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Crack growth in polyconvex materials

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We discuss a model for crack propagation in an elastic body, where the crack path is described a priori. In particular, we develop in the framework of finite-strain elasticity a rate-independent model for crack evolution which is based on the Griffith fracture criterion. Due to the nonuniqueness of minimizing deformations, the energy-release rate is no longer continuous with respect to time and the position of the crack tip. Thus, the model is formulated in terms of the Clarke differential of the energy, generalizing the classical crack evolution models for elasticity with strictly convex energies. We prove the existence of solutions for our model and also the existence of special solutions, where only certain extremal points of the Clarke differential are allowed.

1. Introduction

In this work we discuss a model for crack propagation in an elastic body, where the crack path is prescribed a priori. Typical applications involve a body consisting of two parts that are glued together along a potential crack path. The evolution is assumed to be sufficiently slow such that inertial terms can be neglected, which is the quasistatic setting. Even more, we are interested in the rate-independent limit, which is relevant for cases, where the external loading via time-dependent forces is much slower than internal relaxation times. Thus, this paper also relates to the work in [1–3] where prescribed crack paths are considered for cohesive zone models describing delamination with partially debonded crack surfaces. However, in this work we follow [4–6] and restrict ourselves to brittle fracture, where only the not-yet-opened and the already-opened states are admitted for the crack such that the position of the crack tip determines all information about the crack. The evolution of the crack tip is assumed to follow the Griffith law, namely a crack does not move if the energy-release rate is less than the fracture toughness and it moves if the energy-release rate is larger, cf. e.g. [7–10] for work on Griffith criterion. We refer to [11] and the references therein for the physical background and numerical simulations. In particular our paper provides an existence result for a simplified version of the model in [11].

The novelty of the present work is that we allow for finite-strain elasticity in the bulk of the material. Thus, the elastic energy is nonconvex and for a given crack position there may be several minimizing deformations $\varphi : \Omega \to \mathbb{R}^2$ of the elastic energy. Moreover, the energy functional is no longer continuous on the set of admissible deformations as we impose the local invertibility constraint $\det \nabla\varphi > 0$ almost everywhere in Ω. We exploit the fact that the existence of energy-release rates for this case was established in [12]. However, in contrast to the work in [7–9,5,6,10] we are now faced with the difficulty that the energy-release rate is no longer continuous with respect to the time and the position of the crack tip, since it is defined via a minimization over the set of all possible minimizers for the current time and crack-tip position.

Following [13,1,6,14] we construct solutions for the rate-independent limit by a method of vanishing viscosity. However, our aim is to derive limit equations that describe the occurring limit solutions (also called approximable solutions) as precisely as

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possible. In this work we will obtain solutions called *local energetic solutions* which are the same as the *BV solution* defined in [14], except that here we are in a unidirectional setting ($\dot{s} \geq 0$) while there symmetric dissipation distances are used. Because of the jumps occurring it is useful to introduce *parameterized solutions* as used also in [13,14] (called parameterized metric solutions in the latter work). Since the present work allows for nonconvex elasticity the underlying (reduced) energy functional will only be Lipschitz continuous with points of non-differentiability that are locally nonconvex.

Thus, the above-mentioned local energetic model is formulated using the Clarke differential, which is the largest one among the different choices for the differentials at our disposal. We will also define corresponding *special local energetic* and *special parameterized solutions*, where only certain extremal points in the Clarke differential are allowed.

To be more specific, the set $\Omega \subset \mathbb{R}^2$ is the reference configuration of the elastic body, which is assumed to be a bounded Lipschitz domain. We denote by $t \in [0, T]$ the process time and by $s \in [s_0, s_1] \subset [0, L]$ the position of the crack tip. Here $\gamma : [0, T] \to \overline{\Omega}$ is the prescribed crack path in arc-length parameterization and we assume $\gamma \in \mathrm{C}^{2,1}([0, L]; \mathbb{R}^2)$. For a given crack position s the set of admissible deformations is $\mathrm{W}^{1,p}(\Omega_s; \mathbb{R}^2)$, where $\Omega_s = \Omega \setminus \{\gamma(\sigma) \mid \sigma \in [0, s]\}$. We define the *reduced energy functional* $\mathcal{I} : [0, T] \times [s_0, s_1] \to \mathbb{R}$ by minimizing the full energy functional with respect to the elastic deformation:

$$\mathcal{I}(t, s) = \min\left\{ \int_{\Omega_s} W(\nabla\varphi)\,\mathrm{d}x - \langle \ell(t), \varphi\rangle \mid \varphi \in \mathrm{W}^{1,p}(\Omega_s; \mathbb{R}^2),\, (\varphi - g_{\mathrm{Dir}})\big|_{\Gamma_{\mathrm{Dir}}} = 0 \right\}.$$

Under suitable technical assumptions we show that the mapping $(t, s) \mapsto \mathcal{I}(t, s) - \frac{\lambda}{2}(t^2 + s^2)$ is concave for a suitable $\lambda > 0$. Thus, for each point (t_*, s_*) all directional derivatives exist and determine the Clarke differential completely. In fact, in the λ-concave case there is a close relation between different notions of subdifferentials like the Clarke differential, the Fréchet differential and the subdifferential from convex analysis, cf. Section 3.1. In [12] it was shown that the total energy-release rate

$$\mathcal{G}(t, s) := -\partial_s^+\mathcal{I}(t, s) \geq 0$$

exists for all t and s, but we need additional one-sided continuity properties and semi-continuities of the one-sided partial derivatives $\partial_s^\pm\mathcal{I}$ and $\partial_t^\pm\mathcal{I}$. The concavity implies for the negative of the energy-release rates the estimates $\partial_s^+\mathcal{I}(t, s) \leq \partial_s^-\mathcal{I}(t, s)$, where inequality occurs due to different elastic minimizers. We define

$$\mathcal{G}^-(t, s) := -\partial_s^-\mathcal{I}(t, s)$$

satisfying

$$0 \leq \mathcal{G}^-(t, s) = \lim_{\delta\searrow 0}\mathcal{G}(t, s - \delta) \leq \mathcal{G}(t, s).$$

The fracture toughness is encoded in the continuous function $\kappa : [s_0, s_1] \to]0, \infty[$. Since our solutions will be non-decreasing the left-hand limit $s(t^-)$ and the right-hand limit $s(t^+)$ exist for all t and we define the continuity set $C(s) = \{t \in [0, T] \mid s(t^-) = s(t) = s(t^+)\}$. With this we obtain the jump set $J(s)$ and the differentiability set $D(s)$ as follows

$$J(s) = [0, T] \setminus C(s), \qquad D(s) = \{t \in [0, T] \mid \dot{s}(t) \text{ exists}\}.$$

A *local energetic solution* to the crack problem is a function $s \in \mathrm{BV}([0, T]; [s_0, s_1])$ that satisfies for all $t \in [0, T]$ the following conditions

(a) s is non-decreasing;
(b) if $t \notin J(s)$, then $\kappa(s(t)) + \partial_s^-\mathcal{I}(t, s(t)) \geq 0$;
(c) if $\kappa(s(t)) + \partial_s^+\mathcal{I}(t, s(t)) > 0$, then $t \in D(s)$ and $\dot{s}(t) = 0$;
(d) for all $t_* \in J(s)$ and all $s_* \in [s(t_*^-), s(t_*^+)]$ we have $\kappa(s_*) + \partial_s^+\mathcal{I}(t_*, s_*) \leq 0$.

Condition (a) is the unidirectionality (sometimes called irreversibility). Condition (b) is a kind of stability condition for rate-independent systems, namely $\mathcal{G}^-(t, s(t)) = -\partial_s^-\mathcal{I}(t, s(t)) \leq \kappa(s(t))$. This means that the smallest possible energy release cannot be bigger than the fracture toughness since otherwise the crack would have already moved further. Condition (c) is one part of the Griffith criterion, namely that the crack does not move if the release rate $\mathcal{G}(t, s(t))$ is less than the toughness $\kappa(s(t))$. Condition (d) states that along a jump the energy-release rate is at least as big as the toughness.

We will show in Section 4 that limits from vanishing-viscosity, time-incremental problems are in fact local energetic solutions. Actually the discrete solutions for the incremental problems are strictly related with the special local energetic solutions (cf. formula (4.4)), but in order to perform the limit passage as the time step goes to zero we have to also involve $\partial_s^-\mathcal{I}$ and therefore are able to derive local energetic solutions. However, as indicated via the example discussed in Section 4.3 there may still be too many solutions of this type. In fact, we conjecture that the limits constructed are always *special local energetic solutions*, which differ from the general local energetic solutions by replacing (b) by the stronger condition

(b_s) if $t \notin J(s)$, then $\kappa(s(t)) + \partial_s^+\mathcal{I}(t, s(t)) \geq 0$.

This leads to the exact Griffith criterion $\mathcal{G}(t, s(t)) = \kappa(s(t))$ along slowly moving cracks.

In Section 5 we finally show that special local energetic solutions exist. For this we use corresponding parameterized solutions. Moreover, for these solutions we establish the energy balance

$$\mathcal{I}(t_2, s(t_2)) + \int_{s(t_1)}^{s(t_2)} \kappa(\sigma)\,\mathrm{d}\sigma + \mu(s, [t_1, t_2]) = \mathcal{I}(t_1, s(t_1)) + \int_{t_1}^{t_2} \partial_t^-\mathcal{I}(\tau, s(\tau))\,\mathrm{d}\tau$$

where, as in [6], $\mu(s, [t_1, t_2])$ denotes the extra energy losses along jumps at times $t \in [t_1, t_2]$, see (2.8).

Finally we emphasize that our local energetic solutions are quite different from the energetic solutions discussed in [7,8,15], as the energetic solutions always satisfy a global stability condition which is stronger than (b) and (c), but in return the jumps are considered as true jumps and nothing is said about the curve connecting the points $s(t^-)$ and $s(t^+)$ and (d) is not valid. However, the global stability enforces the energy balance (2.9) with $\mu \equiv 0$. See also the discussion in [6].

2. Set up of the model

In this section we collect all the assumptions on the data that will be satisfied throughout this paper.

The reference configuration is a bounded open subset of the plane, $\Omega \subset \mathbb{R}^2$, with Lipschitz boundary $\partial\Omega$. We assume that $\partial\Omega$ is the union of two disjoint subsets Γ_{D} and Γ_{N}, with $\mathscr{H}^1(\Gamma_{\mathrm{D}}) > 0$, where $\mathscr{H}^1$ denotes the one-dimensional Hausdorff measure. On the Dirichlet part of the boundary Γ_{D} we impose a time-dependent boundary deformation $g_{\mathrm{Dir}}(t)$, while on the Neumann part Γ_{N} we prescribe surface forces $h(t)$.

The prescribed crack path is represented by a simple $\mathrm{C}^{2,1}$-path (i.e., the second derivative is Lipschitz continuous) $\mathcal{C} \subset \overline{\Omega}$ with $\mathscr{H}^1(\mathcal{C}) =: L$ and let $\gamma : [0, L] \to \mathcal{C}$ be its arc-length parameterization. We assume that for every $s \in \,]0, L[$ we have

$\gamma(s) \in \overline{\Omega} \setminus \partial\Omega$, while the endpoints of $\mathcal{C}$, that is $\gamma(0)$ and $\gamma(L)$, can meet the boundary $\partial\Omega$.

Let us fix $0 < s_0 < s_1 < L$ and for each $s \in [s_0, s_1]$ we define the admissible crack set by $\mathcal{C}_s := \{\gamma(\sigma) \mid 0 \le \sigma \le s\}$, i.e., the admissible crack $\mathcal{C}_s$ is uniquely determined by its length s. The cracked domain is denoted by the set $\Omega_s := \Omega \setminus \mathcal{C}_s$.

In this paper we will not model crack initiation and we will also not discuss the case of a crack separating the body into two disconnected parts. We refer to [10] for a discussion of crack initiation. Since in this paper we use monotonicity arguments in several places, further analysis is needed to model cracks which are described by more than one geometrical parameter. On the other hand, our analysis is applicable e.g. to describe the evolution of a single penny-shaped crack in three dimensions.

In Section 2.1 we provide all the notations which we need to define our crack model. For the precise mathematical assumptions we refer to Section 3. In Section 2.2 we give different notions of solutions, while in Section 2.3 we formulate our main results.

2.1. Polyconvex materials and release rate

We consider finite-strain elasticity and assume that the stored energy density $\widetilde{W} : \mathbb{R}^{2\times 2} \to [0, \infty]$ is polyconvex and coercive. In addition we assume that $\widetilde{W}$ satisfies the *multiplicative stress control estimate*

$$\left|F^\top \mathrm{D}\widetilde{W}(F)\right| \le c_1(\widetilde{W}(F) + C_0),$$

whenever the deformation gradient $F = \nabla\varphi$ satisfies that $\det F > 0$, while for $\det F \le 0$ we assume $\widetilde{W}(F) = \infty$. The exact conditions are listed and discussed in Section 3.2. At this point we want to remark that this type of stress control is compatible with polyconvexity and frame indifference, see [16,17]. It will be crucial to deal with time-dependent boundary conditions as well as interior variations to handle the changing domain due to crack growth.

We denote by $f : [0, T] \times \Omega \to \mathbb{R}^2$ the applied volume force density and by $h : [0, T] \times \Gamma_{\mathrm{N}} \to \mathbb{R}^2$ the applied surface forces. For shortness, we put

$$\langle \ell(t), \psi\rangle := \int_\Omega f(t)\cdot\psi \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} h(t)\cdot\psi \,\mathrm{d}\mathscr{H}^1 \tag{2.1}$$

for every deformation field ψ. We assume that the Dirichlet datum is the restriction to Γ_{D} of a function $g_{\mathrm{Dir}} : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$.

We look for an elastic deformation $\varphi(t) : \Omega_{s_1} \to \mathbb{R}^2$ such that $\varphi(t) = g_{\mathrm{Dir}}(t)$ on Γ_{D}. As already observed, e.g. in [18], in the finite-strain case the appropriate split is the multiplicative one, that is of the form $\varphi(t) = g_{\mathrm{Dir}}(t) \circ \psi$ with ψ belonging to the space

$$\mathrm{W}^{1,p}_{\Gamma_{\mathrm{D}}}(\Omega_{s_1}; \mathbb{R}^2) := \{\psi \in \mathrm{W}^{1,p}(\Omega_{s_1}; \mathbb{R}^2) \mid \psi = \mathrm{id} \text{ on } \Gamma_{\mathrm{D}}\},$$

where the equality is understood in the sense of traces and $p > 2$ will be fixed in Section 3.2. For given $t \in [0, T]$, $\psi \in \mathbb{R}^2$ and $F \in \mathbb{R}^{2\times 2}$ we set

$$W(t, \psi, F) := \widetilde{W}(\nabla g_{\mathrm{Dir}}(t, \psi)F).$$

The energy functional $\mathcal{E} : [0, T] \times \mathrm{W}^{1,p}_{\Gamma_{\mathrm{D}}}(\Omega_{s_1}; \mathbb{R}^2) \times [s_0, s_1] \to \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ is defined by

$$\mathcal{E}(t, \varphi, s) := \int_{\Omega_s} W(t, \varphi(x), \nabla\varphi(x))\,\mathrm{d}x - \langle \ell(t), g_{\mathrm{Dir}}(t, \varphi)\rangle$$

if $\varphi \in \mathrm{W}^{1,p}_{\Gamma_{\mathrm{D}}}(\Omega_s; \mathbb{R}^2)$ and $\mathcal{E}(t, \varphi, s) := \infty$ otherwise.

For given $t \in [0, T]$ and $s \in [s_0, s_1]$ we define

$$\Phi(t, s) := \mathrm{Argmin}\{\mathcal{E}(t, \varphi, s) \mid \varphi \in \mathrm{W}^{1,p}_{\Gamma_{\mathrm{D}}}(\Omega_{s_1}; \mathbb{R}^2)\}$$

as the set of all minimizers of the energy functional for fixed t and s (in general in the finite-strain context there is no uniqueness of minimizers, see, e.g., [19–22]). As in [6] we also introduce the reduced energy $\mathcal{I} : [0, T] \times [s_0, s_1] \to \mathbb{R}$, defined as the minimal value of the energy functional for given t and s, namely,

$$\mathcal{I}(t, s) := \mathcal{E}(t, \varphi_{t,s}, s) \quad \text{for } \varphi_{t,s} \in \Phi(t, s). \tag{2.2}$$

For fixed time t and crack length s the total energy-release rate $\mathcal{G}(t, s)$ is defined by

$$\mathcal{G}(t, s) := -\lim_{\delta\searrow 0}\frac{1}{\delta}\big(\mathcal{I}(t, s+\delta) - \mathcal{I}(t, s)\big) = -\partial_s^+\mathcal{I}(t, s). \tag{2.3}$$

It is shown in [12], see also Theorem 3.2 from [6], that the total energy-release rate is well-defined. Introducing the local energy-release rate

$$G(t, \varphi, s) := -\lim_{\delta\searrow 0}\frac{1}{\delta}\Big(\mathcal{E}(t, \varphi\circ T_{s,\delta}^{-1}, s+\delta) - \mathcal{E}(t, \varphi, s)\Big), \tag{2.4}$$

where $\varphi \in \mathrm{W}^{1,p}_{\Gamma_{\mathrm{D}}}(\Omega_s)$ is an admissible deformation field and $T_{s,\delta}$ is a diffeomorphism between Ω_s and $\Omega_{s+\delta}$, the identities

$$\mathcal{G}(t, s) = \max\{G(t, \varphi_{t,s}, s) \mid \varphi_{t,s} \in \Phi(t, s)\}, \tag{2.5}$$

$$\mathcal{G}^-(t, s) = \min\{G(t, \varphi_{t,s}, s) \mid \varphi_{t,s} \in \Phi(t, s)\} \tag{2.6}$$

are valid. The energy-release rates are related with the partial Clarke generalized gradient through (see Section 3.1)

$$\partial_s^{\mathrm{Cl}}\mathcal{I}(t, s) = [\partial_s^+\mathcal{I}(t, s), \partial_s^-\mathcal{I}(t, s)] = [-\mathcal{G}(t, s), -\mathcal{G}^-(t, s)].$$

Explicit formulas are provided in [12].

In Proposition 3.1 we show that $\mathcal{I}$ is λ-concave, i.e., it can be written as a Lipschitz-continuous, concave function plus $\lambda(t^2 + s^2)/2$. From this we deduce upper and lower semicontinuity properties for $\partial_s^\pm\mathcal{I}$ and $\partial_t^\pm\mathcal{I}$. Moreover, we can immediately derive the chain rule from the corresponding one working for the standard subdifferential of convex functions.

As in our previous paper [6], the motion of the crack tip is associated with the dissipation of energy via a dissipation potential $\mathcal{R}$. Let $\kappa \in \mathrm{C}^0([0, L])$ be positive and define the dissipation potential

$$\mathcal{R}(s, \dot{s}) := \begin{cases}\kappa(s)\dot{s} & \text{if } \dot{s} \ge 0\\ \infty & \text{else.}\end{cases} \tag{2.7}$$

Thus, the function κ gives the fracture toughness of the material. The condition $\mathcal{R} = \infty$ for $\dot{s} < 0$ will imply unidirectionality, also called irreversibility e.g. in [8,1,23–25,3,26].

2.2. Notion of solution

Since for fixed (t, s) the set of minimizers might not be single-valued, the local energy-release rates might take different values on $\Phi(t, s)$. For this reason we give the definition of our notion of solution only in terms of the reduced functional $\mathcal{I}$. All our definitions avoid the usage of the derivative $\dot{s}$ and hence are formulated to hold for *all* $t \in [0, T]$.

Definition 2.1. A *local energetic solution* to the rate-independent problem associated with $\mathcal{I}$ and $\mathcal{R}$ is a function $s \in \mathrm{BV}([0, T]; [s_0, s_1])$ such that for every $t \in [0, T]$ we have

(a) s is non-decreasing;
(b) if $t \notin J(s)$, then $\kappa(s(t)) + \partial_s^-\mathcal{I}(t, s(t)) \ge 0$;
(c) if $\kappa(s(t)) + \partial_s^+\mathcal{I}(t, s(t)) > 0$, then $t \in D(s)$ and $\dot{s}(t) = 0$;
(d) for all $t_* \in J(s)$ and all $s_* \in [s(t_*^-), s(t_*^+)]$ we have $\kappa(s_*) + \partial_s^+\mathcal{I}(t_*, s_*) \le 0$,

where $J(s)$ and $D(s)$ denote the jump set and the set of differentiability, respectively.

This notion of solution is the counterpart of the notion of local energetic solution in the small-strain case proposed in [6], where we had $\partial_s^- \mathcal{I} = \partial_s^+ \mathcal{I}$. We provide in Section 4.3 an explicit example showing that this notion allows too many solutions in this context. Hence we also introduce a subclass of local energetic solutions, called special local energetic solutions, which, in our opinion, selects the most interesting solutions.

Definition 2.2. The function $s \in \mathrm{BV}([0,T];[s_0,s_1])$, is a *special local energetic solution* for $\mathcal{R}$ and $\mathcal{I}$ if for every $t \in [0,T]$ the following conditions hold true.

- (a) s is non-decreasing;
- (b_s) if $t \notin J(s)$, then $\kappa(s(t)) + \partial_s^+ \mathcal{I}(t,s(t)) \geq 0$;
- (c) if $\kappa(s(t)) + \partial_s^+ \mathcal{I}(t,s(t)) > 0$, then $t \in D(s)$ and $\dot{s}(t) = 0$;
- (d) for all $t_* \in J(s)$ and all $s_* \in [s(t_*^-), s(t_*^+)]$ we have $\kappa(s_*) + \partial_s^+ \mathcal{I}(t_*, s_*) \leq 0$.

For proving existence of special local energetic solutions and for establishing an energy balance for local energetic solutions we also need a parameterized version of the previous definition. Such parameterized solutions are also used in [13,14]. We will prove in Section 5.1 the equivalence of both definitions up to a natural non-uniqueness at jump points. Let $R \in\ ]0, T + (s_1 - s_0)]$.

Definition 2.3. The pair $(t_\mathrm{p}, s_\mathrm{p}) \in \mathrm{C}^{\mathrm{Lip}}([0,R];[0,T]\times[s_0,s_1])$ is a *parameterized solution* associated with $\mathcal{R}$ and $\mathcal{I}$, if for all $\rho \in [0,R]$ the following conditions hold:

- (a_p) t_p and s_p are non-decreasing, $t_\mathrm{p}(\rho) + s_\mathrm{p}(\rho) - \rho = \text{const.}$;
- (b_p) if $\rho \notin \mathcal{J}_\mathrm{p}$, then $\kappa(s_\mathrm{p}(\rho)) + \partial_s^- \mathcal{I}(t_\mathrm{p}(\rho), s_\mathrm{p}(\rho)) \geq 0$ (not jumping);
- (c_p) if $\rho \notin \mathcal{S}_\mathrm{p}$, then $\kappa(s_\mathrm{p}(\rho)) + \partial_s^+ \mathcal{I}(t_\mathrm{p}(\rho), s_\mathrm{p}(\rho)) \leq 0$ (not sticking);

with jump set

$$\mathcal{J}_\mathrm{p} := \{\rho \mid \exists \delta > 0 : t_\mathrm{p} \text{ is constant on } B_\delta(\rho) \cap [0,R]\}$$

and sticking set

$$\mathcal{S}_\mathrm{p} := \{\rho \mid \exists \delta > 0 : s_\mathrm{p} \text{ is constant on } B_\delta(\rho) \cap [0,R]\}.$$

Note that conditions (b_p) and (c_p) imply

(d_p) $\rho \notin \mathcal{J}_\mathrm{p} \cup \mathcal{S}_\mathrm{p} \implies \exists \widetilde{\sigma} \in \partial_s^{\mathrm{Cl}} \mathcal{I}(t_\mathrm{p}(\rho), s_\mathrm{p}(\rho))$ with $\kappa(s_\mathrm{p}(\rho)) + \widetilde{\sigma} = 0$

i.e., $-\kappa(s_\mathrm{p}(\rho)) \in \partial_s^{\mathrm{Cl}} \mathcal{I}(t_\mathrm{p}(\rho), s_\mathrm{p}(\rho))$ (sliding).

Again, we define a special subclass by strengthening the condition for "not jumping".

Definition 2.4. The pair $(t_\mathrm{p}, s_\mathrm{p}) \in \mathrm{C}^{\mathrm{Lip}}([0,R];[0,T]\times[s_0,s_1])$ is a *special parameterized solution* for $\mathcal{R}$ and $\mathcal{I}$, if for every $\rho \in [0,R]$ the following conditions hold:

- (a_p) t_p and s_p are non-decreasing, $t_\mathrm{p}(\rho) + s_\mathrm{p}(\rho) - \rho = \text{const.}$;
- (b_sp) if $\kappa(s_\mathrm{p}(\rho)) + \partial_s^+ \mathcal{I}(t_\mathrm{p}(\rho), s_\mathrm{p}(\rho)) < 0$, then there exists $\delta > 0$ such that $]\rho, \rho+\delta[\ \subset \mathcal{J}_\mathrm{p}$;
- (c_p) if $\kappa(s_\mathrm{p}(\rho)) + \partial_s^+ \mathcal{I}(t_\mathrm{p}(\rho), s_\mathrm{p}(\rho)) > 0$, then $\rho \in \mathcal{S}_\mathrm{p}$.

Note that conditions (b_sp) and (c_p) imply

(d_sp) $\Big(\rho \notin \mathcal{S}_\mathrm{p} \text{ and } \forall \delta > 0 :\ ]\rho, \rho+\delta[\ \not\subset \mathcal{J}_\mathrm{p}\Big) \implies \kappa(s_\mathrm{p}(\rho)) + \partial_s^+ \mathcal{I}(t_\mathrm{p}(\rho), s_\mathrm{p}(\rho)) = 0.$

2.3. Statement of the existence results

To state the main results of this paper, we follow [6, Eq. (2.18)] and introduce the nonnegative *jump functional* $\mu(s,\cdot)$, which is defined on closed subintervals of $[0,T]$ via

$$\mu(s,[t_1,t_2]) := \Delta(t_1, s(t_1), s(t_1^+)) + \Delta(t_2, s(t_2^-), s(t_2)) + \sum_{t \in]t_1,t_2[\cap J(s)} \Delta(t, s(t^-), s(t^+)), \tag{2.8}$$

where $\Delta(t,\sigma_1,\sigma_2) := \int_{\sigma_1}^{\sigma_2} -(\kappa(\sigma) + \partial_s^+ \mathcal{I}(t,\sigma))\,\mathrm{d}\sigma$ denotes the difference of the energy release and the dissipated energy in a jump from σ_1 to σ_2 at time t. Note that in this definition we could replace $\partial_s^+ \mathcal{I}$ by $\partial_s^- \mathcal{I}$ without any effect. We assume that the conditions (H1)–(H4), which are formulated in Section 3.2, and that conditions (4.1) and (4.2) are satisfied.

Theorem 2.5. *There exists a local energetic solution $s \in \mathrm{BV}([0,T])$ to the rate-independent problem associated with $\mathcal{I}$ and $\mathcal{R}$ such that $s(0) = s_0$.*

Moreover, $s(\cdot)$ satisfies the following energy balance: there exists a measurable function $\widetilde{\alpha} : (0,T) \to \mathbb{R}$ with $\widetilde{\alpha}(t) \in \partial_t^{\mathrm{Cl}} \mathcal{I}(t,s(t)) = [\partial_t^+ \mathcal{I}(t,s(t)), \partial_t^- \mathcal{I}(t,s(t))]$ a.e. in $[0,T]$ such that for $0 \leq t_1 < t_2 \leq T$ we have

$$\mathcal{I}(t_2, s(t_2)) + \int_{s(t_1)}^{s(t_2)} \kappa(\sigma)\,\mathrm{d}\sigma + \mu(s,[t_1,t_2]) = \mathcal{I}(t_1, s(t_1)) + \int_{t_1}^{t_2} \widetilde{\alpha}(\tau)\,\mathrm{d}\tau. \tag{2.9}$$

Existence of a local energetic solution is proved in Section 4 using a time-incremental minimization procedure involving a vanishing-viscosity limit, where the viscosity is coupled to the step size. In fact, we conjecture that the obtained limits are even special local energetic solutions, but we were not able to prove this. Hence we state a second existence result for special solutions below, for the proof we refer to Section 5. The energy balance (2.9) will be established in Section 6 by using the parameterized solutions and a chain-rule argument. The refined energy balance (2.10) for special solutions then follows from a careful investigation of the corresponding Clarke differential, see Section 6.

Theorem 2.6. *There exists a special local energetic solution $t \mapsto s(t)$ to the rate-independent problem associated with $\mathcal{I}$ and $\mathcal{R}$ such that $s(0) = s_0$.*

Moreover, $s(\cdot)$ satisfies a refined energy balance: for $0 \leq t_1 < t_2 \leq T$ we have (2.9) with $\widetilde{\alpha}(t) = \partial_t^- \mathcal{I}(t,s(t))$, i.e.

$$\mathcal{I}(t_2, s(t_2)) + \int_{s(t_1)}^{s(t_2)} \kappa(\sigma)\,\mathrm{d}\sigma + \mu(s,[t_1,t_2]) = \mathcal{I}(t_1, s(t_1)) + \int_{t_1}^{t_2} \partial_t^- \mathcal{I}(\tau, s(\tau))\,\mathrm{d}\tau. \tag{2.10}$$

Finally, we want to state an existence result for the original, non-reduced problem involving the deformations φ as well. This formulation looks almost the same as in [6] but we have to take care of the choice of the minimizer $\varphi(t)$, which is no longer unique. For the statement below it was essential to replace the condition (b) for local energetic solutions by the stronger condition (b_s), since in general there is no $\varphi(t) \in \Phi(t,s(t))$ satisfying $-\partial_s^- \mathcal{I}(t,s(t)) = G(t,\varphi(t),s(t)) = -\partial_s^+ \mathcal{I}(t,s(t))$. Note that conditions (a)–(d) of Theorem 2.7 are exactly those of a special local energetic solution and contain the Griffith criterion, while the conditions $(\varphi 1)$ and $(\varphi 2)$ relate to the elastic deformation.

Theorem 2.7. *Under the above assumptions there exists a bounded, measurable map* $(\varphi, s) : [0, T] \to W^{1,p}_{\Gamma_D}(\Omega_{s_1}; \mathbb{R}^2) \times [s_0, s_1]$ *(defined everywhere), such that for all* $t \in [0, T]$ *the following holds*

(φ1) $\varphi(t) \in \Phi(t, s(t)) := \operatorname{Argmin} \mathcal{E}(t, \cdot, s(t))$;
(φ2) $\mathcal{G}(t, s(t)) = G(t, \varphi(t), s(t))$;
(a) *s is non-decreasing;*
(b_s) *if* $t \notin J(s)$, *then* $\kappa(s(t)) \geq \mathcal{G}(t, s(t))$;
(c) *if* $\kappa(s(t)) > \mathcal{G}(t, s(t))$, *then* $t \in D(s)$ *and* $\dot{s}(t) = 0$;
(d) *for all* $t_* \in J(s)$ *and all* $s_* \in [s(t_*^-), s(t_*^+)]$ *we have* $\kappa(s_*) \leq \mathcal{G}(t_*, s_*)$.

Moreover, for $0 \leq t_1 < t_2 \leq T$ *we have the energy balance*

$$\mathcal{E}(t_2, \varphi(t_2), s(t_2)) + \int_{s(t_1)}^{s(t_2)} \kappa(\sigma)\,\mathrm{d}\sigma + \mu(s, [t_1, t_2])$$
$$= \mathcal{E}(t_1, \varphi(t_1), s(t_1)) + \int_{t_1}^{t_2} \partial_t \mathcal{E}(\tau, \varphi(\tau), s(\tau))\,\mathrm{d}\tau.$$

Proof. The result is a consequence of the existence Theorem 2.6, which gives the mapping $s \in \mathrm{BV}([0, T])$. Let (t_p, s_p) be a parameterized solution constructed from s according to (5.1). Propositions 3.7 and 3.6 in combination with Theorem 8.2.4 from [27, Union and Intersection] guarantee that there is a measurable function $\varphi_p : [0, R] \to W^{1,p}_{\Gamma_D}(\Omega_{s_1})$ having the properties $\varphi_p(\rho) \in \Phi(t_p(\rho), s_p(\rho))$, $\mathcal{G}(t_p(\rho), s_p(\rho)) = G(t_p(\rho), \varphi_p(\rho), s_p(\rho))$ for all $\rho \in [0, R]$ and $\partial_t^- \mathcal{I}(t_p(\rho), s_p(\rho)) = \partial_t \mathcal{E}(t_p(\rho), \varphi_p(\rho), s_p(\rho))$ for almost every ρ. The function $\varphi(t) := \varphi_p(t, s_p(\widetilde{\rho}(t)))$ with $\widetilde{\rho}(t) = t + s(t)$ is measurable and satisfies (φ1), (φ2), and (a)–(d). The energy balance follows now directly from (2.10). □

3. Properties of the reduced functional

We will first provide some more differentiability and continuity properties of abstract reduced energies depending on a finite number of parameters in Section 3.1. Furthermore, we discuss the relation between the Clarke generalized gradients and partial derivatives of $\mathcal{I}$ and $\mathcal{E}$. This extends the results from [6]. In Section 3.2 we will present sufficient conditions on the polyconvex energy density $\widetilde{W}$, on the Dirichlet datum $g_{\mathrm{Dir}}(t)$ and on the applied forces $\ell(t)$ such that the abstract results are valid for the reduced functional of our crack problem.

3.1. Properties of the reduced energy $\mathcal{I}$

We recall the abstract assumptions from [6, Section 3]. Let V be a topological Hausdorff space and $\Sigma := [\sigma_1^1, \sigma_2^1] \times \cdots \times [\sigma_1^m, \sigma_2^m] \subset \mathbb{R}^m$ a set of parameters. For the energy functional $\mathcal{E}_0 : \Sigma \times V \to \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ we define

$$\mathcal{I}(\sigma) := \inf\{\mathcal{E}_0(\sigma, v) \mid v \in V\},$$
$$\Phi(\sigma) := \operatorname{Argmin}\mathcal{E}_0(\sigma, \cdot) := \{v \in V \mid \mathcal{E}_0(\sigma, v) := \mathcal{I}(\sigma)\}.$$

The following assumptions are imposed on $\mathcal{E}_0$, cf. [18] and [6].

Compactness of energy sublevels:
$\forall \sigma \in \Sigma\ \exists E \in \mathbb{R} :\ L_{\sigma,E} := \{u \in V \mid \mathcal{E}_0(\sigma, u) \leq E\}$ is not empty. Furthermore, $L_{\sigma,E}$ is sequentially compact for every $\sigma \in \Sigma$ and $E \in \mathbb{R}$. (E1)

This assumption implies that $\mathcal{I} : \Sigma \to \mathbb{R}$ is well-defined.

Uniform control of $\partial_\sigma \mathcal{E}_0$:
$\exists c_0 \in \mathbb{R}\ \exists c_1 > 0\ \forall (\widetilde{\sigma}, u) \in \Sigma \times V$ with $\mathcal{E}_0(\widetilde{\sigma}, u) < \infty$:
$\mathcal{E}_0(\cdot, u) \in C^1(\Sigma)$ and $|\partial_\sigma \mathcal{E}_0(\sigma, u)| \leq c_1(c_0 + \mathcal{E}_0(\sigma, u))$ for all $\sigma \in \Sigma$. (E2)

Conditions (E1) and (E2) guarantee that $\mathcal{I}$ is Lipschitz continuous on Σ, see Proposition 3.1 of [6]. The next condition is a stronger version than (E3) in [6].

Local Lipschitz estimate for $\partial_\sigma \mathcal{E}_0$:
$\forall E \in \mathbb{R}\ \exists \lambda_E > 0\ \forall \sigma_1, \sigma_2 \in \Sigma, \forall \varphi \in V$ with $\mathcal{E}_0(\sigma_i, \varphi) \leq E$: (E3)
$|\partial_\sigma \mathcal{E}_0(\sigma_1, \varphi) - \partial_\sigma \mathcal{E}_0(\sigma_2, \varphi)| \leq \lambda_E |\sigma_1 - \sigma_2|$.

From assumptions (E1)–(E3) we deduce on the basis of [18, Prop. 3.3] that the following implication holds

$$\sigma_n \in \Sigma, u_n \in \Phi(\sigma_n) \text{ with } \sigma_n \to \sigma, u_n \to u \in \Phi(\sigma)$$
$$\Rightarrow \partial_\sigma \mathcal{E}_0(\sigma_n, u_n) \to \partial_\sigma \mathcal{E}_0(\sigma, u). \tag{3.1}$$

Assumption (E1) and the convergence principle (3.1) imply that for fixed σ the set $\{\partial_\sigma \mathcal{E}_0(\sigma, \varphi) \mid \varphi \in \Phi(\sigma)\}$ is sequentially compact.

Proposition 3.1. *Let (E1)–(E3) be satisfied. Then the reduced functional* $\mathcal{I}$ *is* λ*-concave, i.e., there exists* $\lambda > 0$ *such that the function* $\mathcal{I}_\lambda : \Sigma \to \mathbb{R}, \sigma \mapsto \mathcal{I}(\sigma) - \frac{\lambda}{2}|\sigma|^2$ *is concave. Moreover,* $-\mathcal{I}$ *is regular in the sense of* [28, *Def. 2.3.4*].

Proof. Note that $\sup\{\mathcal{E}_0(\sigma, \varphi) \mid \sigma \in \Sigma, \varphi \in \Phi(\sigma)\} =: E$ is finite. Therefore, by (E3), there exists $\lambda_E > 0$ such that $|\partial_\sigma \mathcal{E}_0(\sigma_1, \varphi) - \partial_\sigma \mathcal{E}_0(\sigma_2, \varphi)| \leq \lambda_E |\sigma_1 - \sigma_2|$ for every $\sigma_1, \sigma_2 \in \Sigma$ and $\varphi \in \cup_{\sigma \in \Sigma} \Phi(\sigma)$. This implies that for every fixed $\varphi \in \cup_{\sigma \in \Sigma} \Phi(\sigma)$ the function $\mathcal{E}_{\lambda_E}(\cdot, \varphi) : \Sigma \to \mathbb{R}, \sigma \mapsto \mathcal{E}_0(\sigma, \varphi) - \frac{\lambda_E}{2}|\sigma|^2$ is concave. Let now $\theta \in [0, 1]$, $\sigma_0, \sigma_1 \in \Sigma$, $\sigma_\theta := \theta\sigma_0 + (1-\theta)\sigma_1$ and choose $\varphi_\theta \in \Phi(\sigma_\theta)$. Then, by the concavity of $\mathcal{E}_{\lambda_E}(\cdot, \varphi_\theta)$, it follows

$$\mathcal{I}_{\lambda_E}(\sigma_\theta) = \mathcal{E}_{\lambda_E}(\sigma_\theta, \varphi_\theta) \geq \theta\mathcal{E}_{\lambda_E}(\sigma_0, \varphi_\theta) + (1-\theta)\mathcal{E}_{\lambda_E}(\sigma_1, \varphi_\theta)$$
$$\geq \theta\mathcal{I}_{\lambda_E}(\sigma_0) + (1-\theta)\mathcal{I}_{\lambda_E}(\sigma_1).$$

Thus $\mathcal{I}$ is λ_E-concave. Since $-\mathcal{I}$ is λ-convex, [28, Prop. 2.3.6] gives the last assertion. □

For $\tau \in \mathbb{R}^m \setminus \{0\}$ and $\sigma \in \Sigma$ the right and left directional derivatives of $\mathcal{I}$ are denoted by

$$\partial_\tau^+ \mathcal{I}(\sigma) = \lim_{h \searrow 0} \frac{1}{h}\big(\mathcal{I}(\sigma + h\tau) - \mathcal{I}(\sigma)\big) \text{ and}$$
$$\partial_\tau^- \mathcal{I}(\sigma) = \lim_{h \searrow 0} \frac{1}{h}\big(\mathcal{I}(\sigma) - \mathcal{I}(\sigma - h\tau)\big).$$

If (E1)–(E3) are satisfied, which means that $\mathcal{I}$ is λ-concave, then the right and left derivatives of $\mathcal{I}$ exist and are given by

$$\partial_\tau^+ \mathcal{I}(\sigma) = \min\{\partial_\sigma \mathcal{E}_0(\sigma, u) \cdot \tau \mid u \in \Phi(\sigma)\}, \tag{3.2}$$
$$\partial_\tau^- \mathcal{I}(\sigma) = -\partial_{-\tau}^+ \mathcal{I}(\sigma) = \max\{\partial_\sigma \mathcal{E}_0(\sigma, v) \cdot \tau \mid v \in \Phi(\sigma)\}, \tag{3.3}$$

see also [6, Thm. 3.2]. Moreover, $\partial_\tau^+ \mathcal{I}(\sigma)$ and $\partial_\tau^- \mathcal{I}(\sigma)$ coincide for almost every $\sigma \in \Sigma$. The derivatives have the following upper and lower semicontinuity properties:

Theorem 3.2. *Let (E1)–(E3) be satisfied and* $\tau \in \mathbb{R}^m \setminus \{0\}$. *Then* $\partial_\tau^+ \mathcal{I}$ *is lower semicontinuous, while* $\partial_\tau^- \mathcal{I}$ *is upper semicontinuous. Moreover, if* $h_n > 0$ *with* $h_n \searrow 0$, *then*

$$\partial_\tau^+ \mathcal{I}(\sigma + h_n\tau) \to \partial_\tau^+ \mathcal{I}(\sigma),\ \partial_\tau^- \mathcal{I}(\sigma - h_n\tau) \to \partial_\tau^- \mathcal{I}(\sigma). \tag{3.4}$$

Proof. Let $\tau \in \mathbb{R}^m \setminus \{0\}$ and $\sigma_0, \sigma_n \in \Sigma$ with $\sigma_n \to \sigma_0$. For every n there exists $u_n \in \Phi(\sigma_n)$ such that $\partial_\tau^- \mathcal{I}(\sigma_n) = \partial_\sigma \mathcal{E}_0(\sigma_n, u_n) \cdot \tau$. Furthermore, there exists a subsequence $(u_{n'})_{n'}$ and an element $u_0 \in \Phi(\sigma_0)$ with $u_{n'} \to u_0$. From (3.1) and (3.3) we conclude that

$$\partial_\tau^- \mathcal{I}(\sigma_{n'}) = \partial_\sigma \mathcal{E}_0(\sigma_{n'}, u_{n'}) \cdot \tau \to \partial_\sigma \mathcal{E}_0(\sigma_0, u_0) \cdot \tau \leq \partial_\tau^- \mathcal{I}(\sigma_0)$$

for $n' \to \infty$. A proof by contradiction shows finally that $\limsup_{n \to \infty} \partial_\tau^- \mathcal{I}(\sigma_n) \leq \partial_\tau^- \mathcal{I}(\sigma_0)$. The convergence properties in (3.4) are proved in [6]. □

An appropriate notion for describing generalized gradients of Lipschitz continuous functions is the Clarke generalized gradient $\partial^{\text{Cl}}\mathcal{I} \subset \mathbb{R}^m$. We refer to [28] for a definition. By $\partial^{\text{Cl}}_\tau\mathcal{I} \subset \mathbb{R}$ we denote the partial Clarke generalized gradient with respect to $\tau \in \mathbb{R}^m\setminus\{0\}$. Let us remark that in the λ-concave case the Clarke generalized gradient coincides with the Fréchet subdifferential in the following sense: $\partial^{\text{Cl}}\mathcal{I} = -\partial^{\text{F}}(-\mathcal{I})$.

Lemma 3.3. *Let (E1)–(E3) be satisfied and let $\mathcal{I}$ be the corresponding reduced energy.*

(a) *For every $\sigma \in \Sigma$ and $\tau \in \mathbb{R}^m \setminus \{0\}$ it holds $\partial^{\text{Cl}}_\tau\mathcal{I}(\sigma) = [\partial^+_\tau\mathcal{I}(\sigma), \partial^-_\tau\mathcal{I}(\sigma)]$.*
(b) $\partial^{\text{Cl}}\mathcal{I}(\sigma) = \text{co}\{\partial_\sigma\mathcal{E}_0(\sigma,\varphi) \mid \varphi \in \Phi(\sigma)\}$.
(c) *The set-valued function $\partial^{\text{Cl}}\mathcal{I} : \Sigma \to \mathcal{P}(\mathbb{R}^m)$ is measurable in the sense of Definition 8.1.1 in [27].*

Proof. Part (a) is a consequence of Theorem 2.5.1 in [28], while part (c) follows from [28, Prop. 2.1.5], [27, Cor. 1.4.17] and [27, Thm. 8.1.4].

Proof of (b): Since $\mathcal{I}$ is λ-concave, it follows from Proposition 2.2.7 in [28] that $\partial^-_\tau\mathcal{I}(\sigma) = \mathcal{I}^\circ(\sigma;\tau)$ for every $\sigma \in \Sigma$, $\tau \in \mathbb{R}^m \setminus \{0\}$. Here, $\mathcal{I}^\circ(\sigma;\tau)$ stands for the Clarke generalized directional derivative. From (3.3) and part (a) it follows that $\mathcal{I}^\circ(\sigma;\tau) \geq \partial_\sigma\mathcal{E}_0(\sigma,\varphi)\cdot\tau$ for every $\varphi \in \Phi(\sigma)$ and every τ. The previous considerations imply that with $\mathcal{M}(\sigma) := \{\partial_\sigma\mathcal{E}_0(\sigma,\varphi) \mid \varphi \in \Phi(\sigma)\}$ we have $\text{co}\mathcal{M}(\sigma) \subset \partial^{\text{Cl}}\mathcal{I}(\sigma)$. The reverse relation follows with a proof by contradiction using the Hahn–Banach separation Theorem taking into account that $\mathcal{M}(\sigma)$ is compact and using the identities $\mathcal{I}^\circ(\sigma;\tau) = \partial^-_\tau\mathcal{I}(\sigma) = \partial_\sigma\mathcal{E}_0(\sigma;\varphi_\tau)\cdot\tau$, with suitable $\varphi_\tau \in \Phi(\sigma)$, and the definition of $\partial^{\text{Cl}}\mathcal{I}(\sigma)$. □

The next lemma provides a selection principle for Clarke generalized gradients.

Lemma 3.4. *Assume that (E1)–(E3) are satisfied and let τ_i be the unit normal vector of the i-th coordinate direction. Let $\sigma \in \text{C}^{\text{Lip}}([0,R];\Sigma)$ and assume that $\beta : [0,R] \to \mathbb{R}$ is measurable with $\beta(\rho) \in \partial^{\text{Cl}}_{\tau_1}\mathcal{I}(\sigma(\rho))$ for every $\rho \in [0,R]$.*

Then there exists a measurable function $\alpha : [0,R] \to \mathbb{R}^{m-1}$ with $\alpha_i(\rho) \in \partial^{\text{Cl}}_{\tau_i}\mathcal{I}(\sigma(\rho))$, $2 \leq i \leq m$, such that $(\beta(\rho),\alpha(\rho)) \in \partial^{\text{Cl}}\mathcal{I}(\sigma(\rho))$ for every $\rho \in [0,R]$.

Proof. Since $-\mathcal{I}$ is regular, see Proposition 3.1, and since $-\partial^{\text{Cl}}\mathcal{I} = \partial^{\text{Cl}}(-\mathcal{I})$, we deduce from [29, Cor. 10.11] and [28, Prop. 2.3.15] that the following holds true: For every $\beta \in \partial^{\text{Cl}}_{\tau_1}\mathcal{I}(\sigma)$ exists $\alpha \in \prod_{i=2}^m \partial^{\text{Cl}}_{\tau_i}\mathcal{I}(\sigma)$ such that $(\beta,\alpha) \in \partial^{\text{Cl}}\mathcal{I}(\sigma)$. The proof of the lemma now relies on [27, Thm. 8.2.9]. Let $\beta : [0,R] \to \mathbb{R}$ be a measurable selection of the set-valued function $\rho \mapsto \partial^{\text{Cl}}_{\tau_1}\mathcal{I}(\sigma(\rho))$. The mappings $F : [0,R] \to \mathbb{R}\times\mathcal{P}(\mathbb{R}^{m-1})$, $\rho \mapsto \{\beta(\rho)\} \times \prod_{i=2}^m \partial^{\text{Cl}}_{\tau_i}\mathcal{I}(\sigma(\rho))$ and $G : [0,R] \to \mathcal{P}(\mathbb{R}^m)$, $\rho \mapsto \partial^{\text{Cl}}\mathcal{I}(\sigma(\rho))$ are measurable and have closed images. Moreover, in view of the first part of this proof it holds for all $\rho \in [0,R]$ that $F(\rho) \cap G(\rho) \neq \emptyset$. Theorem 8.2.9 in [27] therefore provides a measurable selection η of $F \cap G$. The choice $\alpha(\rho) := (\eta_2(\rho),\ldots,\eta_m(\rho))$ finishes the proof. □

Since the functional $-\mathcal{I}$ is λ-convex and Lipschitz, the following lemma is an immediate consequence of a corresponding chain rule for convex functions.

Lemma 3.5 (*Chain Rule*)**.** *Assume that (E1)–(E3) are satisfied. For all $\sigma \in \text{C}^{\text{Lip}}([0,R];\Sigma)$ and $\eta \in \text{L}^\infty([0,R];\mathbb{R}^m)$ with $\eta(\rho) \in \partial^{\text{Cl}}\mathcal{I}(\sigma(\rho))$ a.e. in $[0,R]$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\mathcal{I}(\sigma(\rho)) = \eta(\rho)\cdot\frac{\mathrm{d}}{\mathrm{d}\rho}\sigma(\rho), \tag{3.5}$$

for every $\rho \in [0,R]$ such that $\eta(\rho) \in \partial^{\text{Cl}}\mathcal{I}(\sigma(\rho))$ and $\frac{\mathrm{d}}{\mathrm{d}\rho}\sigma(\rho)$ exists.

From the chain rule we can derive more information about the structure of the Clarke gradient. We assume now that $m = 2$ and $\Sigma \subset \mathbb{R}^2$.

Proposition 3.6. *Let $\sigma \equiv (t,s) \in \text{C}^{\text{Lip}}([0,R];\Sigma)$ and assume that both components are non-decreasing and that $t(\rho)+s(\rho)-\rho$ is constant on $[0,R]$. Moreover let (E1)–(E3) be satisfied. Then for almost every $\rho \in [0,R]$ with $t'(\rho) > 0$ or $s'(\rho) > 0$ we have*

$$\partial^{\text{Cl}}\mathcal{I}(\sigma(\rho)) = \text{co}\left\{\begin{pmatrix}\partial^+_t\mathcal{I}(\sigma(\rho))\\ \partial^-_s\mathcal{I}(\sigma(\rho))\end{pmatrix}, \begin{pmatrix}\partial^-_t\mathcal{I}(\sigma(\rho))\\ \partial^+_s\mathcal{I}(\sigma(\rho))\end{pmatrix}\right\} \tag{3.6}$$

and there exist elements $\varphi_1(\rho), \varphi_2(\rho) \in \Phi(\sigma(\rho))$ such that

$$\partial_\sigma\mathcal{E}_0(\sigma(\rho),\varphi_1(\rho)) = \begin{pmatrix}\partial^+_t\mathcal{I}(\sigma(\rho))\\ \partial^-_s\mathcal{I}(\sigma(\rho))\end{pmatrix}$$

$$\partial_\sigma\mathcal{E}_0(\sigma(\rho),\varphi_2(\rho)) = \begin{pmatrix}\partial^-_t\mathcal{I}(\sigma(\rho))\\ \partial^+_s\mathcal{I}(\sigma(\rho))\end{pmatrix}. \tag{3.7}$$

Proof. From the assumptions on σ it follows that for a.e. $\rho \in [0,R]$ the derivative $\sigma'(\rho)$ exists and belongs to the convex cone $\mathbb{R}_+e_1 + \mathbb{R}_+e_2$ with $\mathbb{R}_+ = (0,\infty)$. In particular, σ' vanishes at most on a set of measure zero. The chain rule (3.5) therefore implies that for almost every $\rho \in [0,R]$ there exist $\eta_1(\rho), \eta_2(\rho) \in \mathbb{R}^2$ with $\partial^{\text{Cl}}\mathcal{I}(\sigma(\rho)) = \text{co}\{\eta_1(\rho),\eta_2(\rho)\}$ and $\sigma'(\rho)\cdot(\eta_1(\rho)-\eta_2(\rho)) = 0$. It remains to show that $\eta_1(\rho)$ and $\eta_2(\rho)$ coincide with the extremal points given in (3.6). Let $\varphi^+_s(\rho), \varphi^-_t(\rho) \in \Phi(\sigma(\rho))$ with $\partial^+_s\mathcal{I}(\sigma(\rho)) = \partial_s\mathcal{E}_0(\sigma(\rho),\varphi^+_s(\rho))$ and $\partial^-_t\mathcal{I}(\sigma(\rho)) = \partial_t\mathcal{E}_0(\sigma(\rho),\varphi^-_t(\rho))$. From Lemma 3.3(b) we conclude that $\widehat{\eta}_1$ and $\widehat{\eta}_2$ given by

$$\widehat{\eta}_1 := (\partial_t\mathcal{E}_0(\sigma(\rho),\varphi^+_s(\rho)), \partial^+_s\mathcal{I}(\sigma(\rho)))^\top$$
$$\widehat{\eta}_2 := (\partial^-_t\mathcal{I}(\sigma(\rho)), \partial_s\mathcal{E}_0(\sigma(\rho),\varphi^-_t(\rho)))^\top$$

belong to $\partial^{\text{Cl}}\mathcal{I}(\sigma(\rho))$. Moreover, by (3.5) we have $\sigma'(\rho)\cdot(\widehat{\eta}_1(\rho)-\widehat{\eta}_2(\rho)) = 0$. Since $\partial_t\mathcal{E}_0(\sigma(\rho),\varphi^+_s(\rho)) - \partial^-_t\mathcal{I}(\sigma(\rho)) \leq 0$ and $\partial^+_s\mathcal{I}(\sigma(\rho)) - \partial_s\mathcal{E}_0(\sigma(\rho),\varphi^-_t(\rho)) \leq 0$, see part (a) of Lemma 3.3, it follows that $\partial_t\mathcal{E}_0(\sigma(\rho),\varphi^+_s(\rho)) = \partial^-_t\mathcal{I}(\sigma(\rho))$ if $t'(\rho) > 0$ and that $\partial_s\mathcal{E}_0(\sigma(\rho),\varphi^-_t(\rho)) = \partial^+_s\mathcal{I}(\sigma(\rho))$ if $s'(\rho) > 0$. Thus, in both cases there exists $\varphi_2 \in \Phi(\sigma(\rho))$ satisfying $\partial_\sigma\mathcal{E}_0(\sigma(\rho),\varphi_2) = (\partial^-_t\mathcal{I}(\sigma(\rho)), \partial^+_s\mathcal{I}(\sigma(\rho)))^\top \in \partial^{\text{Cl}}\mathcal{I}(\sigma(\rho))$. In the same way we conclude that there exists a minimizer $\varphi_1 \in \Phi(\sigma(\rho))$ which satisfies the first relation in (3.7). Part (b) of Lemma 3.3 implies that

$$\text{co}\left\{\begin{pmatrix}\partial^+_t\mathcal{I}(\sigma(\rho))\\ \partial^-_s\mathcal{I}(\sigma(\rho))\end{pmatrix}, \begin{pmatrix}\partial^-_t\mathcal{I}(\sigma(\rho))\\ \partial^+_s\mathcal{I}(\sigma(\rho))\end{pmatrix}\right\} \subset \partial^{\text{Cl}}\mathcal{I}(\sigma(\rho)) = \text{co}\{\eta_1(\rho),\eta_2(\rho)\}.$$

The identity (3.6) therefore follows taking into account that by Lemma 3.3(a)

$$\partial^{\text{Cl}}\mathcal{I} \subset \partial^{\text{Cl}}_t\mathcal{I}\times\partial^{\text{Cl}}_s\mathcal{I} = [\partial^+_t\mathcal{I}(\sigma(\rho)), \partial^-_t\mathcal{I}(\sigma(\rho))] \times [\partial^+_s\mathcal{I}(\sigma(\rho)), \partial^-_s\mathcal{I}(\sigma(\rho))]. \quad \square$$

For the last assertion in this abstract part we assume that V is a separable, reflexive Banach space. The goal is to prove that there exists a measurable selection of minimizers $\varphi(\sigma)$, which for every σ realize $\partial^+_\tau\mathcal{I}(\sigma)$.

Proposition 3.7. *Let V be a separable reflexive Banach space and assume that for $\mathcal{E}_0 : \Sigma\times V \to \mathbb{R}_\infty$ the conditions (E1)–(E3) are satisfied with respect to the weak topology. Then for every fixed $\tau \in \mathbb{R}^m \setminus \{0\}$ the mapping $H : \text{int}\,\Sigma \to \mathcal{P}(V)$, which is defined by*

$$H(\sigma) = \{\varphi \in \Phi(\sigma) \mid \partial_\sigma\mathcal{E}_0(\sigma,\varphi)\cdot\tau = \partial^+_\tau\mathcal{I}(\sigma)\},$$

is $\mathcal{L}(\Sigma)$-$\mathcal{B}(V)$ measurable and hence possesses a measurable selection.

Proof. The proof is an application of Theorem 8.2.9 from [27, Inverse Image]. We define $F : \Sigma \to \mathcal{P}(V)$ by $F(\sigma) = \Phi(\sigma)$. From the convergence principle (3.1) it follows that the sets $\Phi(\sigma)$ are compact and that the graph of F is closed. Hence, by characterization Theorem 8.1.4 in [27], the mapping F is measurable. Moreover, $\partial_\tau^+\mathcal{I} : \operatorname{int}\Sigma \to \mathbb{R}$ is measurable since it is lower semicontinuous. Finally we define $g : \Sigma \times V \to \mathbb{R}_\infty$ through

$$g(\sigma, \varphi) = \begin{cases} \partial_\sigma \mathcal{E}_0(\sigma, \varphi) \cdot \tau & \text{if } \varphi \in \Phi(\sigma), \\ \infty & \text{else.} \end{cases}$$

It follows again from the convergence principle (3.1) that g is lower semicontinuous on $\Sigma \times V$ and hence g is $(\mathcal{L}(\Sigma) \times \mathcal{B}(V))$-$\mathcal{B}(\mathbb{R}_\infty)$ measurable (since it has closed sublevels). Thus, Theorem 8.2.9 in [27] gives the measurability of H. □

3.2. Application to the crack problem

The goal of this subsection is to provide assumptions on the elastic energy density and the data such that the corresponding reduced energy functional from (2.2) has the properties discussed in the previous subsection.

We consider finite-strain elasticity and assume that the stored energy density $\widetilde{W} : \mathbb{R}^{2\times 2} \to [0, \infty]$ is polyconvex and coercive. More precisely, we assume the following (see [12]).

(H1) $\widetilde{W} : \mathbb{R}^{2\times 2} \to [0, \infty]$ is frame indifferent and polyconvex, that is there exists a lower semicontinuous and convex map $g : \mathbb{R}^5 \to [0, \infty]$ with $\widetilde{W}(F) = g(F, \det F)$, for every $F \in \mathbb{R}^{2\times 2}$. Moreover, $\widetilde{W}(F) = \infty$ if $\det F \le 0$.

(H2) There exist constants $p \in\ ]2, \infty[$, $c_0 \in \mathbb{R}$ and $c_1 > 0$ such that

$$\widetilde{W}(F) \ge c_0 + c_1 |F|^p \quad \forall F \in \mathbb{R}^{2\times 2}. \tag{3.8}$$

(H3) Multiplicative stress control condition: $\widetilde{W}$ is differentiable on the set of 2×2 matrices with positive determinant, here denoted by $\mathbb{R}^{2\times 2}_+$, and there exists a constant $c_3 > 0$ such that for every $F \in \mathbb{R}^{2\times 2}_+$

$$\left|F^\top \mathrm{D}\widetilde{W}(F)\right| \le c_3(\widetilde{W}(F) + 1). \tag{3.9}$$

(H4) (H3) is satisfied and there exist constants $c_4 \in \mathbb{R}$ and $\gamma > 0$ such that for all $F \in \mathbb{R}^{2\times 2}_+$ and all $N \in \mathcal{N}_\gamma := \{N \in \mathbb{R}^{2\times 2} \mid |N - \mathbb{I}| < \gamma\}$ we have

$$\left|F^\top \mathrm{D}\widetilde{W}(F) - (FN)^\top \mathrm{D}\widetilde{W}(FN)\right| \le c_4 |N - \mathbb{I}| (\widetilde{W}(F) + 1). \tag{3.10}$$

Growth condition (H3) and frame indifference entail a similar estimate for $\left|\mathrm{D}\widetilde{W}(F)F^\top\right|$, see [16,17]. We refer to [12] for an example of an energy density satisfying the above assumptions.

For the Dirichlet datum we assume

$$\begin{aligned} &g_{\mathrm{Dir}} \in \mathrm{C}^{1,1}([0, T] \times \mathbb{R}^2; \mathbb{R}^2), \\ &\nabla g_{\mathrm{Dir}} \in \mathrm{BC}^1([0, T] \times \mathbb{R}^2; \mathrm{Lin}(\mathbb{R}^2; \mathbb{R}^2)), \\ &\left|\nabla g_{\mathrm{Dir}}(t, x)^{-1}\right| \le C \text{ for all } (t, x) \in [0, T] \times \mathbb{R}^2, \\ &\int_\Omega \widetilde{W}(\nabla g_{\mathrm{Dir}}(t, x))\mathrm{d}x < \infty, \end{aligned} \tag{3.11}$$

where BC stands for bounded and continuous, see also [18]. The applied forces shall satisfy

$$\begin{aligned} &f \in \mathrm{C}^{1,1}([0, T] \times \overline{\Omega}; \mathbb{R}^2), \\ &h \in \mathrm{C}^{1,1}([0, T]; L^q(\Gamma_N; \mathbb{R}^2)), \end{aligned} \tag{3.12}$$

with $p^{-1} + q^{-1} = 1$. The functional ℓ is defined as in (2.1).

For given $t \in [0, T]$, $\psi \in \mathbb{R}^2$ and $F \in \mathbb{R}^{2\times 2}$ we set

$$W(t, \psi, F) := \widetilde{W}(\nabla g_{\mathrm{Dir}}(t, \psi)F)$$

and define the energy functional

$$\mathcal{E} : [0, T] \times \mathrm{W}^{1,p}_{\Gamma_\mathrm{D}}(\Omega_{s_1}; \mathbb{R}^2) \times [s_0, s_1] \to \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$$

by

$$\mathcal{E}(t, \varphi, s) := \int_{\Omega_s} W(t, \varphi(x), \nabla\varphi(x))\, \mathrm{d}x - \langle \ell(t), g_{\mathrm{Dir}}(t, \varphi)\rangle \tag{3.13}$$

for $\varphi \in \mathrm{W}^{1,p}_{\Gamma_\mathrm{D}}(\Omega_s; \mathbb{R}^2)$, and $\mathcal{E}(t, \varphi, s) := \infty$ else.

In order to study the continuity and differentiability properties of the corresponding reduced functional, we locally transform energies from domains with different crack lengths to a domain with a fixed crack length. For $s \in [s_0, s_1]$ and $|\delta| \le \delta_0$ (small enough) let $T_{s,\delta} : \mathbb{R}^2 \to \mathbb{R}^2$ be a family of diffeomorphisms having the properties

(T1) $T_{s,\cdot} \in \mathrm{C}^2([-\delta_0, \delta_0] \times \mathbb{R}^2; \mathbb{R}^2)$ and for every $|\delta| \le \delta_0$ the mapping $T_{s,\delta}$ is a $\mathrm{C}^{2,1}$-diffeomorphism on $\mathbb{R}^2$ with $T_{s,\delta}(\Omega_s) = \Omega_{s+\delta}$, $T_{s,\delta}(\gamma(s)) = \gamma(s + \delta)$, $T_{s,\delta}(\mathcal{C}_s) = \mathcal{C}_{s+\delta}$ and $T_{s,\delta}(x) = x$ for every $x \in \mathbb{R}^2 \setminus B_{r_0}(\gamma(s))$ and some $r_0 > 0$.

We recall that $s \mapsto \gamma(s)$ is the arc-length parameterization of the crack $\mathcal{C}$.

(T2) $\sup_s \left\|T_{s,\cdot}\right\|_{\mathrm{C}^2([-\delta_0,\delta_0]\times\mathbb{R}^2)} + \left\|T_{s,\cdot}^{-1}\right\|_{\mathrm{C}^2([-\delta_0,\delta_0]\times\mathbb{R}^2)} < \infty$ and there exist constants $c_1, c_2 > 0$ such that for every $x \in \mathbb{R}^2$, s and δ we have $c_1 \le \det \nabla T_{s,\delta}(x) \le c_2$.

Furthermore, we define $\varrho_s(x) := \partial_\delta(T_{s,\delta}(x))|_{\delta=0}$ and assume that

(T3) $\partial_\delta(\det \nabla T_{s,\delta})|_{\delta=0} = \operatorname{div} \varrho_s$, $\partial_\delta(\nabla T_{s,\delta})^{-1}|_{\delta=0} = -\nabla\varrho_s$.

We refer to [6] for an explicit construction of the family $T_{s,\delta}$ for a $\mathrm{C}^{2,1}$-smooth crack. Furthermore, we use the following abbreviations

$$x_\delta(y) = T_{s,\delta}(y), \qquad q_\delta(y) = \det \nabla T_{s,\delta}(y),$$
$$B_\delta(y) = (\nabla T_{s,\delta}(y))^{-1}.$$

For fixed s and elements $\varphi \in \mathrm{W}^{1,p}_{\Gamma_\mathrm{D}}(\Omega_s)$ we define the energy

$$\begin{aligned} \mathcal{E}_s(t, \varphi, \delta) := &\int_{\Omega_s} q_\delta(y) W(t, \varphi(y), \nabla\varphi(y)B_\delta(y))\mathrm{d}y \\ &- \langle \ell_\delta(t), g_{\mathrm{Dir}}(t, \varphi(\cdot))\rangle, \end{aligned} \tag{3.14}$$

with

$$\langle \ell_\delta(t), \psi\rangle = \int_{\Omega_s} q_\delta f(t, x_\delta(y)) \cdot \psi(y)\mathrm{d}y + \int_{\Gamma_\mathrm{N}} h(t) \cdot \psi(y)\, \mathrm{d}\mathscr{H}^1.$$

The definition of $\mathcal{E}_s$ is chosen in such a way that for $\varphi \in \mathrm{W}^{1,p}_{\Gamma_\mathrm{D}}(\Omega_s)$ we have

$$\mathcal{E}(t, \varphi \circ T_{s,\delta}^{-1}, s + \delta) = \mathcal{E}_s(t, \varphi, \delta). \tag{3.15}$$

For the reduced functional $\mathcal{I}$ corresponding to $\mathcal{E}$ we find

$$\mathcal{I}(t, s + \delta) = \min\{\mathcal{E}_s(t, \psi, \delta) \mid \psi \in \mathrm{W}^{1,p}_{\Gamma_\mathrm{D}}(\Omega_s)\},$$

and φ minimizes $\mathcal{E}(t, \cdot, s + \delta)$ if and only if $\varphi \circ T_{s,\delta}$ minimizes $\mathcal{E}_s(t, \cdot, \delta)$. Thus the properties of the functional $\mathcal{E}_s$ determine the continuity and differentiability properties of $\mathcal{I}$.

Proposition 3.8. *Let* (H1)–(H4) *be satisfied, assume that* (3.11) *and* (3.12) *are valid and let the crack path be* $\mathrm{C}^{2,1}$*-smooth. Then for every* $s \in [s_0, s_1]$ *the energy functional* $\mathcal{E}_s$ *from* (3.14) *satisfies* (E1)–(E3) *on* $\Sigma = [0, T] \times [-\delta_0, \delta_0]$ *and with* $V = \mathrm{W}^{1,p}_{\Gamma_\mathrm{D}}(\Omega_s)$ *equipped with the weak topology.*

Corollary 3.9. *Let* (H1)–(H4) *be satisfied. Then the reduced energy functional* $\mathcal{I}$ *is Lipschitz continuous on* $[0, T] \times [s_0, s_1]$*, has left and right derivatives with respect to t and s for every* $(t, s) \in [0, T] \times [s_0, s_1]$ *and satisfies the properties stated in Section* 3.1.

Explicit formulas for the derivatives of $\mathcal{I}$ are derived in [12].

Proof of Proposition 3.8. Due to the identity (3.15) it is sufficient to show that for every $(t, s) \in [0, T] \times [s_0, s_1]$ the energy $\mathcal{E}(t, \cdot, s)$ satisfies (E1) on $W^{1,p}_{\Gamma_D}(\Omega_s)$. But this follows from (H1), (H2) and the assumptions on g_{Dir} with similar arguments as in Section 5.1 of [30].

Using the growth properties (H3) and (H4) it can be shown with similar arguments as in [12] that $\partial_\delta \mathcal{E}_s(t, \varphi, \delta)$ exists for every $\varphi \in W^{1,p}_{\Gamma_D}(\Omega_s)$ for which $\mathcal{E}_s(t, \varphi, \delta) < \infty$ and is given by

$$\begin{aligned}\partial_\delta \mathcal{E}_s(t, \varphi, \delta) = \int_{\Omega_s} & \partial_\delta q_\delta W(t, \varphi, \nabla\varphi B_\delta) \\ & + q_\delta \nabla\varphi^\top \mathrm{D}_F W(t, \varphi, \nabla\varphi B_\delta) : \partial_\delta B_\delta \,\mathrm{d}y \\ & - \int_{\Omega_s} g_{\mathrm{Dir}}(t, \varphi) \cdot \partial_\delta (q_\delta f(t, x_\delta))\,\mathrm{d}y.\end{aligned}$$

For $\delta = 0$ we have in particular

$$\begin{aligned}\partial_\delta \mathcal{E}_s(t, \varphi, 0) &= \int_{\Omega_s} \big(W(t, \varphi, \nabla\varphi)\mathbb{I} - \nabla\varphi^\top \mathrm{D}_F W(t, \varphi, \nabla\varphi)\big) : \nabla \varrho_s \,\mathrm{d}y \\ &\quad - \int_{\Omega_s} g_{\mathrm{Dir}}(t, \varphi) \cdot \operatorname{div} f(t) \otimes \varrho_s \,\mathrm{d}y, \qquad (3.16)\end{aligned}$$

which is the well-known Griffith formula. The derivative with respect to t exists as well and is given by (see [30])

$$\begin{aligned}\partial_t \mathcal{E}_s(t, \varphi, \delta) = \int_{\Omega_s} & q_\delta \mathrm{D}_F W(t, \varphi, \nabla\varphi B_\delta)(\nabla\varphi B_\delta)^\top \\ & : \nabla g_{\mathrm{Dir}}^{-1}(t, \varphi)\partial_t \nabla g_{\mathrm{Dir}}(t, \varphi)\,\mathrm{d}y \\ & - \int_{\Omega_s} q_\delta \partial_t \big(f(t, x_\delta) \cdot g_{\mathrm{Dir}}(t, \varphi)\big)\,\mathrm{d}y \\ & - \int_{\Gamma_N} \partial_t \big(h(t) \cdot g_{\mathrm{Dir}}(t, \varphi)\big)\,\mathrm{d}\mathscr{H}^1.\end{aligned}$$

Here, we use that $AB : C = B : A^\top C$ and $\mathrm{D}_F W(t, \psi, F) = \nabla g_{\mathrm{Dir}}(t, \psi)^\top \mathrm{D}\widetilde{W}(\nabla g_{\mathrm{Dir}}(t, \psi)F)$.

The estimate in (E2) follows in the same way as in [30] taking into account that assumption (H3) implies a similar estimate for $\big|\mathrm{D}\widetilde{W}(F)F^\top\big|$, see [16].

Property (E3) follows with similar arguments as in the proof of Theorem 5.3 in [30]. Here one needs the strong differentiability assumptions on the data f and on the smoothness of the crack path. □

As already mentioned in Section 2.1, the quantity $\mathcal{G}(t, s) = -\partial_s^+ \mathcal{I}(t, s)$ is the total energy-release rate, whereas the quantities

$$G(t, \varphi, s) = -\partial_\delta \mathcal{E}_s(t, \varphi, 0), \quad \varphi \in \Phi(t, s),$$

with $\partial_\delta \mathcal{E}_s(t, \varphi, 0)$ from (3.16), can be interpreted as local energy-release rates. Assume that for some (t, s) there exist minimizers $\varphi_1, \varphi_2 \in \Phi(t, s)$ with $G(t, \varphi_1, s) < \kappa(s) < G(t, \varphi_2, s) = \mathcal{G}(t, s)$. The special local energetic model, which is based on the total energy-release rate $\mathcal{G}(t, s)$, would predict an immediate jump of the crack tip, whereas in the (non-special) local energetic model it is also possible that the body stays in the configuration φ_1 and that the crack does not grow. In this sense, the special local energetic model predicts the worst case. However, the construction of an example for a body with crack having at least two minimizers with $G(t, \varphi_1, s) \neq G(t, \varphi_2, s)$ remains a challenging open problem.

4. Construction of a local energetic solution

In this section, we prove the existence of a local energetic solution by using a vanishing-viscosity approach (see, e.g., [13,6,14]). Here, we make a particular choice of the viscous parameter ν in terms of the discrete time step τ, namely $\nu = \sqrt{\tau}$. Thus, by passing to the limit as τ goes to zero, we obtain directly a local energetic solution. The same choice for the viscous parameter was recently considered in [31] for a related problem. Moreover, we note that for our analysis any choice $\nu = \widehat{\nu}(\tau)$ would be possible as long as we have $\widehat{\nu}(\tau) + \tau/\widehat{\nu}(\tau) \to 0$ for $\tau \to 0$. In [4] the choice $\widehat{\nu}(\tau) = \lambda\tau$, with $\lambda > 0$, was used.

We observe that one could also use [27, Thm. 10.1.4] to obtain the existence of a *viscous solution* $s^\nu \in \mathrm{H}^1([0, T])$ for

$$0 \in \partial_{\dot s} \mathcal{R}(s^\nu(t), \dot s^\nu(t)) + \partial_s^{\mathrm{Cl}} \mathcal{I}(t, s^\nu(t)) + \nu \dot s^\nu(t) + \partial \chi_{[s_0, s_1]}(s^\nu(t))$$

first, and then prove that the limit of a (sub)sequence of s^ν as $\nu \to 0$ is a local energetic solution. However, we prefer to present the construction of a local energetic solution via incremental problems and the choice $\nu = \sqrt{\tau}$, since this approach was not previously applied to this context, and it seems to open up more general applications.

For the remaining part of the paper, we will assume (H1)–(H4) from Section 3.2 in order to have Corollary 3.9 satisfied. In addition, we will assume

$$\kappa(s_1) > \max_{(t,s)} \big[-\partial_s^+ \mathcal{I}(t, s)\big] =: G_{\max}. \qquad (4.1)$$

This condition will prevent the evolution $s(t)$ from reaching the endpoint s_1.

On the other hand, in order to obtain a nontrivial evolution, we will assume

$$\kappa(s_0) < G_{\max}. \qquad (4.2)$$

4.1. Incremental minimum problems

We define a discrete version of the local energetic solution and derive the main estimates in order to pass to the limit in the next subsection. This part follows essentially the same lines of [6, Section 4.1], by substituting ν with $\sqrt{\tau}$, and therefore the similar proofs are only sketched. The main difference is that here we deal with the set-valued mapping $\partial_s^{\mathrm{Cl}} \mathcal{I}$.

For $N \in \mathbb{N} \setminus \{0\}$ we define the time step $\tau = T/N$ and a partition of $[0, T]$ by means of $t^k := k\tau$ for $k = 0, 1, \ldots, N$. We define by induction s^k by setting $s^0 := s_0$, while for $k \geq 1$ the value s^k is defined by

$$s^k \in \operatorname{Argmin}\left\{ \mathcal{I}(t^k, \widetilde{s}) + \tau \mathcal{R}\Big(s^{k-1}, \frac{\widetilde{s} - s^{k-1}}{\tau}\Big) + \frac{\big|\widetilde{s} - s^{k-1}\big|^2}{2\sqrt{\tau}} \,\Big|\, \widetilde{s} \in [s_0, s_1] \right\}.$$

The existence of s^k is an easy consequence of the direct method in the calculus of variations, since $s \mapsto \mathcal{I}(t^k, s)$ is (Lipschitz) continuous and $s \mapsto \mathcal{R}(s^{k-1}, \frac{s - s^{k-1}}{\tau})$ is lower semicontinuous. Moreover, s^k satisfies

$$\begin{aligned}0 \in\ & \partial_{\dot s} \mathcal{R}\Big(s^{k-1}, \frac{s^k - s^{k-1}}{\tau}\Big) \\ & + \partial_s^{\mathrm{Cl}} \mathcal{I}(t^k, s^k) + \frac{s^k - s^{k-1}}{\sqrt{\tau}} + \partial \chi_{[s_0, s_1]}(s^k), \qquad (4.3)\end{aligned}$$

for $k = 1, \ldots, N$. But we even have the stronger version

$$0 \in \partial_{\dot{s}}\mathcal{R}\Big(s^{k-1}, \frac{s^k - s^{k-1}}{\tau}\Big) + \partial_s^+\mathcal{I}(t^k, s^k) + \frac{s^k - s^{k-1}}{\sqrt{\tau}} + \partial\chi_{[s_0,s_1]}(s^k), \tag{4.4}$$

for every $k \in \{1, \ldots, N\}$ with $s^k < s_1$. Indeed, if $s^k = s^{k-1}$, then

$$\mathcal{I}(t^k, \widetilde{s}) + \tau\mathcal{R}\Big(s^{k-1}, \frac{\widetilde{s} - s^{k-1}}{\tau}\Big) + \frac{\big|\widetilde{s} - s^{k-1}\big|^2}{2\sqrt{\tau}} \geq \mathcal{I}(t^k, s^{k-1})$$

for every $\widetilde{s} \in [s_0, s_1]$. The nontrivial case happens when $\widetilde{s} \geq s^{k-1}$, and this implies relation (4.4). On the other hand, if $s^k > s^{k-1}$, then s^k is a point of differentiability for $\mathcal{I}(t^k, \cdot)$ and the generalized Clarke gradient reduces to $\partial_s^+\mathcal{I}(t^k, s^k) = \partial_s^-\mathcal{I}(t^k, s^k)$.

If $s^k < s_1$, then by (4.4) the following relation holds

$$\Big(\kappa(s^{k-1}) + \partial_s^+\mathcal{I}(t^k, s^k) + \sqrt{\tau}\frac{s^k - s^{k-1}}{\tau}\Big)\frac{s^k - s^{k-1}}{\tau} = 0. \tag{4.5}$$

This follows by applying the same argument proving formula (4.3) in [6].

We continue by defining the interpolants associated with (t^k, s^k). We set $\bar{t}_\tau : [0, T] \to [0, T]$ by

$$\bar{t}_\tau(0) := 0, \qquad \bar{t}_\tau(t) := t^k \quad \text{for } t \in \,]t^{k-1}, t^k].$$

We define $\bar{s}_\tau$ and $\underline{s}_\tau$ as the left continuous and right continuous piecewise constant interpolants of s^k such that $\bar{s}_\tau(t^k) = \underline{s}_\tau(t^k) = s^k$, i.e., for $k = 1, \ldots, N$, $t \in]t^{k-1}, t^k]$

$$\bar{s}_\tau(t) := s^k \qquad \underline{s}_\tau(t) := s^{k-1}. \tag{4.6}$$

Moreover we introduce the piecewise affine interpolant $\hat{s}_\tau$ given by

$$\hat{s}_\tau(t) := s^{k-1} + \frac{t - t^{k-1}}{\tau}(s^k - s^{k-1}) \quad \forall t \in \,]t^{k-1}, t^k]. \tag{4.7}$$

Hence, the time-incremental problem (4.4) can be rewritten in terms of the interpolants by

$$0 \in \partial_{\dot{s}}\mathcal{R}(\underline{s}_\tau(t), \dot{\hat{s}}_\tau(t)) + \partial_s^+\mathcal{I}(\bar{t}_\tau(t), \bar{s}_\tau(t)) + \sqrt{\tau}\dot{\hat{s}}_\tau(t) + \partial\chi_{[s_0,s_1]}(\bar{s}_\tau(t)). \tag{4.8}$$

In the next result we collect suitable a priori bounds.

Lemma 4.1. *There exists a positive constant C such that for every $\tau > 0$ the following estimates hold:*

$$\|\underline{s}_\tau\|_{L^\infty(0,T)}, \|\bar{s}_\tau\|_{L^\infty(0,T)} \leq C; \tag{4.9}$$

$$\int_0^T \Big(\mathcal{R}(\underline{s}_\tau(t), \dot{\hat{s}}_\tau(t)) + \frac{\sqrt{\tau}}{2}|\dot{\hat{s}}_\tau(t)|^2\Big)\,dt \leq C; \tag{4.10}$$

$$\|\dot{\hat{s}}_\tau\|_{L^2(0,T)} \leq \frac{C}{\tau^{1/4}}; \tag{4.11}$$

$$\|\bar{s}_\tau - \hat{s}_\tau\|_{L^\infty(0,T)}, \|\underline{s}_\tau - \hat{s}_\tau\|_{L^\infty(0,T)} \leq C\tau^{1/4}. \tag{4.12}$$

Moreover there exists $\tau_0 > 0$ such that for $\tau \in \,]0, \tau_0[$ we have

$$\bar{s}_\tau(t) < s_1 \quad \forall t \in [0, T]. \tag{4.13}$$

Proof. It is sufficient to follow the arguments in the proof of [6, Lem. 4.1], by taking $\nu = \sqrt{\tau}$. □

From now on we will consider $\tau < \tau_0$ so that, thanks to (4.13) the time-incremental problem (4.8) becomes

$$0 \in \partial_{\dot{s}}\mathcal{R}(\underline{s}_\tau(t), \dot{\hat{s}}_\tau(t)) + \partial_s^+\mathcal{I}(\bar{t}_\tau(t), \bar{s}_\tau(t)) + \sqrt{\tau}\dot{\hat{s}}_\tau(t). \tag{4.14}$$

4.2. Limit as $\tau \to 0$

We pass to the limit in τ and prove existence of a local energetic solution.

Theorem 4.2. *There exist a function $s \in \mathrm{BV}([0, T]; [s_0, s_1])$ and a subsequence of τ (not relabeled) such that*

$$\hat{s}_\tau \overset{*}{\rightharpoonup} s \quad \text{in } \mathrm{BV}([0, T]; [s_0, s_1]) \tag{4.15}$$

$$\hat{s}_\tau(t) \to s(t) \quad \text{for all } t \in [0, T]. \tag{4.16}$$

Moreover, the limit function s is a local energetic solution (cf. Definition 2.1) for $\mathcal{R}$ and $\mathcal{I}$ satisfying $s(0) = s_0$.

Proof. An application of the classical Helly selection theorem provides the existence of a subsequence of τ and of a non-decreasing function $s \in \mathrm{BV}([0, T]; [s_0, s_1])$ satisfying the convergence conditions (4.15) and (4.16) and the monotonicity in (a) of Definition 2.1.

In order to prove the remaining conditions (b)–(d) we pass to the limit in the formulation (4.14).

We observe that a priori bound (4.11) implies

$$\sqrt{\tau}\dot{\hat{s}}_\tau \to 0 \quad \text{in } L^2([0, T]). \tag{4.17}$$

From (4.14) and for every $\psi \in L^2([0, T])$ with $\psi \geq 0$ it follows that

$$0 \leq \int_0^T \psi(t)\big[\kappa(\underline{s}_\tau(t)) + \partial_s^+\mathcal{I}(\bar{t}_\tau(t), \bar{s}_\tau(t)) + \sqrt{\tau}\dot{\hat{s}}_\tau(t)\big]\,dt$$
$$\leq \int_0^T \psi(t)\big[\kappa(\underline{s}_\tau(t)) + \partial_s^-\mathcal{I}(\bar{t}_\tau(t), \bar{s}_\tau(t)) + \sqrt{\tau}\dot{\hat{s}}_\tau(t)\big]\,dt,$$

where in the last inequality we used the characterization $\partial_s^{Cl}\mathcal{I} = [\partial_s^+\mathcal{I}, \partial_s^-\mathcal{I}]$ given by part (a) of Lemma 3.3. Thanks to the upper semicontinuity of $\partial_s^-\mathcal{I}$ (provided by Theorem 3.2) and (4.17), we can pass to the limit and (by using Fatou's lemma) obtain an integral version of condition (b), namely for all $\psi \in L^2([0, T])$, $\psi \geq 0$ we have

$$\int_0^T \psi(t)\big(\kappa(s(t)) + \partial_s^-\mathcal{I}(t, s(t))\big)\,dt \geq 0.$$

Then, $\kappa(s(t)) + \partial_s^-\mathcal{I}(t, s(t)) \geq 0$ for a.e. $t \in [0, T]$. In particular, the inequality is true for every t in which the map s is continuous, and therefore condition (b) is proved.

We continue by proving condition (d). Let us fix $t_* \in J(s)$ and $s(t_*^-) < s^a < s^b < s(t_*^+)$. From the continuity of the map $t \mapsto \hat{s}_\tau(t)$ we deduce that for every sufficiently small τ there exist t_τ^a and t_τ^b such that

$$t_\tau^a < t_\tau^b, \quad t_\tau^a \to t_*, \quad t_\tau^b \to t_*, \quad \hat{s}_\tau(t_\tau^a) \equiv s^a, \quad \hat{s}_\tau(t_\tau^b) \equiv s^b.$$

Condition (4.5) implies that for every $\varphi \in L^2([s_0, s_1])$ with $\varphi \geq 0$ we have

$$0 \geq \int_{t_\tau^a}^{t_\tau^b} \big[\kappa(\underline{s}_\tau(t)) + \partial_s^+\mathcal{I}(\bar{t}_\tau(t), \bar{s}_\tau(t))\big]\varphi(\hat{s}_\tau(t))\dot{\hat{s}}_\tau(t)\,dt, \tag{4.18}$$

since $\sqrt{\tau}|\dot{\hat{s}}_\tau(t)|^2 \geq 0$. Now, as in the proof of [6, Thm. 5.2] we change variables. If $\sigma := \hat{s}_\tau(t)$ and $\hat{t}_\tau(\sigma) := \min\{t \in [t_\tau^a, t_\tau^b] \mid \hat{s}_\tau(t) = \sigma\}$ then inequality (4.18) becomes

$$\int_{s^a}^{s^b} \big[\kappa(\underline{s}_\tau(\hat{t}_\tau(\sigma))) + \partial_s^+\mathcal{I}(\bar{t}_\tau(\hat{t}_\tau(\sigma)), \bar{s}_\tau(\hat{t}_\tau(\sigma)))\big]\varphi(\sigma)\,d\sigma \leq 0$$

for every $\varphi \in L^2([s_0, s_1])$, $\varphi \geq 0$. In order to pass to the limit as $\tau \to 0$, we observe first that $t_\tau(\hat{t}_\tau(\sigma)) \to t_*$, and secondly that

$$|\underline{s}_\tau(\hat{t}_\tau(\sigma)) - \sigma| \leq C\tau^{1/4}, \quad \text{and} \quad |\bar{s}_\tau(\hat{t}_\tau(\sigma)) - \sigma| \leq C\tau^{1/4}$$

by using the fact that $\sigma = \hat{s}_\tau(\hat{t}_\tau(\sigma))$ and (4.12). Thanks to the lower semicontinuity of $\partial_s^+\mathcal{I}$ (provided by Theorem 3.2), by applying Fatou's lemma we pass to the limit as $\tau \to 0$ and get

$$\int_{s^a}^{s^b} \big(\kappa(\sigma) + \partial_s^+\mathcal{I}(t_*, \sigma)\big)\varphi(\sigma)\,\mathrm{d}\sigma \le 0.$$

Therefore, since $\partial_s^+\mathcal{I}(t_*, \cdot)$ is lower semicontinuous, we conclude $\kappa(s_*) + \partial_s^+\mathcal{I}(t_*, s_*) \le 0$ for every $s_* \in [s^a, s^b]$ and, by the fact that s^a and s^b were arbitrarily chosen in $[s(t_*^-), s(t_*^+)]$, we finally obtain condition (d).

We are left with condition (c). Let t be such that $\kappa(s(t)) + \partial_s^+\mathcal{I}(t, s(t)) =: \eta > 0$. Then by condition (d) $t \notin J(s)$ so that the map s is continuous in t. The continuity of κ and the lower semicontinuity of $\partial_s^+\mathcal{I}$ imply (e.g., arguing by contradiction) that there exist $\varepsilon, \delta > 0$ (and independent of τ) such that for all $(\tilde{t}, \tilde{s})$ with $|\tilde{s} - s(t)| \le \varepsilon$ and $|\tilde{t} - t| \le \delta$ we have $\kappa(\tilde{s}) + \partial_s^+\mathcal{I}(\tilde{t}, \tilde{s}) \ge \eta/2 > 0$.

Since $\|\bar{s}_\tau - \underline{s}_\tau\|_{L^\infty(0,T)} \le C\tau^{1/4}$ (by using (4.12)), there exists $\tilde{\tau} = \tilde{\tau}(\varepsilon)$ such that for every $0 < \tau < \tilde{\tau}$ and every $|\tilde{t} - t| \le \delta$ we have $\kappa(\underline{s}_\tau(\tilde{t})) + \partial_s^+\mathcal{I}(\bar{t}_\tau(\tilde{t}), \bar{s}_\tau(\tilde{t})) > 0$. Then $\bar{s}_\tau = \underline{s}_\tau = \hat{s}_\tau =$ const. on $[t - \delta, t + \delta]$ for all $0 < \tau < \tilde{\tau}$. Therefore, the limit map s is constant on $[t-\delta, t+\delta] \cap [0, T]$, so that $t \in D(s)$ and $\dot{s}(t) = 0$. This concludes the proof of condition (c) and the theorem is proved. □

4.3. *An example with many local energetic solutions*

Here we provide an explicit example showing that the notion of local energetic solution may allow for too many solutions. Let $q(t) := -2 + \frac{t}{2}$ and assume

$$\mathcal{E}(t, \varphi, s) := \begin{cases} \frac{1}{2}s^2 - st + q(t)t & \text{if } \varphi = 0 \\ \frac{1}{2}s^2 & \text{if } \varphi = 1. \end{cases}$$

Then the reduced energy functional $\mathcal{I}$ is given by

$$\mathcal{I}(t, s) := \min\{\mathcal{E}(t, 0, s), \mathcal{E}(t, 1, s)\} = \begin{cases} \mathcal{E}(t, 1, s) & \text{if } s \le q(t), \\ \mathcal{E}(t, 0, s) & \text{if } s \ge q(t), \end{cases}$$

while the partial Clarke gradient $\partial_s^{\mathrm{Cl}}\mathcal{I}$ is given by

$$\partial_s^{\mathrm{Cl}}\mathcal{I}(t, s) = \begin{cases} \{s\} & \text{if } s < q(t) \\ [q(t) - t, q(t)] & \text{if } s = q(t) \\ \{s - t\} & \text{if } s > q(t). \end{cases}$$

Assume $\kappa(s) \equiv 3$. We are in the following situation. The crack tip sticks (i.e., no motion of crack tip, represented with horizontal arrows in Fig. 4.1) in the region above the line $\kappa + \partial_s\mathcal{E}(t, 0, s(t)) = 0$. The crack tip jumps (represented with vertical arrows in Fig. 4.1) in the region bounded by the lines $\kappa + \partial_s\mathcal{E}(t, 0, s(t)) = 0$ and $s = q(t)$. Again we have sticking in the region bounded by the lines $s = q(t)$ and $\kappa + \partial_s\mathcal{E}(t, 1, s(t)) = 0$, and jumping in the region below the line $\kappa + \partial_s\mathcal{E}(t, 1, s(t)) = 0$.

Now we make explicit the local energetic solutions associated with $\mathcal{I}$ and $\mathcal{R}$ in dependence of the choice of the initial value s_0. If $-3 \le s_0 < -1$, then the only local energetic solution is $s(t) \equiv s_0$. If $s_0 > -1$, then the function $s(\cdot)$ defined by

$$s(t) := \begin{cases} s_0 & \text{if } t \in [0, 3 + s_0], \\ t - 3 & \text{if } t \ge 3 + s_0; \end{cases}$$

is the only local energetic solution with initial condition $s(0) = s_0$ (see the upper figure in Fig. 4.2). In the case $s_0 = -1$ there are several local energetic solutions satisfying the initial condition $s(0) = s_0$ (see the lower figure in Fig. 4.2). Indeed there are both the constant solution $s_1(t) \equiv -1$, and

$$s_2(t) := \begin{cases} -1 & \text{if } t \in [0, 2], \\ t - 3 & \text{if } t \ge 2, \end{cases}$$

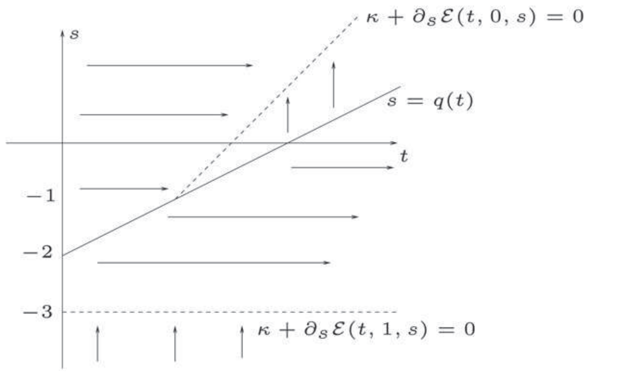

Fig. 4.1. The behavior of a local energetic solution: sticking above the line $\kappa + \partial_s\mathcal{E}(t, 0, s) = 0$, and jumping in the region between $\kappa + \partial_s\mathcal{E}(t, 0, s) = 0$ and $s = q(t)$. Again sticking in the region between $s = q(t)$ and $\kappa + \partial_s\mathcal{E}(t, 1, s) = 0$, and jumping below the line $\kappa + \partial_s\mathcal{E}(t, 1, s) = 0$.

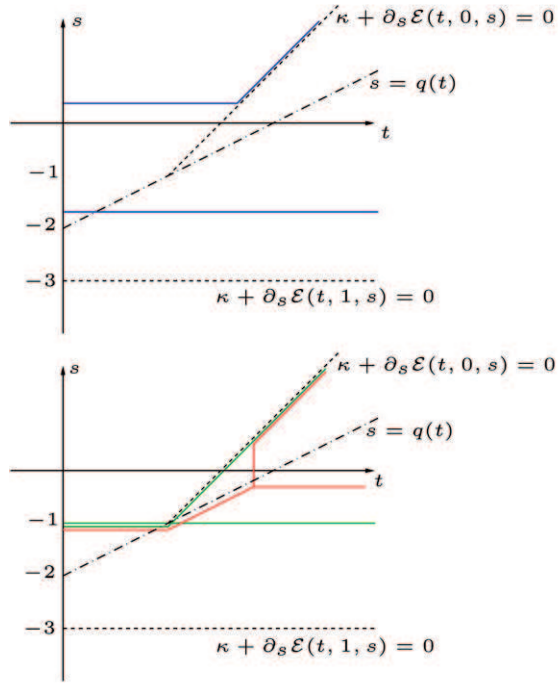

Fig. 4.2. Graph of local energetic solutions. More precisely, in the upper figure, the blue lines correspond to local energetic solutions with initial condition $s_0 > -1$ and $s_0 < -1$. In the lower figure, we depicted some solutions starting from $s_0 = -1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

and, additionally, for $j > 2$, also the following ones are local energetic solutions:

$$s_j(t) := \begin{cases} -1 & \text{if } t \in [0, 2], \\ q(t) & \text{if } t \in [2, j[\,, \\ w_j^{\pm}(t) & \text{if } t \ge j, \end{cases}$$

where $w_j^-(t) := \frac{j}{2} - 2$ and $w_j^+(t) := \begin{cases} \beta & \text{if } t = j, \\ t - 3 & \text{if } t > j, \end{cases}$ and $\beta \in [\frac{j}{2} - 2, j - 3]$ is arbitrary.

We believe that only one of the above solutions is relevant. The solutions s_j move along the highly unstable state $s(t) = q(t)$ for $t \in\ ]2, j[$. Under slight perturbations the system will prefer the solutions that avoid this state. These will be the special local energetic solutions discussed in Section 5.

5. Construction of special local energetic solutions

In the above setting it was easy to pass directly to the limit of τ and to obtain local energetic solutions. In a more general (non-scalar) setting this is more delicate. In such a situation it is advantageous to use an equivalent formulation using parameterized solutions, cf. [13,14], where time and state variable are Lipschitz functions of the arc-length parameter ρ. Thus, this technique has its justification in its own right. However, even in the present scalar case, it is essential to obtain information on the energy balance. For this, it turns out that the parameterized solutions are again very useful.

5.1. Equivalence of local energetic and parameterized solutions

For the equivalence between the local energetic solutions (LES) and the parameterized solution we have to identify the LES that coincide on a dense subset of $[0, T]$. This means that all LES are identified that differ only at jump times. Clearly, if we change a LES s only on its jump set $J(s)$ and still $s(t) \in [s(t^-), s(t^+)]$, then the new function is again a LES. We will show that the thus identified LES give rise to the same parameterized solution and we will describe how one can reconstruct the family of LES from a given parameterized solution.

The proofs of the propositions formulated in this Subsection are straightforward, but technical and are postponed to the Appendix.

Let $s \in \mathrm{BV}([0, T])$ be a non-decreasing function. We put $R = T + s(T)$ and define

$$\begin{aligned} &\widetilde{\rho} : [0, T] \to [0, R], \widetilde{\rho}(t) := t + s(t), \\ &t_p(\rho) := \sup\{t \mid \widetilde{\rho}(t) \le \rho\}, s_p(\rho) := \rho - t_p(\rho), \\ &\mathcal{J}_p := \Big\{\rho \in [0, R] | \exists \delta > 0 \text{ such that} \\ &\qquad t_p \text{ is constant on } B_\delta(\rho) \cap [0, R]\Big\}. \end{aligned} \tag{5.1}$$

Proposition 5.1. *Let $s \in \mathrm{BV}([0, T])$ be a local energetic solution according to Definition 2.1. Then the pair $(t_p, s_p) : [0, R] \to \mathbb{R}^2$ is a parameterized solution in the sense of Definition 2.3.*

In particular the Proposition shows that parameterized solutions exist. Now we start from a parameterized solution (t_p, s_p) and construct the corresponding LES. Let $(t_p, s_p) \in \mathrm{C}^{\mathrm{Lip}}([0, R], \mathbb{R}^2)$ be a parameterized solution in the sense of Definition 2.3 satisfying $T = t_p(R)$. Choose any function

$$\widehat{\rho} : [0, T] \to [0, R] \quad \text{with } t_p(\widehat{\rho}(t)) = t \quad \text{and} \quad \widehat{\rho}(0) = 0.$$

Such a function exists, since $t_p : [0, R] \to [0, T]$ is a continuous and monotone function, and hence is surjective. The function $\widehat{\rho} : [0, T] \to [0, R]$ is strictly increasing. For $t \in [0, T]$ we define s via

$$s(t) := s_p(\widehat{\rho}(t)),$$

which is a non-decreasing function with $s(0) = s_0$. Since $\widehat{\rho}$ is not unique if t_p has plateaus, we see that there may be many LES solutions for one parameterized solution.

Proposition 5.2. *If $(t_p, s_p) \in \mathrm{C}^{\mathrm{Lip}}([0, R]; [0, T] \times [s_0, s_1])$ is a parameterized solution with $t_p(0) = 0$ and $t_p(R) = T$, the function s defined above is a local energetic solution in the sense of Definition 2.1. Moreover,*

$$J(s) = J(\widehat{\rho}) = \{t \in [0, T] \mid \exists \rho \in \mathcal{J}_p \text{ with } t_p(\rho) = t\}$$

and for $t \in J(s)$ we have $[s(t^-), s(t^+)] = \mathrm{cl}\{s_p(\rho) \mid \rho \in \mathcal{J}_p, t = t_p(\rho)\}$.

If the pair (t_p, s_p) is a special parameterized solution, then s is a special local energetic solution in the sense of Definition 2.2.

Remark 5.3. Assume that (t_p, s_p) are non-decreasing Lipschitz continuous functions with $t_p(\rho) + s_p(\rho) = \rho$. Let $s \in \mathrm{BV}([0, T])$ be constructed as above. We define $(\widetilde{t}_p, \widetilde{s}_p)$ as in (5.1). Then $t_p = \widetilde{t}_p$ and $s_p = \widetilde{s}_p$.

5.2. Existence of special parameterized solutions

We are going to construct a special parameterized solution for $\mathcal{R}$ and $\mathcal{I}$, see Definition 2.4. For any $\rho \in [0, R]$ we define

$$\begin{aligned} t_m(\rho) &:= \inf\{t_p(\rho) | \exists s_p : (t_p, s_p) \text{ is a parameterized} \\ &\qquad \text{solution for } \mathcal{R} \text{ and } \mathcal{I} \text{ with } (t_p(0), s_p(0)) = (0, s_0)\} \\ s_m(\rho) &:= \sup\{s_p(\rho) | \exists t_p : (t_p, s_p) \text{ is a parameterized} \\ &\qquad \text{solution for } \mathcal{R} \text{ and } \mathcal{I} \text{ with } (t_p(0), s_p(0)) = (0, s_0)\}. \end{aligned} \tag{5.2}$$

Proposition 5.4. *The pair (t_m, s_m) is a parameterized solution (cf. Definition 2.3) for $\mathcal{R}$ and $\mathcal{I}$ satisfying the initial condition $(t_m(0), s_m(0)) = (0, s_0)$.*

Proof. For simplicity, during the proof we will assume $s_0 = 0$. We observe that by definition the pair (t_m, s_m) satisfies the initial condition and is non-decreasing. In order to complete the proof of condition (a_p) we need to show

$$t_m(\rho) + s_m(\rho) = \rho \tag{5.3}$$

for every $\rho \in [0, R]$. Indeed, for fixed $\rho \in [0, R]$, let (t_k, s_k) be a parameterized solution with $s_k(\rho) \nearrow s_m(\rho)$ as $k \to \infty$. Then $t_k(\rho) = \rho - s_k(\rho) \searrow \rho - s_m(\rho)$, and, by definition, $t_m(\rho) \le \rho - s_m(\rho)$. Similarly, using $t_j(\rho) \searrow t_m(\rho)$ implies $s_m(\rho) \ge \rho - t_m(\rho)$; therefore (5.3) holds, and the proof of (a_p) is complete.

Monotonicity and (5.3) imply that $t_m, s_m \in \mathrm{C}^{\mathrm{Lip}}([0, R])$ with Lipschitz constant $L \le 1$. To prove condition (b_p), let us suppose that $\kappa(s_m(\rho)) + \partial_s^- \mathcal{I}(t_m(\rho), s_m(\rho)) < 0$. We have to show that $\rho \in \mathcal{J}_m$. The continuity of $\kappa(\cdot)$ and the upper semicontinuity of $\partial_s^- \mathcal{I}(\cdot, \cdot)$ (provided by Theorem 3.2) imply that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\kappa(s) + \partial_s^- \mathcal{I}(t, s) \le -\varepsilon < 0 \quad \forall (t, s) \in B_\delta(t_m(\rho), s_m(\rho)) \tag{5.4}$$

where $B_\delta(t_m(\rho), s_m(\rho)) = [t_m(\rho) - \delta, t_m(\rho) + \delta] \times [s_m(\rho) - \delta, s_m(\rho) + \delta]$.

For any parameterized solution (t_p, s_p) with initial condition $(0, 0)$, we claim that the following holds true.

$$\begin{aligned} &(t_p(\rho), s_p(\rho)) \in B_{\frac{\delta}{2}}(t_m(\rho), s_m(\rho)) \\ &\quad \Rightarrow (t_p(\widetilde{\rho}), s_p(\widetilde{\rho})) \in B_\delta(t_m(\rho), s_m(\rho)) \quad \forall \widetilde{\rho} \in B_{\frac{\delta}{2}}(\rho). \end{aligned} \tag{5.5}$$

This is a consequence of the equality $t_p(\widetilde{\rho}) + s_p(\widetilde{\rho}) = \widetilde{\rho}$. Indeed by assumption the pair $(t_p(\rho), s_p(\rho))$ belongs to the dashed line in Fig. 5.1 and it is contained in the (smallest) square of radius $\delta/2$ centered at $(t_m(\rho), s_m(\rho))$. From $|t_p(\widetilde{\rho}) - t_p(\rho) + s_p(\widetilde{\rho}) - s_p(\rho)| = |\widetilde{\rho} - \rho| \le \delta/2$ for any $\widetilde{\rho} \in B_{\frac{\delta}{2}}(\rho)$, it follows that $|t_p(\widetilde{\rho}) - t_p(\rho)| \le \delta/2$ and $|s_p(\widetilde{\rho}) - s_p(\rho)| \le \delta/2$. Using then the triangular inequality, we obtain (5.5).

We note that by assumption, the worst possible case would be $(t_p(\rho), s_p(\rho))$ belonging to the boundary of the set $B_{\frac{\delta}{2}}(t_m(\rho), s_m(\rho))$. In this case, it turns out that for $\widetilde{\rho} \in [\rho - \frac{\delta}{2}, \rho]$ the pair $(t_p(\widetilde{\rho}), s_p(\widetilde{\rho}))$ belongs to the shaded triangle on the left of the line $t + s = \rho$, while it belongs to the shaded triangle to the right of $t + s = \rho$ for $\widetilde{\rho} \in [\rho, \rho + \frac{\delta}{2}]$, see Fig. 5.1.

Consider a sequence (t_ℓ, s_ℓ) of parameterized solutions with initial condition $(0, 0)$ such that $s_\ell(\rho - \frac{\delta}{6}) \nearrow s_m(\rho - \frac{\delta}{6})$. Consequently, $t_\ell(\rho - \frac{\delta}{6}) \searrow t_m(\rho - \frac{\delta}{6})$. Using (5.4) and (5.5) there exists $\ell_0 > 0$ such that, for every $\ell > \ell_0$,

$$\kappa(s_\ell(\widetilde{\rho})) + \partial_s^- \mathcal{I}(t_\ell(\widetilde{\rho}), s_\ell(\widetilde{\rho})) \le -\varepsilon/2 < 0 \quad \forall \widetilde{\rho} \in B_{\frac{\delta}{3}}\left(\rho - \frac{\delta}{6}\right).$$

Condition (b_p) applied to (t_ℓ, s_ℓ) yields

$$(t_\ell(\widetilde{\rho}), s_\ell(\widetilde{\rho})) = (t_\ell(\rho), s_\ell(\rho) + \widetilde{\rho} - \rho) \quad \forall \widetilde{\rho} \in B_{\frac{\delta}{3}}\left(\rho - \frac{\delta}{6}\right).$$

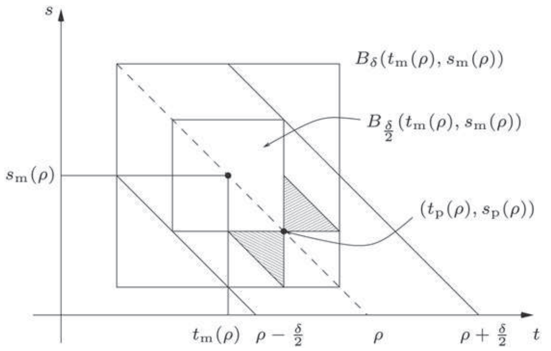

Fig. 5.1. If $(t_p(\rho), s_p(\rho))$ belongs to the boundary of the square $B_{\frac{\delta}{2}}(t_m(\rho), s_m(\rho))$, then for every $\widetilde{\rho} \in [\rho - \frac{\delta}{2}, \rho + \frac{\delta}{2}]$ the pair $(t_p(\widetilde{\rho}), s_p(\widetilde{\rho}))$ belongs to the two shaded triangles, both contained in the largest square $B_\delta(t_m(\rho), s_m(\rho))$.

Taking the limit as $\ell \to \infty$ shows that $t_m(\widetilde{\rho}) \le \lim_\ell t_\ell(\widetilde{\rho}) = t_m(\rho)$, and $s_m(\widetilde{\rho}) \ge \lim_\ell s_\ell(\widetilde{\rho}) = s_m(\rho) + \widetilde{\rho} - \rho$ for every $\widetilde{\rho} \in B_{\frac{\delta}{3}}(\rho - \frac{\delta}{6})$. Since $t_m(\cdot)$ and $s_m(\cdot)$ are non-decreasing, it follows in particular that

$$t_m(\widetilde{\rho}) = t_m(\rho), \qquad s_m(\widetilde{\rho}) = s_m(\rho) + \widetilde{\rho} - \rho$$
$$\forall \widetilde{\rho} \in B_{\frac{\delta}{3}}\left(\rho - \frac{\delta}{6}\right) \cap \left[\rho - \frac{\delta}{6}, R\right] \supset B_{\frac{\delta}{6}}(\rho).$$

This proves $\rho \in \mathcal{J}_m$ and (b_p) is established.

Condition (c_p) can be proved by similar arguments and therefore is omitted. We only note that now one has to use the lower semicontinuity of $\partial_s^+\mathcal{I}(\cdot,\cdot)$ provided by Theorem 3.2. □

Theorem 5.5. *The parameterized solution*

$$(t_m, s_m) \in \mathrm{C}^{\mathrm{Lip}}([0, R]; [0, T] \times [s_0, s_1])$$

defined in (5.1) is a special parameterized solution in the sense of Definition 2.4.

Proof. We are going to show that the pair (t_m, s_m) defined in (5.1) is a special parameterized solution. For simplicity, during the proof we will assume $s_0 = 0$. Thanks to Proposition 5.4, we have only to prove condition (b_{sp}). Let $\rho \in [0, R]$ be such that $\kappa(s_m(\rho)) + \partial_s^+\mathcal{I}(t_m(\rho), s_m(\rho)) < 0$. We want to prove that there exists $\delta > 0$ such that

$$\begin{cases} t_m(\widetilde{\rho}) = t_m(\rho) \\ s_m(\widetilde{\rho}) = s_m(\rho) + \widetilde{\rho} - \rho \end{cases} \quad \forall \widetilde{\rho} \in \,]\rho, \rho + \delta[. \tag{5.6}$$

Let $\varepsilon > 0$ be such that $\kappa(s_m(\rho)) + \partial_s^+\mathcal{I}(t_m(\rho), s_m(\rho)) = -\varepsilon < 0$. The continuity of $\kappa(\cdot)$ and the right continuity of $\partial_s^+\mathcal{I}(t_m(\rho), \cdot)$ (provided by Theorem 3.2) yield the existence of $\delta_* > 0$ such that for all $\widetilde{s} \in [s_m(\rho), s_m(\rho) + \delta_*[\cap [s_0, s_1]$

$$\kappa(\widetilde{s}) + \partial_s^+\mathcal{I}(t_m(\rho), \widetilde{s}) \le -\frac{\varepsilon}{2}. \tag{5.7}$$

Moreover, having fixed $t_m(\rho)$, we have $\partial_s^+\mathcal{I}(t_m(\rho), \widetilde{s}) = \partial_s^-\mathcal{I}(t_m(\rho), \widetilde{s})$ for a.e. $\widetilde{s} \in [s_0, s_1]$. Due to the left continuity of $\partial_s^-\mathcal{I}(t_m(\rho), \cdot)$ (provided by Theorem 3.2) we can conclude that the previous equality holds true for all $\widetilde{s} \in]s_m(\rho), s_m(\rho) + \delta_*] \cap [s_0, s_1]$. Thus,

$$\kappa(\widetilde{s}) + \partial_s^-\mathcal{I}(t_m(\rho), \widetilde{s}) \le -\frac{\varepsilon}{2} \tag{5.8}$$

for all $\widetilde{s} \in]s_m(\rho), s_m(\rho) + \delta_*[\cap [s_0, s_1]$. We then define the functions

$$t_\ell(\widetilde{\rho}) := \begin{cases} t_m(\widetilde{\rho}) & \text{if } \widetilde{\rho} \le \rho \\ t_m(\rho) & \text{if } \rho \le \widetilde{\rho} \le \rho + \delta_* \\ t_p(\widetilde{\rho}) & \text{if } \rho + \delta_* \le \widetilde{\rho} \le R, \end{cases}$$

$$s_\ell(\widetilde{\rho}) := \begin{cases} s_m(\widetilde{\rho}) & \text{if } \widetilde{\rho} \le \rho \\ s_m(\rho) + \widetilde{\rho} - \rho & \text{if } \rho \le \widetilde{\rho} \le \rho + \delta_* \\ s_p(\widetilde{\rho}) & \text{if } \rho + \delta_* \le \widetilde{\rho} \le R, \end{cases}$$

where $(t_p, s_p) \in \mathrm{C}^{\mathrm{Lip}}([\rho + \delta_*, R])$ is a parameterized solution with initial condition

$$(t_p(\rho + \delta_*), s_p(\rho + \delta_*)) = (t_m(\rho), s_m(\rho) + \delta_*).$$

Obviously, the pair (t_ℓ, s_ℓ) is a parameterized solution on $[0, R]$ with initial condition $(0, 0)$. In particular, by definition, $t_m(\widetilde{\rho}) \le t_\ell(\widetilde{\rho}) = t_m(\rho)$ for every $\widetilde{\rho} \in]\rho, \rho + \delta_*[$. Hence, by monotonicity, $t_m(\widetilde{\rho}) \equiv t_m(\rho)$ on $]\rho, \rho + \delta_*[$. This gives (5.6) and therefore the proof is complete. □

We observe that the pair (t_m, s_m) satisfies the following additional property:

$$\text{(P)} \quad \Big(\exists \delta > 0 : \kappa(s) + \partial_s^-\mathcal{I}(t_m(\rho), s) \le 0$$
$$\forall s \in (s_m(\rho), s_m(\rho) + \delta)\Big) \Longrightarrow]\rho, \rho + \delta[\subset \mathcal{J}_m.$$

Indeed, for $\widetilde{\rho} \in]\rho, \rho + \delta[$ we define the pair $(\widetilde{t}, \widetilde{s})$ by $\widetilde{t}(\widetilde{\rho}) := t_m(\rho)$ and $\widetilde{s}(\widetilde{\rho}) := s_m(\rho) + \widetilde{\rho} - \rho$. Then the pair (t_p, s_p) given by

$$t_p(\widetilde{\rho}) := \begin{cases} t_m(\widetilde{\rho}) & \text{if } \widetilde{\rho} \in [0, \rho] \\ \widetilde{t}(\widetilde{\rho}) & \text{if } \widetilde{\rho} \in \,]\rho, \rho + \delta[\end{cases}$$

$$s_p(\widetilde{\rho}) := \begin{cases} s_m(\widetilde{\rho}) & \text{if } \widetilde{\rho} \in [0, \rho] \\ \widetilde{s}(\widetilde{\rho}) & \text{if } \widetilde{\rho} \in \,]\rho, \rho + \delta[\end{cases}$$

is a parameterized solution on the interval $[0, \rho + \delta[$. By the definition (5.1) of (t_m, s_m) it follows easily that $t_p(\widetilde{\rho}) = t_m(\rho)$ for every $\widetilde{\rho} \in \,]\rho, \rho + \delta[$, i.e., $]\rho, \rho + \delta[\,\subset \mathcal{J}_m$.

The same property holds true by substituting $\partial_s^-\mathcal{I}$ with $\partial_s^+\mathcal{I}$.

We end this Subsection by noting that another special parameterized solution for $\mathcal{R}$ and $\mathcal{I}$ is the following one:

$$t_i(\rho) := \sup\{t_p(\rho) | \exists s_p : (t_p, s_p) \text{ is a parameterized}$$
$$\text{solution for } \mathcal{R} \text{ and } \mathcal{I} \text{ with } (t_p(0), s_p(0)) = (0, s_0)\}$$
$$s_i(\rho) := \inf\Big\{s_p(\rho) | \exists t_p : (t_p, s_p) \text{ is a parameterized}$$
$$\text{solution for } \mathcal{R} \text{ and } \mathcal{I} \text{ with } (t_p(0), s_p(0)) = (0, s_0)\Big\}.$$

6. Energy balances

We start by proving an exact energy balance in the parameterized setting, which implies a corresponding formula in the non-parameterized setting, cf. Proposition 6.3.

Proposition 6.1. *Let $\xi = (t, s) \in \mathrm{C}^{\mathrm{Lip}}((0, R); [0, T] \times [s_0, s_1])$ be a parameterized solution according to Definition 2.3. Then there exists a measurable function $\alpha : (0, R) \to \mathbb{R}$ with $\alpha(\rho) \in \partial_t^{\mathrm{Cl}}\mathcal{I}(\xi(\rho))$ a.e. such that for every $0 \le \rho_0 < \rho_1 \le R$ the following energy balance holds true:*

$$\mathcal{I}(\xi(\rho_1)) + \int_{\rho_0}^{\rho_1} \kappa(s(\rho)) s'(\rho)\, \mathrm{d}\rho$$
$$- \int_{\mathcal{J} \cap (\rho_0, \rho_1)} \big(\kappa(s(\rho)) + \partial_s^+\mathcal{I}(\xi(\rho))\big)\, \mathrm{d}\rho$$
$$= \mathcal{I}(\xi(\rho_0)) + \int_{\rho_0}^{\rho_1} \alpha(\rho) t'(\rho)\, \mathrm{d}\rho. \tag{6.1}$$

If ξ is a special parameterized solution for $\mathcal{R}$ and $\mathcal{I}$, then (6.1) holds with $\alpha(\rho) = \partial_t^-\mathcal{I}(\xi(\rho))$.

Remark 6.2. For almost every $\rho \in \mathcal{J} \cup \mathcal{S}$ we have $\partial_s^+\mathcal{I}(\xi(\rho)) = \partial_s^-\mathcal{I}(\xi(\rho))$ and $\partial_t^+\mathcal{I}(\xi(\rho)) = \partial_t^-\mathcal{I}(\xi(\rho))$. Moreover, conditions (b_p) and (c_p) of Definition 2.3 imply that

$$- \int_{\mathcal{J} \cap (\rho_0, \rho_1)} \big(\kappa(s(\rho)) + \partial_s^+\mathcal{I}(\xi(\rho))\big)\, \mathrm{d}\rho \ge 0.$$

Proof of Proposition 6.1. For $\mathcal{I}$ and ξ as in Proposition 6.1, by Corollary 3.9 the results of Section 3.1 hold true. In particular, Lemma 3.5 provides the following integrated version of the chain rule: for every $\rho_0, \rho_1 \in [0, R]$ and for every measurable function $\eta : (\rho_0, \rho_1) \to \mathbb{R}^2$ with $\eta(\rho) \in \partial^{\mathrm{Cl}}\mathcal{I}(\xi(\rho))$ a.e. we have

$$\mathcal{I}(\xi(\rho_1)) - \mathcal{I}(\xi(\rho_0)) = \int_{\rho_0}^{\rho_1} \eta(\rho)\cdot\xi'(\rho)\,\mathrm{d}\rho. \tag{6.2}$$

We make now a specific choice of η in order to obtain (6.1). Let us define first

$$\beta(\rho) := \begin{cases} \partial_s^+\mathcal{I}(\xi(\rho)) & \text{if } \rho \in \mathcal{J}\cup\mathcal{S}, \\ -\kappa(s(\rho)) & \text{else.}\end{cases}$$

Conditions (a) of Lemma 3.3 and (d_{p}) of Section 2.2 imply that $\beta(\rho) \in \partial_s^{\mathrm{Cl}}\mathcal{I}(\xi(\rho))$. By condition (c) of Lemma 3.4, there exists a measurable selection α of $\partial_t^{\mathrm{Cl}}\mathcal{I}(\xi(\cdot))$ such that $\eta(\rho) := (\alpha(\rho), \beta(\rho)) \in \partial^{\mathrm{Cl}}\mathcal{I}(\xi(\rho))$. Plugging this specific choice of η into (6.2) and rearranging the terms will give (6.1).

Assume now that ξ is a special parameterized solution. Proposition 3.6 in combination with (6.2) leads to

$$\mathcal{I}(\xi(\rho_1)) - \int_{\rho_0}^{\rho_1} \partial_s^+\mathcal{I}(\xi(\rho))s'(\rho)\,\mathrm{d}\rho$$
$$= \mathcal{I}(\xi(\rho_0)) + \int_{\rho_0}^{\rho_1} \partial_t^-\mathcal{I}(\xi(\rho))t'(\rho)\,\mathrm{d}\rho.$$

Applying property (d_{sp}) from Section 2.2 to the term $\int_{\rho_0}^{\rho_1}\partial_s^+\mathcal{I}(\xi(\rho))\, s'(\rho)\,\mathrm{d}\rho$ finishes the proof. □

Proposition 6.3. *Let $s \in \mathrm{BV}([0,T])$ be a local energetic solution according to Definition 2.1. Then s satisfies the energy balance (2.9) with $\widetilde{\alpha}(t) \in \partial_t^{\mathrm{Cl}}\mathcal{I}(t, s(t))$ for a.e. $t \in (0,T)$.*

Moreover, if s is a special local energetic solution, then the energy balance (2.9) holds with $\widetilde{\alpha}(t) = \partial_t^-\mathcal{I}(t, s(t))$, i.e., the refined energy balance (2.10) is satisfied.

Proof. Proposition 6.3 is a consequence of Proposition 6.1 and the equivalence between parameterized and non-parameterized solutions provided in Section 5.1. □

Let us consider again the energy functional $\mathcal{I}$ and the dissipation potential $\mathcal{R}$ of the example in Section 4.3 and let us compute the energy balance for some specific local energetic solution.

The partial Clarke gradient $\partial_t^{\mathrm{Cl}}\mathcal{I}$ is given by

$$\partial_t^{\mathrm{Cl}}\mathcal{I}(t,s) = \begin{cases} \{0\} & \text{if } s < -2 + \dfrac{t}{2} \\ \left[0, \dfrac{t}{2}\right] & \text{if } s = -2 + \dfrac{t}{2} \\ \{t - s - 2\} & \text{if } s > -2 + \dfrac{t}{2}. \end{cases}$$

For $j > 2$ and $\beta \in [\frac{j}{2} - 2, j - 3]$ we consider the following local energetic solution with initial condition $s_0 = -1$:

$$s(t) := \begin{cases} -1 & \text{if } t \in [0,2] \\ -2 + \dfrac{t}{2} & \text{if } t \in [2, j[\\ \beta & \text{if } t = j \\ t - 3 & \text{if } t > j. \end{cases}$$

Then, for $2 < t_1 < j < t_2$ the energy balance

$$\mathcal{I}(t_2, s(t_2)) + 3(s(t_2) - s(t_1)) + \int_{s(j-)}^{s(j+)} -(3 + \partial_s\mathcal{I}(j,\sigma))\,\mathrm{d}\sigma$$
$$= \mathcal{I}(t_1, s(t_1)) + \int_{t_1}^{t_2} \widetilde{\alpha}(\tau)\,\mathrm{d}\tau$$

holds true with

$$\widetilde{\alpha}(\tau) := \begin{cases} \dfrac{\tau}{4} + \dfrac{1}{2} & \text{if } \tau \in [t_1, j[\\ 1 & \text{if } \tau \in [j, t_2]. \end{cases}$$

We note that $\widetilde{\alpha}(\tau) \in \partial_t^{\mathrm{Cl}}\mathcal{I}(\tau, s(\tau))$ for every $\tau \in [t_1, t_2]$, and moreover that $\widetilde{\alpha}(\tau) \in\ ]\partial_t^+\mathcal{I}(\tau, s(\tau)), \partial_t^-\mathcal{I}(\tau, s(\tau))[$ for each $\tau \in [t_1, j[$.

Let us list now the special local energetic solutions associated with $\mathcal{I}$ and $\mathcal{R}$, in dependence of the initial datum s_0. Of course, for $s_0 > -1$ it is $s(t) = s_0$ for $t \in [0, 3 + s_0]$ and $s(t) = t - 3$ for $t \geq 3 + s_0$, while for $s_0 < -1$ it is $s(t) \equiv s_0$. For $s_0 = -1$ there are two special local energetic solutions, namely $s_1(t) \equiv -1$ and $s_2(t) = -1$ for $t \in [0,2]$ and $s_2(t) = t - 3$ for $t \geq 2$.

By simple calculation, one obtains that the special local energetic solutions of this example satisfy the following refined energy balance

$$\mathcal{I}(t_2, s(t_2)) + \kappa(s(t_2) - s(t_1)) = \mathcal{I}(t_1, s(t_1)) + \int_{t_1}^{t_2} \partial_t^-\mathcal{I}(t, s(t))\,\mathrm{d}t$$

for every $0 \leq t_1 < t_2 \leq T$.

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Appendix. Proofs for Section 5.1

We collect here the proofs for the statements in Section 5.1. Let $s \in \mathrm{BV}([0,T])$ be non-decreasing and let t_{p} and s_{p} be defined according to (5.1). For proving the equivalence of local energetic solutions and parameterized solutions, the next proposition is useful.

Proposition A.1. *The above defined functions t_{p} and s_{p} are non-decreasing, Lipschitz continuous and satisfy $t_{\mathrm{p}}(\widetilde{\rho}(t)) = t$, $s_{\mathrm{p}}(\widetilde{\rho}(t)) = s(t)$. Moreover, for all (t_i, ρ_i) with $\rho_i \in [\widetilde{\rho}(t_i^-), \widetilde{\rho}(t_i^+)]$ we have the monotonicity estimate $(\rho_1 - \rho_2)(t_1 - t_2) \geq |t_1 - t_2|^2$. The jump set $J(\widetilde{\rho})$ and the set $\mathcal{J}_{\mathrm{p}}$ are related as follows:*

(a) *$t_{\mathrm{p}}(\rho) = t_*$ if and only if $\rho \in [\widetilde{\rho}(t_*^-), \widetilde{\rho}(t_*^+)]$, with $\widetilde{\rho}(0^-) = \widetilde{\rho}(0)$ and $\widetilde{\rho}(R^+) = \widetilde{\rho}(R)$.*

(b) *$t \in J(s) = J(\widetilde{\rho})$ if and only if there exists ρ with $t_{\mathrm{p}}(\rho) = t$ and $\delta > 0$ such that $\emptyset \neq \big(]\rho, \rho + \delta[\cap[0,R]\big) \subset \mathcal{J}_{\mathrm{p}}$ or $\emptyset \neq \big(]\rho - \delta, \rho[\cap[0,R]\big) \subset \mathcal{J}_{\mathrm{p}}$.*

Proof. By definition, the function $\widetilde{\rho}$ is a strictly monotone BV-function with $J(\widetilde{\rho}) = J(s)$. Thus t_{p} is monotone, as well. From the monotonicity of s we conclude furthermore that for every $t_0 \leq t_1$ we have $\widetilde{\rho}(t_1^-) - \widetilde{\rho}(t_0^+) \geq t_1 - t_0$. This gives the monotonicity estimate. Let now $\rho_0 \leq \rho_1$. From the monotonicity of t_{p} and the monotonicity estimate we conclude $0 \leq t_{\mathrm{p}}(\rho_1) - t_{\mathrm{p}}(\rho_2) \leq \widetilde{\rho}(t_{\mathrm{p}}(\rho_1)^-) - \widetilde{\rho}(t_{\mathrm{p}}(\rho_2)^+) \leq \rho_1 - \rho_2$. Together with the monotonicity of t_{p} this estimate implies that t_{p} is Lipschitz continuous with a Lipschitz constant ≤ 1. Consequently, s_{p} is Lipschitz and monotone as well.

Part (b) of Proposition A.1 is an immediate consequence of Proposition A.1 (a). Thus it remains to show that (a) is valid. Let $\rho \in [0,R]$. Let furthermore $(\delta_n)_n$ be a sequence with $\delta_n \searrow 0$. Taking into account the definition of t_{p} we have $\widetilde{\rho}(t_{\mathrm{p}}(\rho) - \delta_n) \leq$

$\rho \le \widetilde{\rho}(t_p(\rho)+\delta_n)$, which implies that $\rho \in [\widetilde{\rho}(t_p(\rho)^-), \widetilde{\rho}(t_p(\rho)^+)]$. This proves the implication "⇒". For the proof of the inverse implication observe first that $t_p(\rho) = t_p(\rho_*)$ for every $\rho \in [\widetilde{\rho}(t_p(\rho_*)^-), \widetilde{\rho}(t_p(\rho_*)^+)]$. Indeed, let $\widetilde{\rho}_\pm := \widetilde{\rho}(t_p(\rho_*)^\pm)$ and let $\delta_n \searrow 0$ be an arbitrary sequence. Then

$$t_p(\widetilde{\rho}_+) = t_p(\widetilde{\rho}(t_p(\rho_*)+\delta_n)) + t_p(\widetilde{\rho}_+) - t_p(\widetilde{\rho}(t_p(\rho_*)+\delta_n)) \le t_p(\widetilde{\rho}(t_p(\rho_*)+\delta_n)) = t_p(\rho_*)+\delta_n,$$

and thus $\widetilde{\rho}(t_p(\rho_*)^+) \le t_p(\rho_*)$. In the same way it follows that $\widetilde{\rho}(t_p(\rho_*)^-) \ge t_p(\rho_*)$. This shows that t_p is constant on $[\widetilde{\rho}(t_p(\rho_*)^-), \widetilde{\rho}(t_p(\rho_*)^+)]$.

Let now $\rho \in [\widetilde{\rho}(t_*^-), \widetilde{\rho}(t_*^+)]$. Since $t_* = t_p(\widetilde{\rho}(t_*))$, it follows from the previous considerations that $t_p(\rho) = t_p(\widetilde{\rho}(t_*)) = t_*$, which finishes the proof. □

Proof of Proposition 5.1. Proposition A.1 implies that the pair (t_p, s_p) satisfies (a_p) of Definition 2.3. Moreover, again by Proposition A.1 it follows that

$$s_p(\rho) \in [s(t_p(\rho)^-), s(t_p(\rho)^+)]. \tag{A.1}$$

Let $\rho_* \in [0, R]$ with

$$\kappa(s_p(\rho_*)) + \partial_s^- \mathcal{I}(t_p(\rho_*), s_p(\rho_*)) < 0. \tag{A.2}$$

The goal is to show that $\rho_* \in \mathcal{J}_p$, which gives (b_p) of Definition 2.3. Since $\partial_s^- \mathcal{I}$ is upper semicontinuous, see Theorem 3.2, there exists $\delta > 0$ such that (A.2) is valid for every $\rho \in B_\delta(\rho_*)$. Let $\rho \in B_\delta(\rho_*)$ and assume that $t_p(\rho) \notin J(s)$. Then, by (A.1), it follows that $s(t_p(\rho)) = s_p(\rho)$ and (b) of Definition 2.1 is valid for $t = t_p(\rho)$. But this is a contradiction to (A.2). Thus $t_p(B_\delta(\rho_*)) \subset J(s)$. Since t_p is continuous, it follows that t_p is constant on $B_\delta(\rho_*)$, and thus $\rho_* \in \mathcal{J}_p$.

Assume now that

$$\kappa(s_p(\rho_*)) + \partial_s^+ \mathcal{I}(t_p(\rho_*), s_p(\rho_*)) > 0. \tag{A.3}$$

The goal is to show that $\rho_* \in \mathcal{S}_p$. From the lower semicontinuity of $\partial_s^+ \mathcal{I}$ it follows that (A.3) is valid on $B_\delta(\rho_*)$ for some $\delta > 0$. A proof by contradiction using (A.1) and part (d) of Definition 2.1 implies that $t_p(B_\delta(\rho_*)) \cap J(s) = \emptyset$. Thus, $s(t_p(\rho)) = s_p(\rho)$ for every $\rho \in B_\delta(\rho_*)$ and from (A.3) and (c) of Definition 2.1 we conclude that $t_p(B_\delta(\rho_*)) \subset D(s)$ and that $\dot{s} = 0$ for every $t \in t_p(B_\delta(\rho_*))$. It follows that s is constant on the interior of $t_p(B_\delta(\rho_*))$. If $\operatorname{int} t_p(B_\delta(\rho_*)) \neq \emptyset$, then s_p is constant on $B_\delta(\rho_*)$ and thus $\rho_* \in \mathcal{S}_p$. We prove now by contradiction that t_p is not constant on $B_\delta(\rho_*)$. Assume that t_p is constant on $B_\delta(\rho_*)$. This implies that $s_p(\rho) = \rho - t_p(\rho_*)$ for every $\rho \in B_\delta(\rho_*)$, and therefore $s_p'(\rho) = 1$. Since we know already that $s(t_p(\rho)) = s_p(\rho)$, the chain rule implies that for a.e. ρ we have $1 = s_p'(\rho) = \frac{d}{d\rho} s(t_p(\rho)) = \dot{s}(t_p(\rho)) t_p'(\rho) = 0$, which is a contradiction.

Proof of Proposition 5.2. Observe first that $\widehat{\rho}$ and hence s are non-decreasing BV-functions. This proves (a) of Definition 2.1.

Before we prove the other conditions of Definition 2.1, we discuss the relations between the jump sets $J(s), J(\widehat{\rho})$ and $\mathcal{J}_p$. Note that $J(\widehat{\rho}) = \{t \in [0, T] \mid \exists \rho \in \mathcal{J}_p \text{ with } t_p(\rho) = t\}$.

We prove now that $J(s) = J(\widehat{\rho})$. Since s_p is continuous, the inclusion $J(s) \subset J(\widehat{\rho})$ is obvious. Assume that $t \in J(\widehat{\rho})$. Then there exists $\delta > 0$ such that $t_p(\rho) = t$ for all $\rho \in (\widehat{\rho}(t), \widehat{\rho}(t)+\delta)$ or for all $\rho \in (\widehat{\rho}(t)-\delta, \widehat{\rho}(t))$. Assume that the first case is valid. Then, $\widehat{\rho}(t^+) \ge \widehat{\rho}(t)+\delta$ and $s_p(\rho) = \rho - t_p(\rho) = \rho - t$ for all $\rho \in (\widehat{\rho}(t), \widehat{\rho}(t)+\delta)$. This implies that for all $\epsilon > 0$ we have

$$\begin{aligned} s(t+\epsilon) = s_p(\widehat{\rho}(t+\epsilon)) &= \widehat{\rho}(t+\epsilon) - t_p(\widehat{\rho}(t+\epsilon)) \\ &\ge \widehat{\rho}(t) + \delta - (t+\epsilon) \\ &= s_p(\widehat{\rho}(t)) + \delta - \epsilon = s(t) + \delta - \epsilon. \end{aligned}$$

Thus, $s(t^+) \ge s(t) + \delta$ and therefore $t \in J(s)$. The other case runs similarly. The identity $[s(t^-), s(t^+)] = \operatorname{cl}\{s_p(\rho) \mid \rho \in \mathcal{J}_p, t = t_p(\rho)\}$ is obvious.

Let $t \in]0, T[\setminus J(s)$ and assume $\kappa(s(t)) + \partial_s^- \mathcal{I}(t, s(t)) < 0$. Since $s(t) = s_p(t)$, condition (b_p) of Definition 2.3 implies that $\widehat{\rho}(t) \in \mathcal{J}_p$, and thus, by the above considerations, $t \in J(s)$, which is a contradiction. Therefore, $\kappa(s(t)) + \partial_s^- \mathcal{I}(t, s(t)) \ge 0$ and we have shown condition (b) of Definition 2.1.

Assume that $\kappa(s(t)) + \partial_s^+ \mathcal{I}(t, s(t)) > 0$. Since $t = t_p(\widehat{\rho}(t))$ and $s(t) = s_p(\widehat{\rho}(t))$, condition (c_p) of Definition 2.3 guarantees that $\widehat{\rho}(t) \in \mathcal{S}$. Consequently, there exists $\delta_0 > 0$ such that s_p is constant on $B_{\delta_0}(\widehat{\rho}(t))$ and $\widehat{\rho}(t+\delta) = \widehat{\rho}(t) + \delta$ for $|\delta| \le \delta_0$. This implies that $s(t+\delta) = s_p(\widehat{\rho}(t))$ for all $|\delta| \le \delta_0$. Thus, $t \in D(s)$ and $\dot{s}(t) = 0$. This proves (c) of Definition 2.1.

If $t \in J(s)$, then from (c_p) of Definition 2.3 we conclude that $\kappa(s_*) + \partial_s^+ \mathcal{I}(t, s_*) \le 0$ for all $s_* \in (s(t^-), s(t^+))$. Using the lower semicontinuity of $\partial_s^+ \mathcal{I}(t, \cdot)$, it follows that the previous inequality is valid also on the closed interval $[s(t^-), s(t^+)]$. This proves part (d) of Definition 2.1.

Let finally (t_p, s_p) be a special parameterized solution. We only have to show that s satisfies (b_s) of Definition 2.2. Let $t \in (0, T)$ with

$$\kappa(s(t)) + \partial_s^+ \mathcal{I}(t, s(t)) < 0. \tag{A.4}$$

We set $\widetilde{\rho}(t) = t + s(t)$. Due to Remark 5.3 it holds that $t_p(\widetilde{\rho}(t)) = t$ and $s_p(\widetilde{\rho}(t)) = s(t)$. Thus, inequality (A.4) and (b_{sp}) of Definition 2.3 imply that there exists $\delta > 0$ such that $]\widetilde{\rho}(t), \widetilde{\rho}(t)+\delta[\subset \mathcal{J}_p$. On the basis of Proposition A.1 we conclude that $t = t_p(\widetilde{\rho}(t)) \in J(s)$. This gives (b) of Definition 2.2. □

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