Applicable Analysis
Vol. 89, No. 9, September 2010, 1493–1512

Taylor & Francis
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Dispersive stability of infinite-dimensional Hamiltonian systems on lattices

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Communicated by A. Pankov

(*Received 7 October 2009; final version received 19 November 2009*)

We derive dispersive stability results for oscillator chains like the FPU chain or the discrete Klein–Gordon chain. If the nonlinearity is of degree higher than four, then small localized initial data decay exactly as in the linear case. For this, we provide sharp decay estimates for the linearized problem using oscillatory integrals and avoiding the nonoptimal interpolation between different ℓ^p spaces.

Keywords: dispersive decay; oscillatory integrals; discrete Klein–Gordon equation; FPU chain

AMS Subject Classifications: 37K60; 34Dxx; 41A60

1. Introduction

The phenomenon of dispersive stability is well-studied for partial differential equations. Usually, one considers a Hamiltonian system where energy conservation prevents the strict spectral stability giving rise to the exponential decay. Typically, the behaviour of small solutions is such that the energy norm is bounded from above and below by constants while the L^∞ norm decays with an algebraic rate of the type $(1+t)^{-\alpha}$. This rate is generated from the fact that initially localized solutions are dispersed by the different group velocities associated with the different wave numbers θ. The fundamental effects derived from the dispersion relation $\hat{\omega} = \omega(\theta)$ of the linear differential operator, where $\hat{\omega}$ is the frequency and $c(\theta) = \nabla_\theta \omega(\theta)$ the group velocity. The dispersion is now related to the fact that c still depends nontrivially on θ, i.e. the second derivative of ω should be nontrivial. We refer to [1–4] for results treating the sine-Gordon, the Klein–Gordon, the nonlinear Schrödinger or the relativistic wave equations. Sometimes, the theory is developed under the name scattering theory for small data. In [5], a recent improvement was made on the lowest-order of nonlinearity for the generalized Korteweg–de Vries equation by a careful combination of sharp estimates for the linear part, obtained via deep harmonic analysis, and careful chain-rule estimates for the fractional derivatives of the nonlinearity.

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ISSN 0003–6811 print/ISSN 1563–504X online

DOI: 10.1080/00036810903517605
http://www.informaworld.com

The same dispersive effects are to be expected in discrete systems, which are infinite ODEs on a lattice $\mathbb{Z}^d$. The difference is that the dispersion relation is now a periodic function in θ, i.e. ω is defined on the torus $\mathbb{T}^d := \mathbb{R}^d/_{(2\pi\mathbb{Z})^d}$. Thus, in contrast to the PDEs, where ω is an algebraic function on $\mathbb{R}^d$, the dispersion relation has necessarily a richer degeneracy structure. As a result, the linear decay estimates for periodic lattices need a more careful analysis, and it is the aim of this work to establish a more general approach to this field.

To describe the work done so far and our contributions we start by highlighting the three major equations treated in this field, namely the Fermi–Pasta–Ulam chain (FPU), the Klein–Gordon chain (dKG) and the discrete nonlinear Schrödinger equation (dNLS):

$$\ddot{x}_j = V'(x_{j+1} - x_j) - V'(x_j - x_{j-1}), \quad j \in \mathbb{Z}; \tag{FPU}$$

$$\ddot{x}_j = x_{j+1} - 2x_j + x_{j-1} + W'(x_j), \quad j \in \mathbb{Z}; \tag{dKG}$$

$$\mathrm{i}\dot{u}_j = u_{j+1} - 2u_j + u_{j-1} + a|u_j|^{\beta-1}u_j, \quad j \in \mathbb{Z}. \tag{dNLS}$$

Here the potentials V and W are assumed to be such that $V'(r) = r + \mathcal{O}(|r|^\beta)$ and $W'(x) = x + \mathcal{O}(|x|^\beta)$. In general, $\beta > 1$ is used to measure the order of the nonlinearity.

A very careful study of the linear FPU equation was given in [6], which highlights the synchronization phenomena in compact domains. In [7] general multidimensional linear lattice systems were studied on the shorter hyperbolic scale, where energy transport along the rays dominates but dispersion is not yet seen. Discrete lattice systems as finite-difference approximations of wave equations are analysed in [8,9], where the proper approximation of dispersion relations is an important point.

Dispersive stability results in the direction of this work are obtained in [10,11]. The latter work provides the dispersive stability of FPU under the assumption that the nonlinearity satisfies $\beta > 5$. In this work, we will improve this result to the case $\beta > 4$. In [10] the dKG and dNLS are studied analytically and numerically; we comment on the result of this article below.

To describe our result we first restrict to FPU, which will be discussed in full detail in Section 3. There we will also treat a generalized FPU chain which allows for any finite number of interactions. Our main result will be that under a suitable stability and nonresonance condition, we have a dispersive stability if the nonlinearity is of order $\beta > 4$. In particular, we will show that the decay of the solution of the nonlinear problem is the same as that of the linear one. The main point in the analysis is that we obtain an improved estimate for the dispersive decay of the linear semigroup. Writing FPU abstractly in the form

$$\dot{\mathbf{z}}(t) = \mathcal{L}\mathbf{z} + \mathcal{K}\mathcal{N}(\mathbf{z})$$

and using the Banach spaces $\ell^p = \ell^p(\mathbb{Z}; \mathbb{R}^2)$ we find, for each $p \in [2, 4) \cup (4, \infty]$, a constant C_p such that

$$\|\mathrm{e}^{\mathcal{L}t}\mathbf{z}_0\|_{\ell^p} \leq \frac{C_p}{(1+t)^{\alpha_p}} \|\mathbf{z}_0\|_{\ell^1}, \quad \|\mathrm{e}^{\mathcal{L}t}\mathcal{K}\mathbf{z}_0\|_{\ell^p} \leq \frac{C_p}{(1+t)^{\tilde{\alpha}_p}} \|\mathbf{z}_0\|_{\ell^1} \quad \text{for } t > 0, \tag{1.1}$$

where the decay rates are given by

$$\alpha_p = \begin{cases} \frac{p-2}{2p} & \text{for } p \in [2,4), \\ \frac{p-1}{3p} & \text{for } p \in (4,\infty], \end{cases} \quad \text{and} \quad \tilde{\alpha}_p = \frac{p-2}{2p}.$$

The operator $\mathcal{K}$ arises from the difference structure of the right-hand side in FPU. The case $p=4$ is excluded in (1.1), since the first estimate holds only with a logarithmic correction (see (3.9b)).

The key observation is that the decay rates for $p \in (2,\infty)$ are strictly better than the ones obtained by interpolating the decay $\alpha_2 = 0$ and $\alpha_\infty = 1/3$, which would lead to $\hat{\alpha}_p = (p-2)/(3p) < \alpha_p$. The main work of Section 3 will be devoted to establish the decay estimates (1.1), which are obtained by analysing the dispersion relation and estimating the resulting oscillatory integrals. The nonlinear stability result is then obtained using standard arguments, which we have collected in an abstract setting in Section 2. We emphasize that all nonlinear decay estimates are of the form that the nonlinear decay is exactly of the order as the linear decay, which is also found numerically (Figure 1). We also show that our decay rates are optimal in the sense that the dispersive decay of the nonlinear system cannot be better than for the linear system.

In Section 4, we will discuss the usage of our method in more general settings such as dKG and dNLS. In particular, we compare our results for dKG with those obtained in [10]. There, for $\beta > 5$ dispersive decay in ℓ^p was proved with the rate $\hat{\alpha}_p = (p-2)/(3p)$, while numerically the values 0.226, 0.267 and 0.292 were obtained for $p=4$, 5 and 6, respectively. We improve the results in a twofold manner: first, we reduce the possible order of nonlinearity to the regime $\beta > 4$ and second, we establish the better (and sharp) decay rate $\alpha_p = (p-1)/(3p)$ matching much better with the numerical values obtained there.

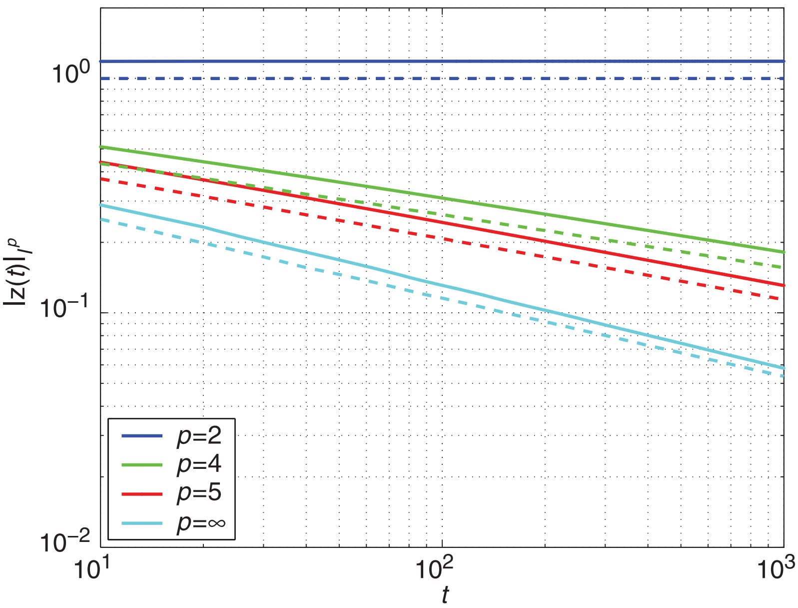

Figure 1. Double-logarithmic plot of ℓ^p norms of the solution to the linear FPU (—) and the nonlinear FPU with $V(r) = r + |r|^4$ (- -) as a function of t.

We conclude with the remark that there is a rich literature on persistent localized solutions in lattices, such as modulated pulses, solitons and breathers (see e.g. [11–14]). From this, it is possible to show that the generalized FPU admits the families of solitary waves of KdV type, for which $\beta<5$ may have an arbitrary small energy. However, the smaller the amplitude of these solitary waves, the broader they are. For the case $\beta<3$, it follows that the dispersive stability cannot hold (Remark 3.3). It remains open what happens in the case $\beta\in[3,4]$.

2. General stability result

In this section, we present a general method to prove the dispersive stability for nonlinear systems. The method, which is based on the weak decay estimates of its linearization, is classical and was established for the dispersive stability in PDE theory (see, for instance [1,2]). See also [15] for a survey in the related theory of diffusive stability in parabolic systems. In the context of lattice models, Giannoulis *et al.* [11] illustrate the ideas in an abstract setting and in [10] these arguments are applied to dKG systems and dNLS equations.

To emphasize the general structure we again use an abstract setting in general Banach spaces, which will be specialized to the spaces $\ell^p(\mathbb{Z}^d, \mathbb{R}^n)$ in the following sections. The general aim is to establish the conditions which guarantee that the nonlinear system still has the same dispersive decay as the linear one. This will be our first result. In the second result, we even go further by showing that the decay of the difference between the solution of the linear systems and the solution of the nonlinear systems is faster than the linear decay.

We start with the general system on a Banach space Z given in the form

$$\dot{\mathbf{z}} = \mathcal{L}\mathbf{z} + \mathcal{K}\mathcal{N}(\mathbf{z}), \tag{2.1}$$

where $\mathcal{L}$, $\mathcal{K}$ are linear and bounded and $\mathcal{N}$ is a nonlinear operator. The operator $\mathcal{L}: Z\to Z$ generates a bounded semi-group $(\mathrm{e}^{\mathcal{L}t})_{t\geq 0}$, that is, there exists a $C_{\mathcal{L}}>0$ with $\|\mathrm{e}^{\mathcal{L}t}\mathbf{z}\|_Z\leq C_{\mathcal{L}}\|\mathbf{z}\|_Z$ for all $t\geq 0$ and $\mathbf{z}\in Z$. Typically, the space Z is chosen such that the solution $\mathbf{z}=0$ is a stable solution of (2.1), i.e.

$$\exists C_E > 0 \quad \forall \text{sln}\, \mathbf{z}(t) \text{ with } \|\mathbf{z}^0\|_Z \leq \varepsilon \quad \forall t > 0 : \|\mathbf{z}(t)\|_Z \leq C_E\|\mathbf{z}^0\|_Z. \tag{2.2}$$

This condition is, in particular, satisfied if the system is Hamiltonian and the energy functional serves as a Lyapunov function. That is, if the energy is bounded from above and below.

However, for proving the dispersive stability we need to choose different spaces and do not rely on (2.2). We consider a scale of Banach spaces $Z_0\subset Z\subset X$ and a space $Z_{\mathcal{N}}\subset Z$ where the embeddings are assumed to be continuous. The space X is used for the estimation of the solutions, Z_0 is taken for the initial conditions, and $Z_{\mathcal{N}}$ measures the nonlinearity. We assume that positive constants C_1, C_2, C_3, α, γ and $\beta>1$ exist such that the following estimates hold for all $\mathbf{z}$ and all $t\geq 0$:

$$\|\mathrm{e}^{\mathcal{L}t}\mathbf{z}\|_X \leq \frac{C_1}{(1+t)^{\alpha}}\|\mathbf{z}\|_{Z_0}, \tag{2.3a}$$

$$\|\mathrm{e}^{\mathcal{L}t}\mathcal{K}\mathbf{z}\|_X \leq \frac{C_2}{(1+t)^{\gamma}}\|\mathbf{z}\|_{Z_{\mathcal{N}}}, \tag{2.3b}$$

$$\|\mathcal{N}(\mathbf{z})\|_{Z_{\mathcal{N}}} \leq C_3\|\mathbf{z}\|_X^{\beta}. \tag{2.3c}$$

The following result is the first simple decay estimate, which we state for the reasons of clarity. It is in fact a special case of the more involved result given below. Hence, we do not provide an independent proof.

THEOREM 2.1 *Let the conditions* (2.3) *hold with* $\min\{\gamma, \alpha\beta, \alpha\beta+\gamma-1\} \geq \alpha$ *and* $\gamma \neq 1 \neq \beta\alpha$. *Then, there exist positive constants* C *and* ε *such that for each* $\mathbf{z}_0 \in Z_0$ *with* $\|\mathbf{z}_0\|_{Z_0} \leq \varepsilon$ *the unique solution* $\mathbf{z}$ *of* (2.1) *with* $\mathbf{z}(0) = \mathbf{z}_0$ *satisfies*

$$\|\mathbf{z}(t)\|_X \leq \frac{C}{(1+t)^\alpha} \|\mathbf{z}(0)\|_{Z_0} \quad \textit{for } t \geq 0.$$

This and the following result rely on the following lemma that is used to estimate the convolution integral occurring in the variation-of-constants formula. The lower bound in the following result is only given to indicate that the provided exponent γ is optimal.

LEMMA 2.2 *For constants* $\alpha_1, \alpha_2 \in [0,1) \cup (1,\infty)$ *there exists a constant* $C > 0$ *such that*

$$\frac{t}{C(1+t)^{\gamma+1}} \leq \int_0^t \frac{1}{(1+t-s)^{\alpha_1}} \frac{1}{(1+s)^{\alpha_2}} \mathrm{d}s \leq \frac{C}{(1+t)^\gamma} \quad \textit{for all } t > 0, \tag{2.4}$$

where $\gamma = \min\{\alpha_1, \alpha_2, \alpha_1+\alpha_2-1\}$.

Proof To obtain the estimate, we split the integral into the two domains $[0, t/2]$ and $[t/2, t]$. In the first interval we estimate $(1+t)/2 \leq 1+t-s \leq 1+t$ and obtain

$$\frac{1}{(1+t)^{\alpha_1}} M_2(t/2) \leq \int_0^{t/2} \frac{1}{(1+t-s)^{\alpha_1}} \frac{1}{(1+s)^{\alpha_2}} \mathrm{d}s \leq \frac{2^{\alpha_1}}{(1+t)^{\alpha_1}} M_2(t/2),$$

where $M_2(r) = \int_0^r (1+s)^{-\alpha_2} \mathrm{d}s$. Evaluating the integral M_2 explicitly, we find a decay estimate with an exponent $\gamma_2 = \min\{\alpha_1, \alpha_1+\alpha_2-1\}$. Treating the interval $[t/2, t]$ similarly, the assertion follows by taking $\gamma = \min\{\gamma_1, \gamma_2\}$. ■

The following result gives a refinement of the above result. It is based on an additional Banach space V which satisfies $Z_0 \subset V \subset X$ with continuous embeddings. It will play the role of an intermediate space in which we have already some information, namely

$$\exists C, \beta_1, \beta_2 > 0 \text{ with } \beta_1+\beta_2 > 1 \quad \forall \mathbf{z} \in Z : \|\mathcal{N}(\mathbf{z})\|_{Z_\mathcal{N}} \leq C \|\mathbf{z}\|_V^{\beta_1} \|\mathbf{z}\|_X^{\beta_2}. \tag{2.5}$$

Such estimates occur naturally by interpolation (see (3.5)).

THEOREM 2.3 *Let the system* (2.1) *satisfy* (2.3) *and* (2.5). *Further assume that there exist positive* δ, C_V, ν *such that for all* $\mathbf{z}_0 \in Z_0$ *with* $\|\mathbf{z}_0\|_{Z_0} \leq \delta$ *the unique solution* $\mathbf{z}$ *of* (2.1) *with* $\mathbf{z}(0) = \mathbf{z}_0$ *satisfies the estimate*

$$\|\mathbf{z}(t)\|_V \leq \frac{C_V}{(1+t)^\nu} \|\mathbf{z}(0)\|_{Z_0} \quad \textit{for all } t \geq 0. \tag{2.6}$$

Let $\rho = \min\{\gamma, \beta_1\nu+\beta_2\alpha, \gamma+\beta_1\nu+\beta_2\alpha-1\}$ *and assume that* $\rho \geq \alpha$, $\beta_1\nu+\beta_2\alpha \neq 1 \neq \gamma$, *then there exist positive* ε *and* C_X *such that for* $\|\mathbf{z}(0)\|_{Z_0} \leq \varepsilon$ *the solutions satisfy*

$$\left.\begin{aligned} \|\mathbf{z}(t)\|_X &\leq \frac{C_X}{(1+t)^\alpha} \|\mathbf{z}(0)\|_{Z_0} \quad \textit{and} \\ \|\mathbf{z}(t) - \mathrm{e}^{\mathcal{L}t}\mathbf{z}(0)\|_X &\leq \frac{C_X}{(1+t)^\rho} \|\mathbf{z}(0)\|_{Z_0}^{\beta_1+\beta_2} \end{aligned}\right\} \quad \textit{for all } t \geq 0. \tag{2.7}$$

Proof We give the proof in such a way that the case $\beta_1=0$ is included, which provides the proof of Theorem 2.1. Then, (2.5) reduces to (2.3c).

We use the variations-of-constants formula

$$\mathbf{z}(t) = \mathrm{e}^{\mathcal{L}t}\mathbf{z}(0) + \int_0^t \mathrm{e}^{\mathcal{L}(t-s)}\mathcal{K}\mathcal{N}(\mathbf{z}(s))\mathrm{d}s$$

and estimate the solution in the space X. Using the assumptions we obtain

$$\|\mathbf{z}(t)\|_X \le \frac{C_1}{(1+t)^\alpha}\|\mathbf{z}(0)\|_{Z_0} + \int_0^t \frac{C_2}{(1+t-s)^\gamma}\frac{C(C_V\|\mathbf{z}(0)\|_{Z_0})^{\beta_1}}{(1+s)^{\nu\beta_1}}\|\mathbf{z}(s)\|_X^{\beta_2}\,\mathrm{d}s.$$

Assuming that $\zeta=\|\mathbf{z}(0)\|_{Z_0}\le\delta$ and introducing $R(t)=\max_{s\in[0,t]}(1+s)^\alpha\|\mathbf{z}(s)\|_X$ and $\mu=\beta_1\nu+\beta_2\alpha$ we find the estimate

$$R(t)\le C_1\zeta + (1+t)^\alpha\int_0^t \frac{1}{(1+t-s)^\gamma}\frac{1}{(1+s)^\mu}\,\mathrm{d}sC_*\zeta^{\beta_1}R(t)^{\beta_2}.$$

Employing Lemma 2.2 we have derived the estimate $R(t)\le C_1\zeta + C^*\zeta^{\beta_1}R(t)^{\beta_2}$. It is now easy to find $\varepsilon>0$ such that for $\zeta\le\varepsilon$ we have $R(t)\le 2C_1\zeta$, which is the first inequality in (2.7).

Reconsidering the variations-of-constants formula once again gives

$$\|\mathbf{z}(t)-\mathrm{e}^{\mathcal{L}t}\mathbf{z}(0)\|_X \le \int_0^t \frac{1}{(1+t-s)^\gamma}\frac{1}{(1+s)^\mu}\,\mathrm{d}sC_*\zeta^{\beta_1}R(t)^{\beta_2},$$

and the second estimate in (2.7) follows by employing Lemma 2.2 and the previous estimate for $R(t)$. ■

3. Dispersive decay for generalized FPU systems

We now apply the general result presented in Section 2 to Hamiltonian systems on a one-dimensional lattice, also called the oscillator chain. Here, we only discuss a generalization of the celebrated FPU chain in detail, while in Section 4 we outline how to treat dKG systems and dNLS equations.

3.1. *The generalized FPU system*

We consider an infinite number of equal particles with a unit mass and interacting with a finite number K of neighbours via potentials $V_1,\dots,V_K$. According to Newton's law, the equations of motion are

$$\ddot{x}_j = \sum_{1\le k\le K}\big(V_k'(x_{j+k}-x_j) - V_k'(x_j-x_{j-k})\big), \quad j\in\mathbb{Z}. \tag{3.1}$$

Here $x_j\in\mathbb{R}$ denotes the displacements. We write $\mathbf{x}:=(x_j)_{j\in\mathbb{Z}}$. For the time being, we only assume that $V_k'(r)=a_kr+V_{\mathrm{nl},k}'(r)$, $V_{\mathrm{nl},k}'(r)=\mathcal{O}(|r|^\beta)_{|r|\to0}$ with $\beta>1$. System (3.1) is Hamiltonian, i.e. $(\dot{\mathbf{x}},\dot{\mathbf{p}})^T=\mathcal{J}_{\mathrm{can}}\mathrm{d}\mathcal{H}_{\mathrm{x}}(\dot{\mathbf{x}},\dot{\mathbf{p}})$ with momentum $\mathbf{p}:=\dot{\mathbf{x}}$, $\mathcal{J}_{\mathrm{can}}$ the Poisson tensor corresponding to the canonical symplectic structure defined by $\langle(\mathbf{x},\mathbf{p}),\mathcal{J}_{\mathrm{can}}(\tilde{\mathbf{x}},\tilde{\mathbf{p}})\rangle_{\ell^2\oplus\ell^2}=\langle\mathbf{x},\tilde{\mathbf{p}}\rangle_{\ell^2}-\langle\tilde{\mathbf{x}},\mathbf{p}\rangle_{\ell^2}$ and Hamiltonian

$$\mathcal{H}_{\mathrm{x}}(\mathbf{x},\mathbf{p})=\sum_{j\in\mathbb{Z}}\left(\frac{1}{2}p_j^2+\sum_{1\le k\le K}V_k(x_{j+k}-x_j)\right).$$

The dispersive decay is driven by the linearized system

$$\ddot{x}_j=\sum_{1\le k\le K}a_k\big(x_{j+k}-2x_j+x_{j-k}\big).$$

The dispersion relation is obtained by looking for plane waves in the form $x_j(t)=\mathrm{e}^{\mathrm{i}(\theta j+\hat{\omega}t)}$. We find the relation

$$\hat{\omega}^2=\Lambda(\theta):=\sum_{1\le k\le K}a_k2\big(1-\cos(k\theta)\big). \tag{3.2}$$

Obviously, we have $\Lambda(0)=0$, which is a consequence of Galilean invariance. By periodicity, it suffices to take $\theta\in[-\pi,\pi]$ and by reflection symmetry we may take $\theta\in[0,\pi]$ only. Throughout, we make the following stability condition:

$$\exists\,\Lambda>0:\quad \Lambda(\theta)\ge\lambda\theta^2\quad\text{for all }\theta\in(0,\pi], \tag{3.3}$$

which certainly holds if all a_k are positive, however more general cases are possible.

An essential feature of the considered model is its Galilean invariance, i.e for all $\xi,c\in\mathbb{R}$ the transformation $(\mathbf{x},\mathbf{p})\mapsto(x_j+\xi+ct,p_j+c)_{j\in\mathbb{Z}}$ leaves (3.1) invariant. Therefore, it is convenient to use distances $\mathbf{r}:=(\partial_+-\mathbf{1})\mathbf{x}:=(x_{j+1}-x_j)_{j\in\mathbb{Z}}$ as new variables instead of the displacements. Introducing $\mathbf{z}:=(\mathbf{r},\ \mathbf{p})^T$ the Hamiltonian turns into

$$\mathcal{H}_{\mathrm{r}}(\mathbf{z})=\frac{1}{2}\langle\mathbf{z},\mathcal{A}_{\mathrm{r}}\mathbf{z}\rangle_{\ell^2}+\mathcal{V}_{\mathrm{nl}}(\mathbf{z})$$

with

$$\langle\mathbf{z},\mathcal{A}_{\mathrm{r}}\mathbf{z}\rangle_{\ell^2}=\sum_{j\in\mathbb{Z}}\left(p_j^2+\sum_{1\le k\le K}a_k\Big|\sum_{0\le l\le k}r_{j+l}\Big|^2\right)$$

and

$$\mathcal{V}_{\mathrm{nl}}(\mathbf{z})=\sum_{j\in\mathbb{Z}}\sum_{1\le k\le K}V_{\mathrm{nl},k}\left(\sum_{0\le l\le k}r_{j+l}\right).$$

The transformed Hamiltonian system (3.1) reads as

$$\dot{\mathbf{z}}=\mathcal{J}_{\mathrm{r}}\mathrm{d}\mathcal{H}_{\mathrm{r}}(\mathbf{z})=\mathcal{L}\mathbf{z}+\mathcal{J}_{\mathrm{r}}\mathcal{N}(\mathbf{z}), \tag{3.4a}$$

where $\mathcal{L}=\mathcal{J}_{\mathrm{r}}\,\mathcal{A}_{\mathrm{r}}$ with $\mathcal{J}_{\mathrm{r}}$, $\mathcal{A}_{\mathrm{r}}$ and $\mathcal{N}$ given by

$$\mathcal{J}_{\mathrm{r}}:=\begin{pmatrix}\mathbf{0} & \partial_+-\mathbf{1}\\ \mathbf{1}-\partial_- & \mathbf{0}\end{pmatrix},\quad \mathcal{A}_{\mathrm{r}}:=\begin{pmatrix}\sum_{|l|<K}\sum_{|l|<k\le K}(k-|l|)a_k\partial_l & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{pmatrix}, \tag{3.4b}$$

$$\mathcal{N}(\mathbf{z}):=\mathrm{d}\mathcal{V}_{\mathrm{nl}}(\mathbf{z})=\begin{pmatrix}\mathbf{0}\\ \Big(\sum_{1\le k\le K}\sum_{0\le m<k}V'_{k,\mathrm{nl}}\big(\sum_{|l+m|\le k}r_{j+l}\big)\Big)_{j\in\mathbb{Z}}\end{pmatrix}, \tag{3.4c}$$

where $(\partial_l \mathbf{z})_j := z_{j+l}$ and $\partial_\pm := \partial_{\pm 1}$. Clearly, $\mathcal{L}\mathbf{z} = \mathcal{J}_{\mathrm{r}}\mathcal{A}_{\mathrm{r}}\mathbf{z}$ gives the linear forces and $\mathcal{J}_{\mathrm{r}}\mathcal{N}(\mathbf{z})$ the nonlinear interaction forces. Here the operator $\mathcal{J}_{\mathrm{r}}$ refers to the push-forward of the Poisson tensor $\mathcal{J}_{\mathrm{can}}$, i.e. $\mathcal{J}_{\mathrm{r}} = \mathcal{T}\ \mathcal{J}_{\mathrm{can}}\ \mathcal{T}^*$ where $\mathcal{T}$ is the linear map defined by $(\mathbf{r}, \mathbf{p})^T = \mathcal{T}(\mathbf{x}, \mathbf{p})^T$. Note that now $\mathcal{J}_{\mathrm{r}}$ is a non-canonical Poisson structure.

3.2. *Nonlinear dispersive stability*

To study the nonlinear system we use the Banach spaces

$$\ell^p(\mathbb{Z}^d; \mathbb{R}^m) \quad \text{with norm } \|\mathbf{z}\|_{\ell^p} := \Big(\sum\nolimits_{J \in \mathbb{Z}^d} |z_J|^p\Big)^{1/p},$$

where $p \in [1, \infty]$. We frequently write ℓ^p to denote $\ell^p(\mathbb{Z}^d; \mathbb{R}^m)$, if the lattice $\mathbb{Z}^d$ and the space $\mathbb{R}^m$ are either irrelevant or clear from the context.

For $1 \leq p_1 < p_2 \leq \infty$ we have the continuous embedding $\ell^{p_1} \subset \ell^{p_2}$ with $\|\mathbf{z}\|_{\ell^{p_2}} \leq \|\mathbf{z}\|_{\ell^{p_1}}$. An essential tool is the interpolation estimate

$$\|\mathbf{z}\|_{\ell^{p_\vartheta}} \leq \|\mathbf{z}\|_{\ell^{p_0}}^{1-\vartheta} \|\mathbf{z}\|_{\ell^{p_1}}^{\vartheta}, \quad \text{where } \frac{1}{p_\vartheta} = \frac{1-\vartheta}{p_0} + \frac{\vartheta}{p_1}, \tag{3.5}$$

$p_0,\ p_1 \in [1, \infty]$ and $\vartheta \in [0, 1]$. This is an easy consequence of Hölder's inequality and plays a crucial role in many estimates concerning dispersive decay. Moreover, we use Young's inequality for convolutions $a * b$ with $(a * b)_J = \sum_{I \in \mathbb{Z}^d} a_{J-I} b_I$. For $r, p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ we have

$$\|a * b\|_{\ell^r} \leq \|a\|_{\ell^p} \|b\|_{\ell^q} \quad \text{for all } a \in \ell^p,\ b \in \ell^q. \tag{3.6}$$

To apply the general result of Section 2 we first provide the *a priori* estimate (2.2). The theory in Section 3.3 shows that (3.3) is equivalent to the existence of a constant $C \geq 1$ such that

$$\frac{1}{C}\|\mathbf{z}\|_{\ell^2}^2 \leq \langle \mathbf{z}, \mathcal{A}_{\mathrm{r}}\mathbf{z}\rangle_{\ell^2} \leq C\|\mathbf{z}\|_{\ell^2}^2 \quad \text{for all } \mathbf{z} \in \ell^2(\mathbb{Z}; \mathbb{R}^2).$$

Using this it is easy to obtain the classical energy stability in $\ell^2(\mathbb{Z}; \mathbb{R}^2)$: there are $C_2 > 0$ and $\varepsilon_0 > 0$ such that for all $\mathbf{z}_0 \in \ell^2$ with $\|\mathbf{z}_0\|_{\ell^2} \leq \varepsilon_0$ the solution $\mathbf{z}$ of (3.4) with $\mathbf{z}(0) = \mathbf{z}_0$ exists globally in time and satisfies

$$\|\mathbf{z}(t)\|_{\ell^2} \leq C\|\mathbf{z}(0)\|_{\ell^2} \quad \text{for all } t \in \mathbb{R}. \tag{3.7}$$

To state the linear decay result, we define the relevant branch $\hat{\omega} = \omega(\theta)$ of the dispersion relation via

$$\omega(\theta) := \sqrt{\Lambda(\theta)} \geq 0.$$

With a slight abuse of notation we simply call ω the dispersion relation. Under the stability assumptions (3.3) we have $\omega \in \mathrm{C}^\infty([0, \pi])$ and we are able to define the set of critical wave numbers as

$$\Theta_{\mathrm{cr}} := \big\{\theta \in [0, \pi] \mid \omega''(\theta) = 0\big\}.$$

Since K in (3.2) is finite, $\Theta_{\rm cr}$ is discrete and contains $\theta=0$. Thus, we have $\Theta_{\rm cr}=\{\theta_0,\dots,\theta_M\}$ with $\theta_0=0<\theta_1<\dots<\theta_M\le\pi$ for some $M\in\mathbb{N}_0$.

The following linear decay results will be proved in Section 3.3.

THEOREM 3.1 *Consider the group* $(\mathrm{e}^{\mathcal{L}t})_{t\in\mathbb{R}}$ *for* $\mathcal{L}=\mathcal{J}_{\rm r}\mathcal{A}_{\rm r}$ *defined in* (3.4b)*. Assume that the dispersion relation* ω *satisfies* (3.3) *and the non-degeneracy condition*

$$\omega'(0)>0 \quad \textit{and} \quad \forall\theta\in\Theta_{\rm cr}:\omega'''(\theta)\neq 0. \tag{3.8}$$

Then, for $p\in[2,4)\cup(4,\infty]$ *there exists* C_p *such that, for all* $t\ge 0$, *we have*

$$\|\mathrm{e}^{\mathcal{L}t}\|_{\ell^1,\ell^p}\le\frac{C_p}{(1+t)^{\alpha_p}},\quad \textit{where}\quad \alpha_p=\begin{cases}\frac{p-2}{2p} & \textit{for } p\in[2,4),\\ \frac{p-1}{3p} & \textit{for } p\in(4,\infty].\end{cases} \tag{3.9a}$$

In the case $p=4$, *there exists* $C_4>0$ *such that*

$$\|\mathrm{e}^{\mathcal{L}t}\|_{\ell^1,\ell^4}\le C_4\Big(\frac{\log(2+t)}{1+t}\Big)^{1/4}\quad \textit{for all } t>0. \tag{3.9b}$$

If furthermore $\Theta_{\rm cr}=\{0\}$, *then for* $p\in[2,\infty]$ *there exists* $\tilde{C}_p$ *such that*

$$\|\mathrm{e}^{\mathcal{L}t}\mathcal{J}_{\rm r}\|_{\ell^1,\ell^p}\le\frac{\tilde{C}_p}{(1+t)^{\tilde{\alpha_p}}}\quad \textit{for all } t>0,\quad \textit{where } \tilde{\alpha_p}=\frac{p-2}{2p}. \tag{3.10}$$

The philosophy of the decay estimate is that the oscillations with wave numbers θ travel along rays $j=c(\theta)t$, where the group velocity is given by $c(\theta)=\omega'(\theta)$. The decay along these rays is like $t^{-1/2}$ if $\omega''(\theta)\neq 0$ and like $t^{-1/3}$ if $\theta\in\Theta_{\rm cr}$. In Figure 2, we plot the dispersion relations ω and the associated solution $r_j(t)$ to display the influence of the critical wave numbers $\theta_j\in\Theta_{\rm cr}$. Thus, the decay like $t^{-1/3}$ in ℓ^∞ is easily obtained. However, for $\theta\approx\theta_n\in\Theta_{\rm cr}$ there is a cross-over between the two different decay rates, which needs to be estimated carefully to obtain the decay rate α_p for $p\in(2,\infty)$.

Because the operator $\mathcal{J}_{\rm r}$ is related to the difference operators ∂_+-1 and $1-\partial_-$, it reduces the amplitudes of very long waves. Thus, in $\mathrm{e}^{\mathcal{L}t}\mathcal{J}_{\rm r}$ the bad decay associated

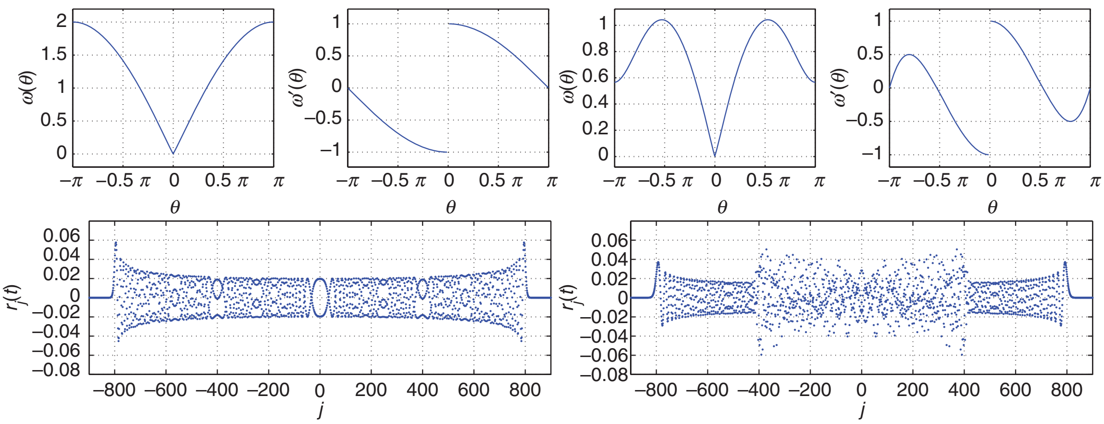

Figure 2. Dispersion relations and time evolutions. Left: classical FPU ($K=1$: $a_1=-1$). Right: generalized FPU ($K=2$: $a_1=0.08$, $a_2=0.23$) with two wave fronts. The upper figures show $\omega(\theta)$ and $\omega'(\theta)$, respectively, and the lower figure shows $r_j(t)$ for $t=800$ and initial condition $(r_j(0),\dot{x}_j(0))=(\delta_{j,0},0)$.

with $\theta_0 = 0 \in \Theta_{\rm cr}$ is reduced but not for any other $\theta_n \in \Theta_{\rm cr}$. Hence, the last statement needs the requirement $\Theta_{\rm cr} = \{0\}$. In this connection, it is interesting to mention that in the case $\Theta_{\rm cr} = \{0\}$ the solutions of $\dot{\mathbf{z}} = \mathcal{L}\mathbf{z}$ globally decay like $t^{-1/2}$ if we restrict the initial conditions to a suitable subspace. Indeed, if we choose $\mathbf{z}^0 \in \mathcal{J}_{\rm r}\ell^1$, this follows from (3.10) and the fact that the operators $\mathcal{J}_{\rm r}$ and $e^{\mathcal{L}t}$ commute.

The following decay result is a direct combination of the abstract results of Section 2 and the above linear decay estimates.

THEOREM 3.2 *Consider the generalized FPU system satisfying the linearized stability condition* (3.3) *and the non-degeneracy condition* (3.8). *Assume that each potential V_k satisfies $V_k'(r) = a_k r + \mathcal{O}(|r|^\beta)$ for $\beta > 4$. Then, for each $p \in [2, 4) \cup (4, \infty]$ there exist C_p and $\omega > 0$ such that all solutions $\mathbf{z}$ of* (3.4) *with $\|\mathbf{z}(0)\|_{\ell^1} \le \varepsilon$ satisfy the estimate*

$$\|\mathbf{z}(t)\|_{\ell^p} \le \frac{C_p}{(1+t)^{\alpha_p}} \|\mathbf{z}(0)\|_{\ell^1} \quad \text{for all } t \ge 0, \tag{3.11}$$

where the decay rate α_p is given in (3.9). *If additionally $\Theta_{\rm cr} = \{0\}$, then*

$$\|\mathbf{z}(t) - e^{\mathcal{L}t}\mathbf{z}(0)\|_{\ell^p} \le \frac{\tilde{C}_p}{(1+t)^{\tilde{\alpha}_p}} \|\mathbf{z}(0)\|_{\ell^1}^\beta \quad \text{for all } t \ge 0, \tag{3.12}$$

where the decay rate $\tilde{\alpha}_p$ is given in (3.10).

We have omitted the case $p = 4$ to avoid a clumsy presentation. For $p = 4$ one can easily obtain an algebraic decay for any $\alpha < 1/4$ by interpolation or a decay as in (3.9b), after generalizing the results in Section 2 to include logarithmic terms.

Proof In a first step we apply Theorem 2.1 with $Z_0 = Z_{\mathcal{N}} = \ell^1 \subset X = \ell^{p_1}$ with $4 > p_1 > 2\beta/(\beta-2)$, where we used $\beta > 4$. Because of $\beta > p_1$ we have $\|\mathcal{N}(\mathbf{z})\|_{\ell^1} \le C\|\mathbf{z}\|_{\ell^{p_1}}^\beta$. We estimate $\mathcal{K} = \mathcal{J}_{\rm r}$ by a constant and use $\alpha = \gamma = \alpha_{p_1}$. The choice of p_1 gives $\alpha < 1 < \alpha\beta$ and $\min\{\gamma, \alpha\beta, \alpha\beta + \gamma - 1\} = \alpha$, which allows us to apply the theorem. We obtain positive C_{p_1} and ε_0 such that (3.11) holds for $p = p_1$. Since the result holds for $p = 2$ by the nonlinear stability estimate (3.7), the interpolation (3.5) shows that the result holds for $p \in [2, p_1]$. Since p_1 can be chosen as close to $p = 4$ as we like, estimate (3.11) is established for all $p \in [2, 4)$.

Next we consider $p \in (4, \beta]$ and see that Theorem 2.3 is applicable with $\nu = \beta_1 = 0$, $\alpha = \alpha_p < \gamma = \tilde{\alpha}_p$ and $Z_0 = Z_{\mathcal{N}} = \ell^1 \subset X = \ell^p$. Thus, (3.11) and (3.12) hold for $p \in (4, \beta]$.

Finally, we treat the case $p = \infty$ by choosing $p_2 \in (2, 4)$ with $p_2 \ge 12 - 2\beta < 4$. Using $\|\mathcal{N}(\mathbf{z})\|_{\ell^1} \le C\|\mathbf{z}\|_{\ell^{p_2}}^{p_2} \|\mathbf{z}\|_{\ell^\infty}^{\beta - p_2}$ we are able to employ Theorem 2.3 with $Z_0 = Z_{\mathcal{N}} = \ell^1 \subset V = \ell^{p_2} \subset X = \ell^\infty$ and $\nu = \alpha_{p_2} < \alpha = 1/3 \le \gamma$, where $\gamma = 1/3$ in the general case and $\gamma = 1/2$ if the additional condition $\Theta_{\rm cr} = \{0\}$ holds (Theorem 3.1). Using $\beta_1 = p_2$ and $\beta_2 = \beta - p_2$ we find $\nu\beta_1 + \alpha\beta_2 > 1$. Hence $\rho = \gamma \ge \alpha$ and the desired estimate (3.11) follows for $p = \infty$. Again, the remaining range $p \in [\beta, \infty]$ follows from interpolation. If the additional condition $\Theta_{\rm cr} = \{0\}$ holds, we can apply the last assertion in Theorem 2.3 and obtain (3.12). ■

So far, we have only derived estimates for $Z_{\mathcal{N}} = \ell^1$. It is however straightforward to obtain results for $Z_{\mathcal{N}} = \ell^q$ for $q \in (1, 2)$, however the decay rates will be lower and one may need higher order of nonlinearity β. To see this, we simply note that the application of the operator $e^{\mathcal{L}t}$ is in fact a convolution with a matrix-valued Green's function $\mathbf{G}(t) \in \ell^1(\mathbb{Z}; \mathbb{R}^{2\times 2})$ (cf. (3.20)). Hence, using Young's inequality (3.6), the operator norm $\|e^{\mathcal{L}t}\|_{\ell^q, \ell^p}$ can be estimated by $\|\mathbf{G}\|_{\ell^s}$ where $1+\frac{1}{p}=\frac{1}{s}+\frac{1}{q}$. In fact, the estimates stated above and proved below are obtained by estimating the ℓ^p norm of $\mathbf{G}(t)$.

We emphasize that for $q=1$ the formula $\|e^{\mathcal{L}t}\|_{\ell^1, \mathcal{L}^p} = \|\mathbf{G}(t)\|_{\ell^p}$ holds, since the upper bound follows from Young's inequality and the lower bound is obtained by using the initial condition $\mathbf{z}=(\delta_j)_{j\in\mathbb{Z}}$. Our estimates for $\mathbf{G}_j(t)$ will also be sharp enough to establish lower bounds $\|\mathbf{G}(t)\|_{\ell^p} \geq c/(1+t)^{\alpha_p}$. Thus, we cannot hope for better estimates for the linear terms. In fact, using that the decay rates $\alpha_2=0$ and $\alpha_\infty=1/3$ are optimal, it suffices to show that the decay rate α_4 cannot be better than $1/4$ (up to the logarithmic term). Then, for no $p\in(2,\infty)$ the decay rate can be better than α_p, because an interpolation would lead to a better decay rate for $p=4$. Below we will show that estimate (3.9b) is indeed optimal.

Figure 3 displays the numerically estimated decay rates, the exact curve α_p and the curve $\hat{\alpha}_p = (p-2)/(3p)$, which is obtained by interpolation between $p=2$ and $p=\infty$ and hence is not optimal. The numerical curves agree well with α_p away from $p=4$. This effect may be due to the logarithmic correction which spoils the convergence.

In the following remark, we argue that the above-mentioned dispersive decay cannot hold for $\beta<3$, because of the existence of solitary waves with an arbitrary small ℓ^1 norm.

Remark 3.3 (Solitary waves) From [12,13] the existence of solitary waves for generalized FPU systems can be deduced under additional global conditions on the interaction potentials V_k. Such waves satisfy $z_j(t)=Z(j-ct)$ for a fixed profile $Z:\mathbb{R}\to\mathbb{R}^2$ and a given wave speed c. In particular, [13] provides the existence of solitary waves for the case $1<\beta<5$ with an arbitrarily small energy, i.e. $\|\mathbf{z}^\delta_{\text{soli}}\|_{\ell^2} = \delta\in(0,\delta_0)$. Our stability result implies that for $\beta>4$ these solution cannot be small in ℓ^1.

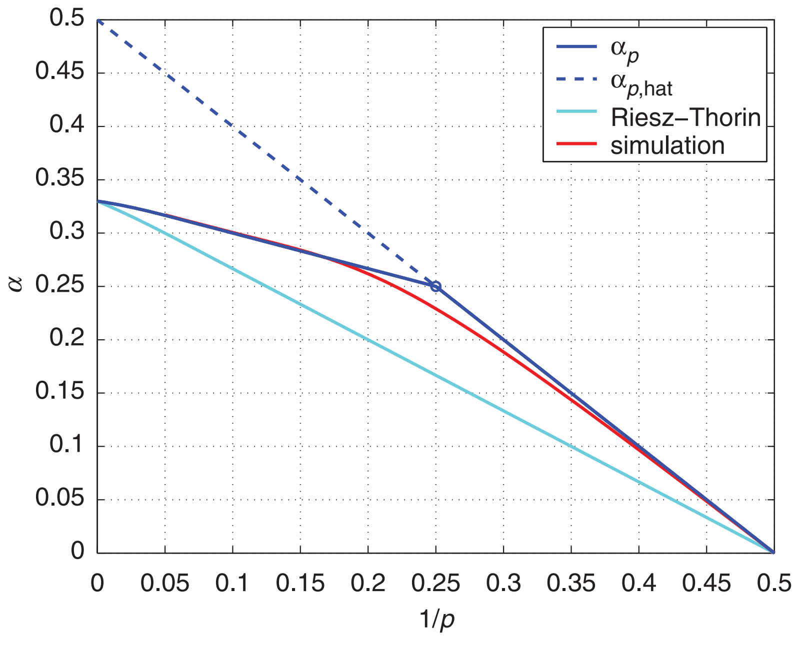

Figure 3. Exact decay rate α_p, interpolation rate $\hat{\alpha}_p$, ℓ^2–ℓ^∞ interpolation rate and numerically estimated rates as functions of $1/p$.

In [12] the case $\beta = 2$ is investigated, and it is shown that $c = \omega'(0) + \mathcal{O}(\varepsilon^2)$. The constructions there can be generalized to our case to provide small-energy solitary waves associated with the generalized KdV limit. Moreover, in [16] it was shown that the solutions of the form $r_j^\varepsilon(t) = \varepsilon^{2/(\beta-1)} R(\varepsilon^3 t, \varepsilon(j+\omega'(0)t)) +$ higher order terms exist, with $R : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfying the generalized KdV equation

$$\partial_\tau R + b_1 \partial_\eta^3 R + b_2 \partial_\eta V'(R) = 0,$$

where $V'(r) = r + \mathcal{O}(|r|^\beta)$. This equation possesses solitary wave solutions with exponentially decaying tales. In terms of the generalized FPU system, these solutions satisfy $\|\mathbf{z}^\varepsilon_{\text{soli}}(t)\|_{\ell^1} \sim \varepsilon^{(3-\beta)/(\beta-1)}$, which shows that for $1 < \beta < 3$ there are solitary waves that are arbitrarily small in ℓ^1. We conclude that the above-mentioned dispersive decay result cannot hold for $\beta < 3$, while the case $\beta \in [3, 4]$ remains open.

3.3. ℓ^p-estimates for the linearized system

We consider the linearization of (3.4) in $\mathbf{z} = 0$, i.e. the case $\mathcal{N}(\mathbf{z}) \equiv 0$. To solve the system explicitly, we use, the Fourier transform $\mathcal{F} : \ell^2(\mathbb{Z}, \mathbb{R}^2) \to L^2(\mathcal{S}^1, \mathbb{R}^2)$ defined by $\hat{z}(\theta) = \sum_{j \in \mathbb{Z}} z_j \mathrm{e}^{-\mathrm{i}j\theta}$. Then $\dot{\mathbf{z}} = \mathcal{J}_\mathrm{r} \mathcal{A}_\mathrm{r} \mathbf{z}$ turns into

$$\begin{pmatrix} \dot{\hat{\mathbf{r}}} \\ \dot{\hat{\mathbf{p}}} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\theta} - 1 \\ 1 - \mathrm{e}^{-\mathrm{i}\theta} & 0 \end{pmatrix} \begin{pmatrix} \omega_\mathrm{r}^2(\theta) & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{p}} \end{pmatrix}, \tag{3.13}$$

where

$$\begin{aligned} \omega_\mathrm{r}^2(\theta) &= \sum_{|l| \le K-1} \sum_{|l| < k \le K} (k - |l|) a_k \mathrm{e}^{\mathrm{i}l\cdot\theta} \\ &= \sum_{0 < k \le K} k a_k + 2 \sum_{0 < l \le K-1} \left(\sum_{l < k \le K} (k - l) a_k \right) \cos(l \cdot \theta). \end{aligned} \tag{3.14}$$

Since (3.13) implies that $\ddot{\hat{\mathbf{r}}} = 2(\cos\theta - 1)\omega_r(\theta)^2 \hat{\mathbf{r}}$ and since the previous subsection gives $\ddot{\hat{\mathbf{r}}} = -\Lambda(\theta)\hat{\mathbf{r}}$, we conclude that

$$\omega(\theta) = 2\left|\sin\frac{\theta}{2}\right| \omega_\mathrm{r}(\theta). \tag{3.15}$$

Using the linear stability condition (3.3), we obtain

$$\exists c_\mathrm{r} > 0 \ \forall \theta \in \mathcal{S}^1 : \omega_\mathrm{r}(\theta) = \omega_\mathrm{r}(-\theta) \ge c_\mathrm{r}. \tag{3.16}$$

The fundamental matrix of the linear system (3.13) is

$$\hat{\mathbf{G}}_\mathrm{r}(\theta, t) = \begin{pmatrix} \cos(\omega(\theta)t) & \frac{\mathrm{e}^{\mathrm{i}\theta} - 1}{\omega(\theta)} \sin(\omega(\theta)t) \\ \frac{-\omega(\theta)}{\mathrm{e}^{-\mathrm{i}\theta} - 1} \sin(\omega(\theta)t) & \cos(\omega(\theta)t) \end{pmatrix}. \tag{3.17}$$

The Green's function of our original problem is given by $\mathbf{G}(t) = \mathcal{F}^{-1}\, \hat{\mathbf{G}}_\mathrm{r}(\theta, t)$, that is $G_j(t) = \frac{1}{2\pi} \int_{\mathcal{S}^1} \hat{G}_\mathrm{r}(\theta, t) \mathrm{e}^{\mathrm{i}j\theta}\, \mathrm{d}\theta$ for $j \in \mathbb{Z}$. Thus the long-time behaviour of solutions to the linearized system is determined by oscillatory integrals. For instance, for the classical FPU, i.e. for $\omega_\mathrm{r}(\theta) \equiv 1$, the components of G_j turn into Bessel functions [6]. Below we apply tools from asymptotic analysis to obtain upper bounds on the solutions of the linearized system. To do so, it turns out that an alternative representation of the above Green's function is more convenient. Using the symmetry of ω_{r} we find that

$$G_j(t) = \frac{1}{2\pi}\int_0^{\pi}\begin{pmatrix} h(\theta, t, \frac{j}{t}) & \frac{1}{\omega_{\mathrm{r}}(\theta)}h\big(\theta, t, \frac{j+1/2}{t}\big) \\ \omega_{\mathrm{r}}(\theta)h\big(\theta, t, \frac{j-1/2}{t}\big) & h(\theta, t, \frac{j}{t}) \end{pmatrix}\mathrm{d}\theta$$
$$\text{with } h(\theta, t, c) = \cos\big(t(\omega(\theta)+\theta c)\big) + \cos\big(t(\omega(\theta)-\theta c)\big). \tag{3.18}$$

The new variable $c \in \mathbb{R}$ roughly characterizes the rays $j = ct$ and is used to remind us to the group velocity $c(\theta) = \omega'(\theta)$.

Thus, we obtained the following representation formula for the solution of the linearized problem.

LEMMA 3.4 (Explicit solution) *Given some initial conditions* $\mathbf{z}^0 = (\mathbf{r}^0, \mathbf{p}^0)^T \in \ell^2(\mathbb{Z}, \mathbb{R}^2)$, *the unique solution of* $\dot{\mathbf{z}} = \mathcal{L}\mathbf{z}$ *with* $\mathcal{L} = \mathcal{J}_{\mathrm{r}}\mathcal{L}_{\mathrm{r}}$ *defined in* (3.4b) *is determined by*

$$\mathbf{z}(t) = \mathrm{e}^{\mathcal{L}t}\mathbf{z}^0, \tag{3.19}$$

where $(\mathrm{e}^{\mathcal{L}t})_{t\in\mathbb{R}}$ *is a differentiable group of bounded operators on* $\ell^2(\mathbb{Z}, \mathbb{R}^2)$ *defined by*

$$\big(\mathrm{e}^{\mathcal{L}t}\mathbf{z}\big)_j = \sum_{k\in\mathbb{Z}} G_k(t)\cdot z_{j-k} \quad \text{for } j \in \mathbb{Z} \tag{3.20}$$

with $G_j(t)$ *defined in* (3.18).

The asymptotic behaviour of (3.18) is determined by the terms of the form

$$g(t, c) = \int_0^{\pi} \psi(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\mathrm{d}\theta \quad \text{with } \phi(\theta, c) = \pm(\omega(\theta) - c\theta) \tag{3.21}$$

with ω defined in (3.15) and $\psi(\theta)$ standing for 1, $1/\omega_{\mathrm{r}}(\theta)$ or $\omega_{\mathrm{r}}(\theta)$. In any case ψ is smooth on $[0, \pi]$.

The main result from the asymptotic analysis which we will use below is van der Corput's lemma (see, e.g. [17]). It states that if ϕ is smooth and $|\phi^{(k)}(\theta)| \geq \lambda > 0$ for $\theta \in (a, b)$ where either $k \geq 2$ or $k = 1$ and ϕ' is monotonic, then

$$\left|\int_a^b \mathrm{e}^{\mathrm{i}t\phi(\theta)}\mathrm{d}\theta\right| \leq C_k(\lambda t)^{-\frac{1}{k}} \quad \text{with} \quad C_k = (5\times 2^{k-1} - 2). \tag{3.22}$$

Note that C_k does neither depend on a and b nor on ϕ explicitly. Writing $F(\theta) = \int_a^{\theta} \mathrm{e}^{\mathrm{i}t\phi(\xi)}\,\mathrm{d}\xi$ and applying integration by parts to $\int_a^b \psi(\theta)F'(\theta)\mathrm{d}\theta$, we obtain

$$\left|\int_a^b \mathrm{e}^{\mathrm{i}t\phi(z)}\psi(z)\mathrm{d}z\right| \leq C_k(\lambda t)^{-\frac{1}{k}}\left(|\psi(b)| + \int_a^b |\psi'(\theta)|\mathrm{d}\theta\right). \tag{3.23}$$

In the following lemmas we provide the decay estimates on $g(t, c)$ required to prove the sharp ℓ^p decay rate of the linear group $\mathrm{e}^{\mathcal{L}t}$. We use the notation

$$C_{\psi} := \max_{\theta\in[0,\pi]} |\psi(\theta)| + \int_0^{\pi} |\psi'(\theta)|\mathrm{d}\theta.$$

Since van der Corput's Lemma only demands assumptions on $|\phi^{(k)}(\theta)|$, the following considerations are indeed independent of the sign of ϕ in (3.21).

The first lemma provides a global upper bound on $g(t,c)$. Using the classical method of stationary phase [18], it is straightforward to check that the result is sharp.

LEMMA 3.5 (Global bound) *Consider the oscillatory integral* (3.21) *with the dispersion relation* ω *satisfying* (3.8) *and* $\psi \in W^{1,1}([0,\pi])$. *Then there exists a constant* $C_\omega > 0$ *depending only on* ω *such that*

$$\forall t \geq 0,\ \ c \in \mathbb{R} : |\, g(t,c)| \leq \frac{C_\omega C_\psi}{(1+t)^{1/3}}. \tag{3.24}$$

Proof Because $\phi(\theta,c) = \omega''(\theta)$, the following considerations are uniform with respect to the group velocity c.

We write $U_\delta(\theta_m) = \{\theta \in [0,\pi] \mid |\theta - \theta_m| < \delta\}$. Due to the non-degeneracy condition (3.8), it is possible to choose $\delta > 0$ such small that $|\omega'''(\theta)| \geq A$ for all $\theta \in \bigcup_{m=0}^M U_\delta(\theta_m)$ for some constant $A > 0$. Since $\omega''(\theta) = 0$ if and only if $\theta \in \Theta_{\text{cr}}$ there exists $B > 0$ with $|\omega''(\theta)| \geq B$ for all $\theta \in [0,\pi] \setminus \bigcup_{m=0}^M U_\delta(\theta_m)$. Now we write

$$g(t,c) = \int_{\bigcup_{m=0}^M U_\delta(\theta_m)} \psi(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta,c)} \mathrm{d}\theta + \int_{[0,\pi]\setminus \bigcup_{m=0}^M U_\delta(\theta_m)} \psi(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta,c)} \mathrm{d}\theta$$

and apply (3.23). Thus

$$|\, g(t,c)| \leq (M+1)\big(18A^{-1/3} + 8B^{-1/2}\big) C_\psi t^{-1/3}$$

holds for $t > 1$. Using $|g(t,c)| \leq \pi \ \max_{\theta \in [0,\pi]} \ |\psi(\theta)|$ in case $0 \leq t \leq 1$ proves the conclusion for $C_\omega = 2\max\{\pi, (M+1)(18A^{-1/3} + 8B^{-1/2})\}$. ■

The next result provides the decay rate $t^{-1/2}$ along noncritical rays. The importance is to characterize the width of the regions around the critical rays with a decay rate $t^{-1/3}$ that has to be excluded. This result provides sharp estimates for cross-over between the two decay rates. Excluding group velocities near the critical ones corresponding to the critical wave numbers, i.e. allowing only for c with $|c| \notin \bigcup_{\theta_n \in \Theta_{\text{cr}}} [|\omega'(\theta_n)| - \varepsilon, |\omega'(\theta_n)| + \varepsilon]$ where $\varepsilon > 0$, (3.25) implies a uniform bound $\sim t^{-1/2}$ on $g(t,c)$. In fact, the result shows that the excluded regions may be taken smaller, namely of width growing like $t^{2/3}$. Using a suitable Airy scaling, it can be shown that this width cannot be decreased (see (3.32) for more details).

LEMMA 3.6 (Envelope function) *Consider the oscillatory integral* (3.21) *with dispersion relation* ω *satisfying* (3.8) *and* $\psi \in W^{1,1}([0,\pi])$. *Then, there exists a constant* $C_\omega > 0$ *depending only on* ω *such that for all* $t > 0$ *and all* $c \in \mathbb{R} \setminus \{c \mid \exists \theta \in \Theta_{\text{cr}} : \| \omega'(\theta)| - |c\| \leq t^{-2/3}\}$ *holds*

$$\big|\, g(t,c)\big| \leq \frac{C_\omega C_\psi}{(1+t)^{1/2}} \left(1 + \sum_{\theta \in \Theta_{\text{cr}}} \frac{1}{|\omega'(\theta)^2 - c^2|^{1/4}}\right). \tag{3.25}$$

Proof For $0 < t \leq 1$ we use $|g(t,\, c)| \leq C_\psi \pi$. Below we assume that $t > 1$.

To simplify the considerations let us first assume that there is only one critical wave number $\theta_0 = 0$. For θ near 0 the phase function of (3.21) behaves like $\phi(\theta,c) = \pm(c_0 - c)\theta \pm \frac{\omega'''(0)}{6}\theta^3 + \mathcal{O}(\theta^5)$ with $c_0 = \omega'(0)$. Now we write

$$g(t,c) = \int_0^{\tilde\delta} \psi(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta,c)} \mathrm{d}\theta + \int_{\tilde\delta}^{\delta} \psi(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta,c)} \mathrm{d}\theta + \int_\delta^\pi \psi(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta,c)} \mathrm{d}\theta. \tag{3.26}$$

Because $\omega'''(0)\neq 0$, there exists $0<\delta<1$ and constants $\underline{A}$, $\bar{A}>0$ such that

$$\forall\theta\in(0,\delta): |\omega''(\theta)|\geq \underline{A}\theta \quad \text{and} \quad |\omega'(\theta)-c_0|\leq \bar{A}\theta^2. \tag{3.27}$$

Then we have, in particular, $|\partial_\theta^2\phi(\theta,c)|=|\omega''(\theta)|\geq \underline{A}\tilde{\delta}$ for all $\theta\in(\tilde{\delta},\delta)$. Since we assumed that $\Theta_{\text{cr}}=\{0\}$ there also exists $B>0$ such that we have $|\partial_\theta^2\phi(\theta,c)|\geq B$. Thus van der Corput's Lemma (3.23) implies that

$$\left|\int_{\tilde{\delta}}^{\delta}\psi(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\mathrm{d}\theta\right|\leq\frac{8C_\psi}{(\underline{A}\tilde{\delta}t)^{1/2}} \quad \text{and} \quad \left|\int_{\delta}^{\pi}\psi(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\mathrm{d}\theta\right|\leq\frac{8C_\psi}{(Bt)^{1/2}}. \tag{3.28}$$

Here δ, $\underline{A}$, $\bar{A}$ and B do not depend on c but only on ω.

If $\tilde{\delta}$ with $0<\tilde{\delta}\leq\delta$ is so small that $\tilde{\delta}^2\leq\frac{1}{\bar{A}+1}|c_0-|c||$, then using (3.27) we obtain $|\partial_\theta\phi(\theta,c)|=|\omega'(\theta)-c|\geq|c_0-|c||-\bar{A}\theta^2\geq\tilde{\delta}^2$ for all $\theta\in(0,\tilde{\delta})$. Hence, again according to (3.23) we obtain

$$\left|\int_{0}^{\tilde{\delta}}\psi(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\mathrm{d}\theta\right|\leq\frac{3C_\psi}{\tilde{\delta}^2 t}. \tag{3.29}$$

Now we distinguish two cases. If $\delta^2\leq\frac{1}{\bar{A}+1}|c_0-|c||$, we choose $\tilde{\delta}:=\delta$. Hence the right-hand side of (3.29) is independent of c. Substituting this bound together with the second estimate in (3.28) and (3.26) gives

$$|g(t,c)|\leq\frac{8C_\psi}{(Bt)^{1/2}}+\frac{3C_\psi}{\delta^2 t}\leq\frac{C_\psi}{t^{1/2}}\left(\frac{8}{\sqrt{B}}+\frac{3}{\delta^2}\right). \tag{3.30}$$

In the case $\frac{1}{\bar{A}+1}|c_0-|c||<\delta^2$ we choose $\tilde{\delta}^2:=\frac{1}{\bar{A}+1}|c_0-|c||$. Then the assumption $|c_0-|c||\geq t^{-2/3}$ yields $\tilde{\delta}^{3/2}t^{1/2}\geq(\bar{A}+1)^{-3/4}$. Thus, combining the upper bound (3.29) with the first estimate in (3.28) leads to

$$\left|\int_{0}^{\delta}\psi(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\mathrm{d}\theta\right|\leq\frac{8C_\psi}{(\underline{\mathrm{A}}\tilde{\delta}t)^{1/2}}+\frac{3C_\psi}{\tilde{\delta}^2 t}\leq\frac{C_\psi}{|c_0-|c||^{1/4}t^{1/2}}\left(\frac{8}{\underline{\mathrm{A}}^{1/2}}+3(\bar{A}+1)^{3/4}\right).$$

Finally, using this together with the second estimate in (3.28) and $|g(t,c)|\leq C_\psi\pi$ for $0<t\leq 1$ yields

$$|g(t,c)|\leq\frac{C_\omega C_\psi}{(1+t)^{1/2}}\left(1+\frac{1}{|c_0^2-c^2|^{1/4}}\right)$$

with $C_\omega>0$ depending only on $\omega(\theta)$. The last estimate also covers (3.30) if we choose C_ω sufficiently large. This completes the proof for $\Theta_{\text{cr}}=\{0\}$.

To prove the general case, assume that we have $\Theta_{\text{cr}}=\{\theta_0,\theta_1,\ldots,\theta_M\}$ with $\theta_0<\theta_1<\cdots<\theta_M$. We decompose the integral defining $g(c,t)$ like

$$g(c,t)=\cdots+\int_{\theta_m}^{\theta_m+\tilde{\delta}_m}\cdots+\int_{\theta_m+\tilde{\delta}_m}^{\theta_m+\delta_m}\cdots+\int_{\theta_m+\delta_m}^{\theta_{m+1}+\delta_{m+1}}\cdots+\cdots$$

with δ_m and δ_{m+1} sufficiently small such that $\omega'''(\theta)\neq 0$ for $\theta\in(\theta_m-\delta_m,\theta_m+\delta_m)\cup(\theta_{m+1}-\delta_{m+1},\theta_{m+1}+\delta_{m+1})$. Then similar estimates like (3.27) hold and we use the same arguments as above to get the upper bound

$$\Big|\int_{\theta_m}^{\theta_{m+1}-\delta_{m+1}} \psi(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\mathrm{d}\theta\Big| \leq \frac{C_{\omega,m}C_\psi}{(1+t)^{1/2}}\bigg(1+\frac{1}{|c_m^2-c^2|^{1/4}}\bigg).$$

Since Θ_{cr} is finite, this implies the statement. ∎

Now we state a result that provides a global decay rate $t^{-1/2}$ under the additional assumption that only $\theta=0$ is a critical wave number and that the function ψ satisfies $\psi(0)=0$. It will be used to estimate $\mathrm{e}^{\mathcal{L}t}\mathcal{J}_{\mathrm{r}}$, where the bad behaviour of the fronts, which relate to long waves (i.e. $\theta=0$) are filtered out by the difference operators $\partial_\pm - 1$ in $\mathcal{J}_{\mathrm{r}}$.

Lemma 3.7 *Consider the oscillatory integral* (3.21) *with dispersion relation* ω *satisfying* (3.8) *and* $\psi\in W^{1,1}([0,\pi])$. *If additionally* $\Theta_{\mathrm{cr}}=\{0\}$ *and* $\psi(0)=0$, *then there exists a constant* $C_\omega>0$ *depending only on* $\omega(\theta)$ *such that*

$$\forall t\geq 0: \big|g(t,c)\big| \leq \frac{C_\omega C_\psi}{(1+t)^{1/2}}. \tag{3.31}$$

The proof relies on an uniform asymptotic expansion of the oscillatory integrals. Since we think that the technical details would dislocate the focus of this article, we forbear to give the full proof but only highlight the main idea. The details can be found in [19].

To see the filter effect of the difference operators $\partial_\pm - 1$, we apply the method of stationary phase (cf. [18]) to $g(t,c)$ for $c=c_0:=\omega'(0)$ and find that it behaves like $t^{-2/3}$. According to [20, 7.7.18], there is a generalization of the classical method of stationary phase which is uniform in terms of the group velocity c. In fact, for $y\in[-\varepsilon,\varepsilon]$ with $\varepsilon>0$ sufficiently small and $c_0:=\omega'(0)$ the following holds:

$$\begin{aligned} g(t,c_0+y) \sim{}& t^{-1/3}\mathrm{Ai}\big(a(y)t^{2/3}\big)\big[u_0(y)+\mathcal{O}(t^{-1})\big] \\ &+ t^{-2/3}\mathrm{Ai}'\big(a(y)t^{2/3}\big)\big[u_1(y)+\mathcal{O}(t^{-1})\big]. \end{aligned} \tag{3.32}$$

Here $\mathrm{Ai}(\cdot)$ refers to the Airy function, and a, u_0 and u_1 are smooth functions with $a(0)=0$. Making these functions explicit, we find that the leading order term cancels. Together with Lemma 3.6 this implies (3.31).

In this connection one should note that there is a smooth cross-over between the different scales. Indeed, employing the asymptotic behaviour of Airy's function (cf. [21]), we obtain for $y<0$ the asymptotic behaviour $t^{-1/3}\mathrm{Ai}(a(y)t^{2/3})\sim C_1 t^{-1/2}$ and $t^{-2/3}\ \mathrm{Ai}'(a(y)t^{2/3})\sim C_2 t^{-1/2}$ as $t\to\infty$. Furthermore, the asymptotic expansion (3.32) implies that the width-scaling of the fronts in Lemma 3.5 is sharp. This holds for $\theta=0$ as well as for general $\theta\in\Theta_{\mathrm{cr}}$, wherein (3.32) occurs as an additional modulating factor $\mathrm{e}^{i\omega(\theta)t}$ (see again [19,20] for details).

Until now we provided the decay rates along critical and noncritical rays, but we did not use that the effective propagation speed is finite. The light cone corresponds to $c\in[-c_*,c_*]$ where $c_*:=\max_{\theta\in\Theta_{\mathrm{cr}}}|\omega'(\theta)|$. Outside this region the decay is faster than algebraic in terms of t as well as in terms of the velocity $c\in(-\infty,-c_*)\cup(c_*,\infty)$.

Applying partial integration, which is the standard argument to see this (cf. [17]), is not straight forward due to the occurring boundary terms. In [6] the exponential decay is proved, for the standard FPU case, using a dilation-analytic argument with respect to Fourier frequency. Using (3.32) and the asymptotic behaviour of Ai as $t \to \infty$, it turn out that the decay is even faster. In any case, one finds for each $\delta > 0$ a decay constant $\kappa_\delta > 0$ such that

$$\forall t \geq 0 \ \forall c \in (-\infty, \, -c_* - \delta] \cup [c_* + \delta, \infty) : |g(t, c)| \leq \mathrm{e}^{-\kappa_\delta(|c| - c_*)t}. \tag{3.33}$$

With the above lemmas we are now prepared to prove the ℓ^p decay rate of $\mathrm{e}^{\mathcal{L}t}$.

Proof of Theorem 3.1 According to Lemma 3.4 the group $\mathrm{e}^{\mathcal{L}t}$ acts as convolution with the matrix-valued Green's function $\mathbf{G}(t) = (\mathbf{G}^{k,m}(t))_{k,m=1,2}$. Using Young's inequality (3.6), we obtain

$$\|\mathrm{e}^{\mathcal{L}t}\mathbf{z}^0\|_{\ell^p} \leq \|\mathbf{G}(t)\|_{\ell^p}\|\mathbf{z}^0\|_{\ell^1}.$$

Thus, it is sufficient to prove the desired decay rates in (3.9) and (3.10), respectively, for the components of $\mathbf{G}(t)$.

We only carry out the details of the proof for $\mathbf{G}^{1,1}(t)$. Let us first consider the case $p \neq 4$. We aim to prove

$$\left\|\mathbf{G}^{1,1}(t)\right\|_{\ell^p} \leq \frac{C_p}{(1+t)^{\alpha_p}} \tag{3.34}$$

for which, according to (3.18) and by introducing the velocity $c = j/t$ as a new variable, it is sufficient to show

$$t\int_{-\infty}^{\infty}\left|\frac{1}{2\pi}\int_0^{\pi} h(\theta, t, c)\mathrm{d}\theta\right|^p \mathrm{d}c = \mathcal{O}(t^{-p\alpha_p}) \quad \text{as } t \to \infty.$$

The left-hand side is bounded by the terms of the form

$$B_p(t) := t\int_{-\infty}^{\infty} |g(t, c)|^p \mathrm{d}c$$

with $g(t, c)$ defined in (3.21), $\phi(\theta, c) = \pm(\omega(\theta) \pm c\theta)$ and $\psi(\theta)$ standing for 1, $1/\omega_\mathrm{r}(\theta)$ or $\omega_\mathrm{r}(\theta)$. Without loss of generality we only consider $\phi(\theta, c) = \omega(\theta) - c\theta$ and may assume that $t > 1$.

To estimate the contributions of each $\theta \in \Theta_{\mathrm{cr}}$, we choose $\varepsilon > 0$ and consider $c \in [\omega'(\theta) - \varepsilon, \, \omega'(\theta) + \varepsilon]$. Using Lemmas 3.5 and 3.6 we find

$$\begin{aligned} B_p(t) &= t\left(\int_{\omega'(\theta)-\varepsilon}^{\omega'(\theta)-t^{-2/3}} + \int_{\omega'(\theta)-t^{-2/3}}^{\omega'(\theta)+t^{-2/3}} + \int_{\omega'(\theta)+t^{-2/3}}^{\omega'(\theta)+\varepsilon}\right)|g(t, c)|^p \mathrm{d}c \\ &\leq \frac{C_\omega C_\psi}{(1+t)^{p/3-1/3}} + \frac{2\tilde{C}_\omega C_\psi}{(1+t)^{p/2-1}}\left(1 + \int_{\omega'(\theta)+t^{-2/3}}^{\omega'(\theta)+\varepsilon}\frac{\mathrm{d}c}{|\omega'(\theta)^2 - c^2|^{p/4}}\right) \\ &\leq \frac{C_\omega C_\psi + 2\tilde{C}_\omega C_\psi C}{(1+t)^{(p-1)/3}} + \frac{2\tilde{C}_\omega C_\psi C}{(1+t)^{(p-2)/2}} \end{aligned} \tag{3.35}$$

with C depending on $\omega'(\theta)$, ε and p. Taking the leading order term we get the decay rate $p\alpha_p$. Thus, using (3.33) and Lemma 3.6 for $c \notin [\omega'(\theta) - \varepsilon, \omega'(\theta) + \varepsilon]$ we obtain

$$B_p(t) \leq 2M\frac{C_\omega C_\psi + 2\tilde{C}_\omega C_\psi C}{(1+t)^{p\alpha_p}} + \mathcal{O}(t^{-(p-2)/2}) + \mathcal{O}(\mathrm{e}^{-\kappa_\varepsilon \varepsilon p t}),$$

which implies (3.34). Hence, the case $p \neq 4$ is established.

In the case $p=4$ the additional factor log t contributing to the leading order term appears on the right-hand side of (3.35). Indeed, we obtain

$$B_4(t) \leq \frac{C_\omega C_\psi + 2\tilde{C}_\omega C_\psi C}{1+t} + \frac{2\tilde{C}_\omega C_\psi C}{1+t}(\log t + \log \varepsilon).$$

This is sufficient to see that $\left\| \mathbf{G}^{1,1}(t) \right\|_{\ell^p} \leq C_p\big((1+t)\log(2+t)\big)^{1/4}$.

For the other components of $\mathbf{G}(t)$ we may use exactly the same arguments. This proves the first statement of Theorem 3.1.

To prove the second statement we proceed like above, but we use the global upper bound Lemma 3.7 instead of Lemmas 3.5 and 3.6. Then, the leading order term behaves like $t^{(2-p)/p}$. ∎

4. Discrete Klein–Gordon and nonlinear Schrödinger equations

Here, we outline how to apply the tools developed in Sections 2 and 3 to other models in one-dimensional chains, namely the dKG and the dNLS (Section 1).

For dKG we have an on-site potential with $W'(x) = bx + \mathcal{O}(|x|^\beta)$. Like in the FPU case, our results are not restricted to the nearest neighbour interaction. Indeed, we may allow for any finite-range interaction as long as the stability condition (3.3) and the non-dependency condition (3.8) are satisfied; but for simplicity we restrict ourselves to the simplest case, where the dispersion relation reads

$$\omega(\theta) = \sqrt{2 + b - 2\cos\theta}.$$

The stability condition immediately implies $b \geq 0$. In Figure 4, we plot the dispersion relation and the time evolution of a prototypical dKG chain. A major difference to FPU is that the propagation fronts do not correspond to the macroscopic wave number $\theta \approx 0$. Hence, the fronts are not monotone but have an Airy expansion as in (3.32) but multiplied with a factor $e^{i\omega(\theta_*)t}$, where $\omega'(\theta_*) = c_*$ and hence $\omega''(\theta_*) = 0$. Now $\theta = 0$ does not lie in Θ_{cr} because the on-site potential W destroys the Galilean invariance.

But, apart from these two difference the results and the approaches to prove these are the same, such as in the FPU case. Using the explicit solution of the linearized system, we may prove the analogue to Theorem 3.1 with the same decay rates. This relies on the fact that the key ingredients for its proof is the representation of the solutions in terms of oscillatory integral of the form (3.21) and quite general conditions on ω, namely (3.8).

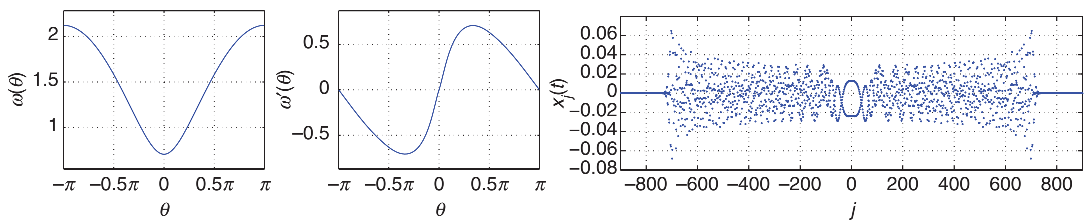

Figure 4. Dispersion relation and time evolution for the prototypical dKG chain ($a_1 = -1$, $b = 0.5$): $\omega(\theta)$, $\omega'(\theta)$ and $x_j(t)$ at $t = 800$ to initial condition $(x_j(0), \dot{x}_j(0)) = (\delta_{j,0}, 0)$.

THEOREM 4.1 *Consider the* dKG *with* $W'(x) = bx + \mathcal{O}(|x|^{\beta})$, $b > 0$ *and* $\beta > 4$. *Then, for each* $p \in [2, 4) \cup (4, \infty]$ *there exist* C_p *and* $\varepsilon > 0$ *such that all solutions* $\mathbf{z} = (\mathbf{x}, \dot{\mathbf{x}})$ *with* $\|\mathbf{z}(0)\|_{\ell^1} \leq \varepsilon$ *satisfy the estimate*

$$\|\mathbf{z}(t)\|_{\ell^p} \leq \frac{C_p}{(1+t)^{\alpha_p}} \|\mathbf{z}(0)\|_{\ell^1} \quad \text{for all } t \geq 0, \tag{4.1}$$

where the decay rate α_p *is given in* (3.9).

Again the case $p = 4$ can be included by adding a suitable logarithmic term.

This theorem improves the result in [10] in a twofold manner, namely in terms of β as well as in terms of the decay rate α_p for $p \in (2, \infty)$. In particular, Theorem 4.1 explains the discrepancy between the numerical simulation and the theoretical decay rate $\hat{\alpha}_p$ in [10]. We see that our decay rate α_p fits the numerics much better.

p	4	5	6
$\hat{\alpha}_p = \frac{p-2}{3p}$	$\frac{1}{6} \approx 0.167$	$\frac{1}{5} = 0.2$	$\frac{2}{9} \approx 0.222$
numerics in [10]	0.226	0.267	0.292
$\alpha_p = \frac{p-1}{3p}$	$\frac{1}{4} = 0.25^{\log}$	$\frac{4}{15} \approx 0.267$	$\frac{5}{18} \approx 0.278$

We recall that for the case $p = 4$ the optimal decay is like $(1+t)^{-1/4} \log(2+t)$. Hence, it is not surprising that the numerical approximation in this case is especially bad.

The above theory can be easily transferred to the dNLS, where the dispersion relation reads $\omega(\theta) = 2 - 2\cos\theta$. Obviously, $\Theta_{\rm cr} = \{\pi/2\}$ and the non-degeneracy condition $\omega'''(\pi) \neq 0$ holds. In this case the ℓ^2 norm is, in fact, a first integral, and hence is preserved exactly along solutions. Using this, it is not difficult to show that for $\beta > 4$ we have dispersive stability with the same decay rates as above.

Acknowledgements

The research was partially supported by the Deutsche Forschungsgemeinsschaft under Mie 459/4-3.

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