}_\varepsilon$. We follow the ideas in Sec. 2.2 in Ref. 26, where the case of Mosco convergence leads to the stronger recovery condition (R3)*. The following result contains first the construction of the *mutual recovery sequence* $(\hat{q}_\varepsilon)_{\varepsilon>0}$, which will be crucial for Step 3 in the proof of Theorem 3.1. Second, we show that under our assumptions every recovery sequence converges strongly.

Proposition 3.1. *Assume* $\mathcal{B}_\varepsilon \xrightarrow{\text{M}} \mathcal{B}_0$ *and the uniform coercivity* (3.9). *Then the following two statements hold*:

(i) *For* $\hat{q}_0 \in \mathbf{D}_0$ *let* $\hat{q}_\varepsilon = A_\varepsilon^{-1}P_\varepsilon^* A_0^+\hat{q}_0 = \operatorname{argmin}\{\mathcal{B}_\varepsilon(q) - \langle A_0^+\hat{q}_0, q\rangle \,|\, q \in \mathbf{Q}\}$; *then*

$$\hat{q}_\varepsilon \to \hat{q}_0, \tag{3.10a}$$

$$\mathcal{B}_\varepsilon(\hat{q}_\varepsilon) \to \mathcal{B}_0(\hat{q}_0), \tag{3.10b}$$

$$\text{if } q_\varepsilon \rightharpoonup q_0 \text{ and } \sup_{\varepsilon\in[0,1]} \mathcal{B}_\varepsilon(q_\varepsilon) < \infty,$$

$$\text{then } \mathcal{B}_\varepsilon(q_\varepsilon + \hat{q}_\varepsilon) - \mathcal{B}_\varepsilon(q_\varepsilon) \to \mathcal{B}_0(q_0 + \hat{q}_0) - \mathcal{B}_0(q_0). \tag{3.10c}$$

(ii) *Every recovery sequence for* q_0 *with* $\mathcal{B}_0(q_0) < \infty$ *converges strongly.*

Proof. Throughout this proof we will use the following simple observation:

$$\forall\, \xi \in \mathbf{Q}^* \forall\, q_\varepsilon \quad \text{with } \mathcal{B}_\varepsilon(q_\varepsilon) < \infty\colon \langle P_\varepsilon^*\xi, q_\varepsilon\rangle = \langle \xi, q_\varepsilon\rangle. \tag{3.11}$$

This follows simply from $q_\varepsilon \in \mathbf{V}_\varepsilon$ giving $P_\varepsilon q_\varepsilon = q_\varepsilon$.

(i) For $\hat{q}_0 \in \mathbf{D}_0$ the essential feature is $A_0^+\hat{q}_0 \in \mathbf{Q}^*$, which implies that $\mathcal{L} : q \mapsto \langle A_0^+\hat{q}_0, q\rangle$ is continuous. Hence, we conclude that $\mathcal{B}_\varepsilon - \mathcal{L} \xrightarrow{\Gamma} \mathcal{B}_0 - \mathcal{L}$, and the classical properties of Γ-convergence provide convergence of minimizers and of the minimum energies, thus we have $\hat{q}_\varepsilon \rightharpoonup \hat{q}_0$ and (3.10b).

To prove $\hat{q}_\varepsilon \to \hat{q}_0$ we use that Mosco convergence implies the existence of at least one strongly converging recovery sequence, let us call it $\tilde{q}_\varepsilon$, i.e. $\tilde{q}_\varepsilon \to \hat{q}_0$ and $\mathcal{B}_\varepsilon(\tilde{q}_\varepsilon) \to \mathcal{B}_0(\hat{q}_0)$. Hence, (3.10a) will follow from $\|\tilde{q}_\varepsilon - \hat{q}_\varepsilon\| \to 0$. Since $\tilde{q}_\varepsilon, \hat{q}_\varepsilon \in \mathbf{V}_\varepsilon$ we have

$$\begin{aligned}\frac{\beta}{2}\|\tilde{q}_\varepsilon - \hat{q}_\varepsilon\|^2 \le \mathcal{B}_\varepsilon(\tilde{q}_\varepsilon - \hat{q}_\varepsilon) &= \mathcal{B}_\varepsilon(\tilde{q}_\varepsilon) - \langle A_\varepsilon \hat{q}_\varepsilon, \tilde{q}_\varepsilon\rangle + \mathcal{B}_\varepsilon(\hat{q}_\varepsilon)\\ &= \mathcal{B}_\varepsilon(\tilde{q}_\varepsilon) - \langle P_\varepsilon^* A_0^+ \hat{q}_0, \tilde{q}_\varepsilon\rangle + \mathcal{B}_\varepsilon(\hat{q}_\varepsilon)\\ &= \mathcal{B}_\varepsilon(\tilde{q}_\varepsilon) - \langle A_0^+ \hat{q}_0, \tilde{q}_\varepsilon\rangle + \mathcal{B}_\varepsilon(\hat{q}_\varepsilon)\\ &\to \mathcal{B}_0(\hat{q}_0) - \langle A_0^+ \hat{q}_0, \hat{q}_0\rangle + \mathcal{B}_0(\hat{q}_0) = 0,\end{aligned}$$

where we used (3.9), the quadratic structure, the definition of $\hat{q}_\varepsilon$, and (3.11) for $\tilde{q}_\varepsilon$. Thus, (3.10a) is established.

To obtain (3.10c) we use (3.11) for q_ε and obtain, for all $\varepsilon \in [0,1]$, the convergence

$$\begin{aligned}\mathcal{B}_\varepsilon(q_\varepsilon + \hat{q}_\varepsilon) - \mathcal{B}_\varepsilon(q_\varepsilon) &= \mathcal{B}_\varepsilon(\hat{q}_\varepsilon) + \langle A_\varepsilon \hat{q}_\varepsilon, q_\varepsilon\rangle = \mathcal{B}_\varepsilon(\hat{q}_\varepsilon) + \langle A_0^+ \hat{q}_0, q_\varepsilon\rangle\\ &\to \mathcal{B}_0(\hat{q}_0) + \langle A_0^+ \hat{q}_0, q_0\rangle = \mathcal{B}_0(q_0 + \hat{q}_0) - \mathcal{B}_0(q_0).\end{aligned}$$

Hence, the desired relation (3.10c) is established.

(ii) To obtain strong convergence of every recovery sequence, we proceed as in the previous part but approximate $q_0 \in \mathbf{V}_0 = \operatorname{dom} \mathcal{B}_0$. We first construct a family $(q^\delta)_{\delta>0}$ such that

$$q^\delta \in \mathbf{D}_0 \quad \text{and} \quad \frac{\beta}{2}\|q^\delta - q_0\|^2 \le \mathcal{B}_0(q^\delta - q_0) \to 0 \quad \text{for } \delta \to 0. \tag{3.12}$$

Next we consider the family $\hat{q}_\varepsilon^\delta = A_\varepsilon^{-1} P_\varepsilon^* A_0^+ q^\delta$. From the above we know that

$$\forall\, \delta > 0 : \ \hat{q}_\varepsilon^\delta \to q^\delta \quad \text{for } \varepsilon \to 0. \tag{3.13}$$

Now consider an arbitrary recovery sequence $q_\varepsilon \rightharpoonup q_0$ such that $\mathcal{B}_\varepsilon(q_\varepsilon) \to \mathcal{B}_0(q_0)$. We estimate the norms via

$$\|q_\varepsilon - q_0\| \le \|q_\varepsilon - \hat{q}_\varepsilon^\delta\| + \|\hat{q}_\varepsilon^\delta - q^\delta\| + \|q^\delta - q_0\|. \tag{3.14}$$

To estimate the first term on the right-hand side we proceed as above and obtain

$$\begin{aligned}\frac{\beta}{2}\|q_\varepsilon - \hat{q}_\varepsilon^\delta\|^2 \le \mathcal{B}_\varepsilon(q_\varepsilon - \hat{q}_\varepsilon^\delta) &= \mathcal{B}_\varepsilon(q_\varepsilon) + \mathcal{B}_\varepsilon(\hat{q}_\varepsilon^\delta) - \langle P_\varepsilon^* A_0^+ q^\delta, q_\varepsilon\rangle\\ &\to \mathcal{B}_0(q_0) + \mathcal{B}_0(q^\delta) - \langle A_0^+ q^\delta, q_0\rangle = \mathcal{B}_0(q^\delta - q_0).\end{aligned} \tag{3.15}$$

Now, by (3.12) for any $\rho > 0$ we can fix $\delta > 0$ such that

$$\|q^\delta - q_0\| < \rho/4 \quad \text{and} \quad \mathcal{B}_0(q^\delta - q_0) < \beta\rho^2/8.$$

By (3.13) and (3.15) we can choose an $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$ we have

$$\|\hat{q}_\varepsilon^\delta - q^\delta\| < \rho/4 \quad \text{and} \quad \|q_\varepsilon - \hat{q}_\varepsilon^\delta\| \le \left(\frac{2}{\beta}\mathcal{B}_0(q^\delta - q_0)\right)^{1/2} + \rho/4 < \rho/2.$$

Inserting this into (3.14), we have shown $\|q_\varepsilon - q_0\| < \rho$ for all $\varepsilon \in]0, \varepsilon_0[$, which is the desired strong convergence. □

3.3. *Proof of Theorem* 3.1

In this section we provide the full proof of the abstract Γ-convergence result for the quadratic rate-independent systems given in Sec. 3.1.

Proof of Theorem 3.1. We first note that the assumptions (3.3a) and (3.3b) imply that for each $\varepsilon \in [0, 1]$ and each stable initial condition q_ε^0, see (3.4a), there is a unique energetic solution (see Theorem 2.1). The convergence result will be a special case of Sec. 3 in Ref. 29. We will follow the established six steps of the existence proof[12,29,28] for the convenience of the reader.

Step 1. A priori *estimates.* Using (3.3c) and (3.4b) we obtain the uniform bounds

$$\|q_\varepsilon\|_{\mathrm{C}^0([0,T];\mathbf{Q})} + \|\dot{q}_\varepsilon\|_{\mathrm{L}^\infty([0,T];\mathbf{Q})} \leq C_1$$

for all $\varepsilon \in [0, 1]$.

Step 2. *Selection of subsequences.* Via the selection principle of Arzela–Ascoli we find a subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$ and a limit function $q_* \in \mathrm{C}^{\mathrm{Lip}}([0, T]; \mathbf{Q})$ such that for all $t \in [0, T]$ we have

$$0 < \varepsilon_k \to 0 \quad \text{and} \quad q_{\varepsilon_k}(t) \rightharpoonup q_*(t) \quad \text{for } k \to \infty. \tag{3.16}$$

The aim is to show that q_* is the unique solution q_0, then we conclude the convergence of the full family $(q_\varepsilon)_{\varepsilon>0}$, without taking a subsequence. By (3.4b) we already know $q_*(0) = q_0^0$, and it remains to show that q_* is an energetic solution.

Step 3. *Stability of the limit.* We want to establish the stability of $q_*(t)$, i.e.

$$\forall\, t \in [0, T]\ \forall\, \tilde{q} \in \mathbf{Q}\colon\ \mathcal{E}_0(t, q_*(t)) \leq \mathcal{E}_0(t, \tilde{q}) + \mathcal{R}_0(\tilde{q} - q_*(t)). \tag{3.17}$$

For this we use the quadratic structure of $\mathcal{E}_\varepsilon$ by employing the results of Sec. 3.2. We fix from now on the time t and write q_0 and q_ε for $q_*(t)$ and $q_\varepsilon(t)$, respectively. Obviously, stability is equivalent to

$$\forall \hat{q} \in \mathbf{Q}\colon 0 \leq \mathcal{B}_\varepsilon(q_\varepsilon + \hat{q}) - \mathcal{B}_\varepsilon(q_\varepsilon) - \langle \ell_\varepsilon, \hat{q}\rangle + \mathcal{R}_\varepsilon(\hat{q}). \tag{3.18}$$

We have this condition for $\varepsilon > 0$ and want to establish it for $\varepsilon = 0$. Clearly, we have $\mathcal{B}_\varepsilon(q_\varepsilon) < \infty$ for all $\varepsilon \in [0, 1]$, and it suffices to check $(3.18)_{\varepsilon=0}$ for $\hat{q} \in \mathbf{V}_0 = \operatorname{dom} \mathcal{B}_0$ only.

For this we restrict first to the case $\hat{q}_0 \in \mathbf{D}_0$ and consider the mutual recovery sequence $\hat{q}_\varepsilon$ constructed in Proposition 3.1. We insert $\hat{q}_\varepsilon$ into (3.18) and see that we can pass to the limit in all terms. The first two terms converge to $\mathcal{B}_0(q_0 + \hat{q}_0) - \mathcal{B}_0(q_0)$ by (3.10c). The third term converges since both factors in the duality product converge strongly, and for the last term we have $\mathcal{R}_\varepsilon(\hat{q}_\varepsilon) \to \mathcal{R}_0(\hat{q}_0)$ using (3.10a) and

(3.3e). Thus, (3.18) with $\varepsilon = 0$ holds for all $\hat{q}_0 \in \mathbf{D}_0$, i.e.

$$\forall\, \hat{q}_0 \in \mathbf{D}_0\colon\ 0 \leq \mathcal{J}(\hat{q}_0) := \mathcal{B}_0(q_0 + \hat{q}_0) - \mathcal{B}_0(q_0) - \langle \ell_0, \hat{q}_0 \rangle + \mathcal{R}_0(\hat{q}_0). \tag{3.19}$$

The functional $\mathcal{J} : \mathbf{V}_0 \to \mathbb{R}$ is coercive and convex. Moreover, it is continuous, if $\mathbf{V}_0$ is considered as the Hilbert space equipped with the norm $\|q\|_{\mathbf{V}_0} = \mathcal{B}_0(q)^{1/2}$. Since $\mathbf{D}_0$ is dense in $\mathbf{V}_0$ (see (3.8)), the minimum of $\mathcal{J}$ over $\mathbf{V}_0$ is equal to the infimum of $\mathcal{J}$ over $\mathbf{D}_0$, which is 0. Thus, we conclude $\mathcal{J}(\hat{q}) \geq 0$ for all $\hat{q} \in \mathbf{V}_0$. As $\mathcal{J}(\hat{q}) = \infty$ for $\hat{q} \in \mathbf{Q}\backslash\mathbf{V}_0$ we have established the desired stability (3.17) of $q_0 = q_*(t)$.

Step 4. *Upper energy estimate.* The energy balance for q_ε reads

$$\mathcal{E}_\varepsilon(t, q_\varepsilon(t)) + \mathrm{Diss}_{\mathcal{R}_\varepsilon}(q_\varepsilon, [0, t]) = \mathcal{E}_\varepsilon(0, q_\varepsilon(0)) - \int_0^t \langle \dot{\ell}_\varepsilon(s), q_\varepsilon(s) \rangle \mathrm{d}s.$$

Using (3.16) we can pass to the limit $\varepsilon_k \to 0$ by employing (3.3d) for the first term, (3.7) for the second term, (3.4b) for the third term, and (3.3d) and (3.16) for the fourth term. This leads to the estimate

$$\mathcal{E}_0(t, q_*(t)) + \mathrm{Diss}_{\mathcal{R}_0}(q_*, [0, t]) \leq \mathcal{E}_0(0, q_0(0)) - \int_0^t \langle \dot{\ell}_0(s), q_*(s) \rangle \mathrm{d}s.$$

Here we use liminf-estimates on the left-hand side (viz. (3.3d) and (3.7)), while convergences hold on the right-hand side.

Step 5. *Lower energy estimate.* The lower energy estimate for q_* follows solely from the stability of q_* derived in Step 3. For any partition $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, the stability of $q_*(t_{j-1})$ implies

$$\begin{aligned}
\mathcal{E}_0(t_{j-1}, q_*(t_{j-1})) &\leq \mathcal{E}_0(t_{j-1}, q_*(t_j)) + \mathcal{R}_0(q_*(t_j) - q_*(t_{j-1})) \\
&= \mathcal{E}_0(t_j, q_*(t_j)) + \mathcal{R}_0(q_*(t_j) - q_*(t_{j-1})) \\
&\quad - \int_{t_{j-1}}^{t_j} \langle \dot{\ell}_0(s), q_*(t_j) \rangle \mathrm{d}s.
\end{aligned}$$

Rearranging and summing over $j \in \{1, \ldots, N\}$ gives

$$\mathcal{E}_0(T, q_*(T)) + \sum_{j=1}^N \mathcal{R}_0(q_*(t_j) - q_*(t_{j-1})) \geq \mathcal{E}_0(0, q_*(0)) - \int_0^T \langle \dot{\ell}_0(s), \overline{q}_*(s) \rangle \mathrm{d}s,$$

where $\overline{q}_*$ is the piecewise constant interpolant with the value $q_*(t_j)$ for $t \in]t_{j-1}, t_j]$. Taking the fineness of the partitions to 0, the right-hand side converges while the left-hand side can be estimated from above giving the desired lower energy estimate

$$\mathcal{E}_0(T, q_*(T)) + \mathrm{Diss}_{\mathcal{R}_0}(q_*, [0, T]) \geq \mathcal{E}_0(0, q_*(0)) - \int_0^T \langle \dot{\ell}_0(s), q_*(s) \rangle \mathrm{d}s. \tag{3.20}$$

Together with the results of Steps 3 and 4 we conclude that q_* is equal to the unique energetic solution q_0 for $(\mathbf{Q}, \mathcal{E}_0, \mathcal{R}_0)$ with $q_0(0) = q_0^0$.

Step 6. *Improved convergence.* We already have established weak convergence instead of the strong convergence stated in (3.5a), see (3.16). Using (3.3d) the convergence (3.5d) follows immediately. The convergences (3.5b) and (3.5c) follow, since we have the obvious liminf estimate for both of them. However, Steps 4 and 5 show that the sum converges to the correct limit. This implies that each of them converges to the desired limit.

Using the energy convergence (3.5b) and the weak convergence $q_\varepsilon(t) \rightharpoonup q_0(t)$ we see that $q_\varepsilon(t)$ is a recovery sequence for $q_0(t)$ for $\mathcal{B}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{B}_0$. Thus, using Proposition 3.1(ii) we conclude the strong convergence $q_\varepsilon(t) \to q_0(t)$ as stated in (3.5a). □

4. Justification of the Elastoplastic Plate Model

In this section we provide the proof of the limit passage stated in Theorem 2.2. The main step is the Mosco convergence of the quadratic forms $\mathcal{B}_\varepsilon$ defined in (2.9) for $\varepsilon > 0$ and (2.17) for $\varepsilon = 0$, respectively. All notions of convergence (weak and strong) as well as the norm will relate to the basic Hilbert space $\mathbf{Q} \subset \mathrm{H}^1(\Omega;\mathbb{R}^3) \times \mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{dev}})$. The proof is similar to the approach devised in Ref. 4, but needs to be repeated as we use a more general material law involving also the plastic variable.

Proposition 4.1. *Under the assumptions of Sec. 2 we have the Mosco convergence* $\mathcal{B}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{B}_0$.

Proof. *Liminf estimate for weak convergence.* For all sequences $Q_\varepsilon \rightharpoonup Q_0$ we have to show $\mathcal{B}_0(Q_0) \leq \liminf_{\varepsilon\to0}\mathcal{B}_\varepsilon(Q_\varepsilon)$.

First consider the case $U_0 \notin \mathbf{U}_{\mathrm{KL}}$. Since $U \mapsto \|\mathbf{E}^3(U)\|^2_{\mathrm{L}^2}$ is convex and lower semicontinuous, we find $\liminf_\varepsilon\|\mathbf{E}^3(U_\varepsilon)\|_{\mathrm{L}^2} > 0$. Using the coercivity of $\mathcal{B}_\varepsilon$ in (2.10) we conclude

$$\mathcal{B}_\varepsilon(Q_\varepsilon) \geq c\|S_\varepsilon\mathbf{E}(U)S_\varepsilon\|^2_{\mathrm{L}^2} \geq \frac{c}{\varepsilon}\|\mathbf{E}^3(U_\varepsilon)\|^2_{\mathrm{L}^2} \to \infty = \mathcal{B}_0(Q_0).$$

For $U_0 \in \mathbf{U}_{\mathrm{KL}}$ we use $\mathcal{B}_0(Q_0) = \overline{\mathcal{B}}(Q_0)$ and employ estimate (2.15) giving $\mathcal{B}_\varepsilon(Q) \geq \overline{\mathcal{B}}(Q)$. Since $\overline{\mathcal{B}}$ is convex and lower semicontinuous, we conclude via

$$\liminf_{\varepsilon\to0}\mathcal{B}_\varepsilon(Q_\varepsilon) \geq \liminf_{\varepsilon\to0}\overline{\mathcal{B}}(Q_\varepsilon) \geq \overline{\mathcal{B}}(Q_0) = \mathcal{B}_0(Q_0).$$

Limsup estimate for strongly converging recovery sequences. The result is trivial for $U_0 \notin \mathbf{U}_{\mathrm{KL}}$ as $\mathcal{B}_0(Q_0) = \infty$, since we may take $Q_\varepsilon = Q_0$.

For $U_0 \in \mathbf{U}_{\mathrm{KL}}$ we have to do a nontrivial construction. We use the splitting of $\mathbb{W}$ into $\overline{\mathbb{W}}$ and $\mathbb{W}_2$ as given in (2.16). This leads to the splitting

$$\mathcal{B}_\varepsilon(Q) = \overline{\mathcal{B}}(Q) + \mathcal{J}\left(\frac{1}{\varepsilon}S_\varepsilon\mathbf{E}^3(U) - N(\mathbf{E}^{1,2}(U),P)\right) \quad \text{with}$$

$$\mathcal{J}(b) = \int_\Omega \mathbb{W}_2(b(x))\mathrm{d}x.$$

Since $\overline{\mathcal{B}}$ is strongly continuous on $\mathbf{Q}$ and $\mathcal{J}$ on $\mathrm{L}^2(\Omega;\mathbb{R}^3)$, it remains to be shown that for each $Q_0 \in \mathbf{Q}$ there exist $Q_\varepsilon \in \mathbf{Q}$ such that

$$\begin{aligned} Q_\varepsilon = (U_\varepsilon, P_\varepsilon) \to Q_0 \quad \text{in } \mathbf{Q} \quad \text{and} \\ \frac{1}{\varepsilon} S_\varepsilon \mathbf{E}^3(U_\varepsilon) - N(\mathbf{E}^{1,2}(U_\varepsilon), P_\varepsilon) \to 0 \quad \text{in } \mathrm{L}^2(\Omega;\mathbb{R}^3). \end{aligned} \tag{4.1}$$

We construct Q_ε in the form

$$Q_\varepsilon = (U_\varepsilon, P_0) \quad \text{with } U_\varepsilon = U_0 + \varepsilon S_\varepsilon^{-1} W^\varepsilon,$$

where W_ε is constructed as follows. Set $n^0 = N(\mathbf{E}^{1,2}(U_0), P_0) \in \mathrm{L}^2(\Omega;\mathbb{R}^3)$ and define the regularization $n^\varepsilon = (n_1^\varepsilon, n_2^\varepsilon, n_3^\varepsilon)$ via the unique solutions of the elliptic equations

$$-\varepsilon \Delta n_j^\varepsilon + n_j^\varepsilon = n_j^0 \ \text{ in } \Omega; \quad n_j^\varepsilon = 0 \ \text{ on } \partial\Omega.$$

Standard elliptic estimates give the *a priori* estimates

$$\varepsilon \|\nabla n_j^\varepsilon\|_{\mathrm{L}^2}^2 + \|n_j^\varepsilon\|_{\mathrm{L}^2}^2 = \|n_j^0\|_{\mathrm{L}^2}^2 \quad \text{and} \quad n^\varepsilon \to n^0 \ \text{ in } \mathrm{L}^2(\Omega;\mathbb{R}^3). \tag{4.2}$$

Now $W^\varepsilon = (W_1^\varepsilon, W_2^\varepsilon, W_3^\varepsilon)$ is defined via integration over x_3, namely

$$W_j^\varepsilon(x) = c_j \int_{-1}^{x_3} n_j^\varepsilon(x_1, x_2, \xi)\mathrm{d}\xi, \quad \text{where } (c_1, c_2, c_3) = (2, 2, 1).$$

This implies $\|W^\varepsilon\|_{\mathrm{L}^2} + \varepsilon^{1/2}\|\nabla W^\varepsilon\|_{\mathrm{L}^2} \le C$, and hence $\|Q_\varepsilon - Q_0\| = \|\varepsilon S_\varepsilon^{-1} W^\varepsilon\|_{\mathrm{H}^1} \le \varepsilon^{1/2} C \to 0$, which is the first condition in (4.1).

Moreover, using $U_0 \in \mathbf{U}_{\mathrm{KL}}$ we find

$$\frac{1}{\varepsilon} S_\varepsilon \mathbf{E}^3(U_\varepsilon) = 0 + \frac{1}{\varepsilon} S_\varepsilon \mathbf{E}^3((\varepsilon W_1^\varepsilon, \varepsilon W_2^\varepsilon, \varepsilon^2 W_3^\varepsilon)^\mathsf{T}) = n^\varepsilon + \frac{\varepsilon}{2}(\partial_{x_1} W_3^\varepsilon, \partial_{x_2} W_3^\varepsilon, 0)^\mathsf{T}.$$

Thus, we obtain the estimate

$$\begin{aligned} &\left\| \frac{1}{\varepsilon} S_\varepsilon \mathbf{E}^3(U_\varepsilon) - N(\mathbf{E}^{1,2}(U_\varepsilon, P) \right\|_{\mathrm{L}^2} \\ &\quad = \left\| n^\varepsilon + \frac{\varepsilon}{2}, (\partial_{x_1} W_3^\varepsilon, \partial_{x_2} W_3^\varepsilon, 0)^\mathsf{T} -, n^0 - N(, \mathbf{E}^{1,2}(\varepsilon S_\varepsilon^{-1}, W^\varepsilon), 0) \right\|_{\mathrm{L}^2} \\ &\quad \le \|n^\varepsilon - n^0\|_{\mathrm{L}^2} + \varepsilon \|\nabla W_3^\varepsilon\|_{\mathrm{L}^2} + \varepsilon |N| \|\nabla W^\varepsilon\|_{\mathrm{L}^2} \le \|n^\varepsilon - n^0\|_{\mathrm{L}^2} + C\varepsilon^{1/2}. \end{aligned}$$

Using (4.2) we obtain the second convergence stated in (4.1). □

Proof of Theorem 2.2. We show that the abstract result presented in Theorem 3.1 can be applied. For this we need to check that the assumptions (3.3) are satisfied for the elastoplastic problem discussed in Sec. 2. Note that for this case we have $\ell_\varepsilon(t) = L(t)$ and $\mathcal{R}_\varepsilon = \mathcal{R}$, since no dependence on ε is present. The conditions (3.3a) and (3.3b) are obviously satisfied. Condition (3.3c) holds after taking (2.10) into account.

The Mosco convergence $\mathcal{B}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{B}_0$ was established in Proposition 4.1, thus (3.3d) holds since $\ell_\varepsilon = L$ is independent of ε.

For the ε-independent dissipation potential $\mathcal{R}_\varepsilon = \mathcal{R}$ the continuous convergence $\mathcal{R}_\varepsilon \overset{c}{\rightsquigarrow} \mathcal{R}_0$ reduces to strong continuity of $\mathcal{R}$, cf. the definition in (3.2). By condition (2.3) the strong continuity of $V \mapsto \mathcal{R}(V)$ for $V \in \mathrm{L}^2(\Omega; \mathbb{R}^{3\times 3}_{\mathrm{dev}})$ is obvious. The Mosco convergence $\mathcal{R}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{R}$, additionally asks for weak lower semicontinuity, which follows immediately from convexity and strong continuity. Hence (3.3e) is established as well.

Thus, Theorem 2.2 follows as a direct consequence of Theorem 3.1. □

5. Discussion of the Elastoplastic Plate Model

In this section we show that the limit model obtained in Sec. 2.3 can be reduced to a two-dimensional elastic problem coupled to plastic effects that can either be described by a three-dimensional model without memory or by a Prandtl–Ishlinskii operator associated to each point $y \in \omega$.

For notational simplicity, we will restrict now to the special isotropic energy $\mathbb{W}$ and its relaxed stored energy density $\overline{\mathbb{W}}$ treated in the Examples 2.1 and 2.2. Thus, we are also in the same framework as Ref. 18, and can compare the results easily.

5.1. *Coupling of stretching, bending, and plasticity*

The main observation is that the Kirchhoff–Love displacements $U \in \mathbf{U}_{\mathrm{KL}}$ can be characterized by functions defined only on the midplane ω. Thus, the energy $\mathcal{E}_0$ can be reduced by integrating over the variable x_3. We will use the letter y to indicate points in ω. From now on, ∇ and Δ will only act on $y \in \omega$, i.e. $\nabla = (\partial_{y_1}, \partial_{y_2})^{\mathsf{T}}$, $\Delta = \partial_{y_1}^2 + \partial_{y_2}^2$, and $\mathrm{D}^2 V_3 \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ denotes the Hessian of $V_3 : \omega \to \mathbb{R}$. Moreover we will use the two-dimensional in-plane strain tensor

$$\mathbf{E}^{1,2}(V) = \frac{1}{2}(\nabla V^{1,2} + (\nabla V^{1,2})^{\mathsf{T}}), \quad \text{where } V^{1,2} = (V_1, V_2)^{\mathsf{T}},$$

which does not depend on V_3.

Concerning the variable $P \in \mathrm{L}^2(\Omega; \mathbb{R}^{3\times 3}_{\mathrm{dev}})$ we will use $\Omega = \omega \times]-1, 1[$ and the identification

$$\mathrm{L}^2(\omega \times]-1, 1[; \mathbb{R}^{3\times 3}_{\mathrm{dev}}) \cong \mathrm{L}^2(\omega; \mathbf{L}) \quad \text{with } \mathbf{L} := \mathrm{L}^2(]-1, 1[; \mathbb{R}^{3\times 3}_{\mathrm{dev}}).$$

Thus, with each point $y \in \omega$ we associate an internal variable $P(y, \cdot) \in \mathbf{L}$.

Using the isomorphism $\mathcal{K}$ between $\mathbf{V}$ and $\mathbf{U}_{\mathrm{KL}}$ (see (2.12)) we find that the RIS $(\mathbf{Q}, \mathcal{E}_0, \mathcal{R}_0)$ is equivalent to the RIS $(\mathbf{Q}_{\mathrm{KL}}, \mathcal{E}_{\mathrm{KL}}, \mathcal{R})$ with

$$\mathbf{Q}_{\mathrm{KL}} := \mathbf{V} \times \mathrm{L}^2(\omega; \mathbf{L}), \quad \mathcal{R}(P) = \int_\omega \int_{-1}^1 R(P(y, x_3))\mathrm{d}x_3 \mathrm{d}y,$$

$$\mathcal{E}_{\mathrm{KL}}(t, V, P) := \int_\omega W(\mathbf{E}^{1,2}(V), \mathrm{D}^2 V_3(y), P(y))\mathrm{d}y - \langle \ell_{\mathrm{KL}}(t), V \rangle, \quad \text{where}$$

$$W(E, H, P) = W_{\mathrm{memb}}(E, [P^{1,2}]_0) + W_{\mathrm{bend}}(H, [P^{1,2}]_1) + W_{\mathrm{plast}}(P),$$

$$
\begin{aligned}
W_{\text{memb}}(E,\Pi) &= \frac{2\lambda\mu}{\lambda+2\mu}\big((\operatorname{tr} E)^2 - \operatorname{tr} E \operatorname{tr}\Pi\big) + 2\mu(|E|^2 - E:\Pi),\\
W_{\text{bend}}(H,\Pi) &= \frac{2\lambda\mu}{\lambda+2\mu}\left(\frac{1}{3}(\operatorname{tr} H)^2 + \operatorname{tr} H \operatorname{tr}\Pi\right) + 2\mu\left(\frac{1}{3}\,|H|^2 + H:\Pi\right),\\
W_{\text{plast}}(P) &= \frac{\lambda\mu}{\lambda+2\mu}\,\|\operatorname{tr} P^{1,2}\|_2^2 + \mu\|P^{1,2}\|_2^2 + \frac{h}{2}\,\|P\|_2^2,\\
\langle \ell_{\text{KL}}(t), V\rangle &= \int_\omega G_{\text{memb}}(t,y)\cdot V^{1,2}(y) + g_{\text{bend}}(t,y)\,V_3(y)\\
&\quad + G_{\text{bend}}(t,y)\cdot\nabla V_3(y)\mathrm{d}y,\\
G_{\text{memb}}(t,y) &= [F^{1,2}_{\text{vol}}(t,y,\cdot)]_0 + F^{1,2}_{\text{surf}}(t,y,1) + F^{1,2}_{\text{surf}}(t,y,-1),\\
g_{\text{bend}}(t,y) &= [F_{\text{vol}\,3}(t,y,\cdot)]_0 + F_{\text{surf}\,3}(t,y,1) + F_{\text{surf}\,3}(t,y,-1),\\
G_{\text{bend}}(t,y) &= F^{1,2}_{\text{surf}}(t,y,-1) - F^{1,2}_{\text{surf}}(t,y,1).
\end{aligned}
$$

Here $V^{1,2} = (V_1, V_2)^\mathsf{T} \in \mathbb{R}^2$, and $E, H, \Pi \in \mathbb{R}^{2\times 2}_{\text{sym}}$ are placeholders for $\mathbf{E}^{1,2}(V)$, $\mathrm{D}^2 V_3$, and $[P^{1,2}]_k$, respectively, and we used the short-hand notations

$$
A:B = \sum_{i,j=1}^{2} A_{ij}B_{ij}, \quad [g]_0 = \int_{-1}^{1} g(x_3)\mathrm{d}x_3,
$$
$$
[g]_1 = \int_{-1}^{1} x_3 g(x_3)\mathrm{d}x_3, \quad \|g\|_2^2 = \int_{-1}^{1} |g(x_3)|^2\mathrm{d}x_3.
$$

The important structure in the form of W is that the membrane strains $E = \mathbf{E}^{1,2}(V)$ only couple to the (even) averages $[P^{1,2}]_0$, while the bending strains $H = \mathrm{D}^2 V_3$ only couple to the (odd) averages $[P^{1,2}]_1$.

To highlight the structure of the derived evolutionary system obtained via the RIS $(\mathbf{Q}_{\text{KL}}, \mathcal{E}_{\text{KL}}, \mathcal{R})$ we now write down the corresponding differential inclusion, cf. (2.7a) versus the energetic formulation (2.7c). It consists of two elliptic systems, one for the membrane part and one for the bending part, and the plastic flow law. Both elliptic systems are nontrivially coupled to the plastic part via the strain tensors

$$
\partial_E W_{\text{memb}}(E,\Pi) = \Sigma_0(2E - \Pi) \in \mathbb{R}^{2\times 2}_{\text{sym}},
$$
$$
\partial_H W_{\text{bend}}(A,p) = \Sigma_0\left(\frac{2}{3}H + \Pi\right) \in \mathbb{R}^{2\times 2}_{\text{sym}},
$$

where $\Sigma_0(E) := \varrho \operatorname{tr} E I_2 + 2\mu E$ with $\varrho := \frac{2\lambda\mu}{\lambda+2\mu}$. The system reads

$$
0 = -\operatorname{div}\big(\Sigma_0(2\mathbf{E}^{1,2}(V)) - [P^{1,2}]_0)\big) - G_{\text{memb}}(t,\cdot) \quad \text{in } \omega, \tag{5.1a}
$$
$$
0 = \operatorname{div}\operatorname{div}\left(\Sigma_0\left(\frac{2}{3}\mathrm{D}^2 V_3 + [P^{1,2}]_1\right)\right) - g_{\text{bend}}(t,\cdot) - \operatorname{div} G_{\text{bend}}(t,\cdot) \quad \text{in } \omega, \tag{5.1b}
$$
$$
0 \in \partial R(\dot{P}) + \operatorname{dev}(\llbracket \Sigma_0(P^{1,2} - \mathbf{E}^{1,2}(V) + x_3\mathrm{D}^2 V_3) \,\|\, 0 \rrbracket) + hP \quad \text{in } \Omega, \tag{5.1c}
$$

where we recall the notation $\llbracket A \,\|\, b \rrbracket \in \mathbb{R}^{3\times 3}_{\text{sym}}$ for $A \in \mathbb{R}^{2\times 2}_{\text{sym}}$ and $b \in \mathbb{R}^3$ from (2.13b).

We see that (5.1a) is a second-order membrane equation for the in-plane displacements $V^{1,2} = (V_1, V_2)$ with the average plastic strains $[P^{1,2}]_0$ acting as plastic strains. The fourth-order equation (5.1b) for the out-of-plane displacement V_3 generalizes Kirchhoff's plate equation, where now the first moments $[P^{1,2}]_1$ (odd averages) act as plastic strains. The flow law (5.1c) is still posed on $\Omega = \omega \times]-1,1[$, but the important point is that the coupling with $\mathbf{E}^{1,2}(V)$ and $\mathrm{D}^2 V_3$ occurs only via special x_3-dependent profiles, namely 1 and x_3, respectively.

5.2. *Nontrivial coupling of the membrane and bending mode via plasticity*

We want to exemplify the nontrivial coupling by comparing two simple examples of loading, which would give the same result, if there is no coupling.

In Example 1, we first load and unload the plate in a uniaxial stretch in direction x_1. Then, we load in a bending mode and unload again. In Example 2 we first do the bending and then the uniaxial stretching.

In both situations we assume strains independent of $y \in \omega$ leading to the displacements

$$V(t,y) = (\delta(t)y_1, \delta(t)y_2, \gamma(t)(y_1^2 + y_2^2)/2)^\mathsf{T}, \tag{5.2}$$

where the scalar stretch δ and the scalar bending $\gamma(t)$ are to be determined. Clearly, we have $\mathbf{E}^{1,2}(V) = \delta(t)I_2$ and $H(t) = \mathrm{D}^2 V_3 = \gamma(t)I_2$.

Assuming that the dissipation potential R is of the form $R(\dot{P}) = r|\dot{P}|$, we can now find a solution P of the flow rule (5.1c) in the form

$$P(t,y,x_3) = \pi(t,x_3)\ \mathrm{diag}(1,1,-2), \quad \pi(t,x_3) = 0. \tag{5.3}$$

For this, it is necessary and sufficient that $\pi(t,x) \in \mathbb{R}$ solves the scalar hysteresis problem given by the differential inclusion

$$0 \in r\,\mathrm{Sign}(\dot{\pi}) + \frac{2}{3}(\varrho + \mu)(\pi - \delta + x_3\gamma) + h\pi. \tag{5.4}$$

To simplify the computations, we choose $r = \frac{2}{3}(\varrho + \mu) = 1$ and $h = 0$.

Using the hat function $\chi(t) = \max\{0, 2 - |2 - t|\}$, Example 1 is given by the loadings $\delta(t) = \chi(t)$ and $\gamma(t) = \chi(t-4)$. The solution is given by

$$\pi^{(1)}(t,x_3) = \begin{cases} \max\{0, \min\{t-1, 1\}\} & \text{for } t \in [0,4], \\ 1 - x_3(t-4) & \text{for } t \in [4,6] \text{ and } x_3 \geq 0, \\ 1 & \text{for } t \in [4,6] \text{ and } x_3 \leq 0, \\ \min\{1, 1-2x_3\} & \text{for } t \in [6,8]. \end{cases}$$

Now we consider Example 2 with stretching after bending, i.e. $\gamma(t) = \chi(t)$ and $\delta(t) = \chi(t-4)$. The solution reads

$$\pi^{(2)}(t,x_3) = \begin{cases} \max\{\min\{1 - tx_3, 0\}, -1 - tx_3\} & \text{for } t \in [0,2], \\ \pi(2,x_3) & \text{for } t \in [2,4], \\ \min\{5 - t, \pi(2,x_3)\} & \text{for } t \in [4,6], \\ -1 & \text{for } t \in [6,8]. \end{cases}$$

By comparing the two solutions, it is clear that the bending and stretching phases behave differently if they start from the unstressed initial condition or from the pre-stressed state left over from the other phase.

We refer to Ref. 23 for more details.

5.3. *Reduction to two-dimensional model via Prandtl–Ishlinskii operators*

We now show how the plastic flow law (5.1c) can be encoded into a hysteresis operator, which does not explicitly show the dependence on the variable x_3. This generalizes the idea in Guenther *et al.*[18]

We first note that (5.1c) can be rewritten in the form

$$0 \in \partial R(\dot{P}) + \mathbb{A}P - \mathbb{L}(t), \quad P(0) = 0, \tag{5.5}$$

where R is as before and $\mathbb{A}$ is a symmetric and positive definite linear operator on $\mathbb{R}^{3\times 3}_{\mathrm{dev}}$, while $\mathbb{L} \in \mathrm{C}^{\mathrm{Lip}}([0,T];\mathbb{R}^{3\times 3}_{\mathrm{dev}})$ is some loading. For simplicity we have added the trivial initial condition $P(0) = 0$ and we will further assume $\mathbb{L}(0) = 0$. (In fact, it would suffice to assume the stability condition $0 \in \partial R(0) + \mathbb{A}P(0) - \mathbb{L}(0)$, cf. Theorem 2.1.)

For each loading $\mathbb{L} \in \mathrm{C}^{\mathrm{Lip}}_0([0,T];\mathbb{R}^{3\times 3}_{\mathrm{dev}})$, where the subscript $_0$ stand for the initial condition $\mathbb{L}(0) = 0$, there is a unique solution $P \in \mathrm{C}^{\mathrm{Lip}}_0([0,T];\mathbb{R}^{3\times 3}_{\mathrm{dev}})$. This defines the (vector-valued) play operator $\mathbb{P}$ (cf. Refs. 38, 5 and 19) associated with $\mathbb{E} = \partial R(0)$ and $\mathbb{A}$ via

$$\mathbb{P}_{\mathbb{E}}[\mathbb{L}](t) := P(t).$$

We define the "parallel extension" of $\mathbb{P}_{\mathbb{E}}$ from $\mathbb{R}^{3\times 3}_{\mathrm{dev}}$ to $\mathbf{L} = \mathrm{L}^2(]-1,1[;\mathbb{R}^{3\times 3}_{\mathrm{dev}})$ via

$$\mathcal{P}_{\mathbb{E}}[L](t,x_3) := \mathbb{P}_{\mathbb{E}}[L(\cdot,x_3)](t),$$

where now $L \in \mathrm{C}^{\mathrm{Lip}}_0([0,T];\mathbf{L})$. We call this a parallel extension as the play operators at different values of x_3 do not interact.

We want to use the play operator $\mathcal{P}_{\mathbb{E}}$ to describe the coupled system (5.1) for the elastoplastic plate. We do this by superimposing such play operators by using the special coupling structure. Recall that the first two equations for V are coupled to $P(t,\cdot) \in \mathbf{L} = \mathrm{L}^2(]-1,1[;\mathbb{R}^{3\times 3}_{\mathrm{dev}})$ only via the averages $[P(t,\cdot)]_0$ and $[P(t,\cdot)]_1$, and additionally the equation for P is forced by $E = \mathbf{E}^{1,2}(V)$ and $H = \mathrm{D}^2 V$ only via the "dual construction".

The forcing via E and H in (5.1c) can be described via the mapping

$$M : \begin{cases} \mathbb{R}^{2\times 2}_{\mathrm{sym}} \times \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbf{L} = \mathrm{L}^2(]-1,1[;\mathbb{R}^{3\times 3}_{\mathrm{dev}}), \\ (E,H) \mapsto \mathrm{dev}\,(\llbracket \Sigma_0(x_3 H - E) \,\|\, 0 \rrbracket). \end{cases}$$

A simple calculation shows that the adjoint mapping takes the form

$$M^* : \begin{cases} \mathbf{L} = \mathrm{L}^2(]-1,1[;\mathbb{R}^{3\times 3}_{\mathrm{dev}}) \to \mathbb{R}^{2\times 2}_{\mathrm{sym}} \times \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \\ P \mapsto (-\Sigma_0([P^{1,2}]_0), \Sigma_0([P^{1,2}]_1)). \end{cases}$$

Note that M^*P contains exactly all the P-dependent terms occurring in (5.1a) and (5.1b).

Thus, we are able to define a *generalized Prandtl–Ishlinskii* operator as a generalized contraction of an infinite family of play operators as follows:

$$\mathfrak{P} : \begin{cases} \mathrm{C}_0^{\mathrm{Lip}}([0,T]; \mathbb{R}^{2\times 2}_{\mathrm{sym}} \times \mathbb{R}^{2\times 2}_{\mathrm{sym}}) \to \mathrm{C}_0^{\mathrm{Lip}}([0,T]; \mathbb{R}^{2\times 2}_{\mathrm{sym}} \times \mathbb{R}^{2\times 2}_{\mathrm{sym}}), \\ (E,H) \mapsto M^*\mathcal{P}_{\mathbb{E}}[M(E,H)]. \end{cases}$$

The usage of M and M^* in the contraction means that the hysteresis operator $\mathfrak{P}$ still has the usual symmetry properties and is compatible with the energetic formulation.

Denoting the components of $\mathfrak{P}$ corresponding to E and H by $\mathfrak{P}_E$ and $\mathfrak{P}_H$, respectively, we are now able to rewrite the coupled system (5.1) for the elastoplastic plate in terms of the elliptic partial differential equations only, since the plastic evolution law is hidden in the hysteresis operator $\mathfrak{P}$:

$$-\mathrm{div}(\Sigma_0(\mathbf{E}^{1,2}(V)) + \mathfrak{P}_E[\mathbf{E}^{1,2}(V), \mathrm{D}^2 V_3](t)) = G_{\mathrm{memb}}(t,\cdot), \tag{5.6a}$$

$$\mathrm{div}\,\mathrm{div}(\Sigma_0(\mathrm{D}^2 V_3) + \mathfrak{P}_H[\mathbf{E}^{1,2}(V), \mathrm{D}^2 V_3](t)) = g_{\mathrm{bend}}(t,\cdot) + \mathrm{div}\, G_{\mathrm{bend}}(t,\cdot). \tag{5.6b}$$

We can restrict to the pure bending case, as was done in Guenther *et al.*[18] if we assume that $G_{\mathrm{memb}} \equiv 0$ and that $P(t,y,\cdot)$ is odd. Then, $V^{1,2} \equiv 0$ for all times. Then, we can forget about E and restrict our attention to H only. This allows us to give $\mathfrak{P}_H$ a more explicit form, which appear to be a true Prandtl–Ishlinskii operator. To see this, we use a classical scaling law for play operators, namely

$$\mathcal{P}_{\mathbb{E}}[\alpha\mathbb{L}] = \alpha\mathcal{P}_{\frac{1}{\alpha}\mathbb{E}}[\mathbb{L}] \quad \text{for every constant } \alpha \neq 0.$$

This leads to the final formula

$$\begin{aligned} \mathfrak{P}_H[H] &= [\Sigma_0(\mathcal{P}_{\mathbb{E}}[x_3\,\mathrm{dev}([\![\Sigma_0(H(\cdot)) \,\|\, 0]\!])])]_1 \\ &= \Sigma_0\left(\int_{-1}^{1} x_3^2\, \mathcal{P}_{\frac{1}{x_3}\mathbb{E}}[\mathrm{dev}([\![\Sigma_0(H(\cdot)) \,\|\, 0]\!])]\mathrm{d}x_3\right). \end{aligned}$$

Thus, our hysteresis operator is just a simple transformation of the operator $\mathcal{F}$ derived at the end of Sec. 2.2 in Ref. 18, namely $\mathcal{P}_{\mathbb{E}} = \Sigma_0 \circ \mathcal{F} \circ \Sigma_0$.

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