Normal Hyperbolicity of Center Manifolds and Saint-Venant's Principle

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Abstract

The concept of normal hyperbolicity of center manifolds is generalized to infinite-dimensional differential equations, in particular, to elliptic problems in cylindrical domains. It is shown that all solutions u staying close to the center manifold for $t \in (-l, l)$ satisfy an estimate of the form $\|u(t) - \tilde{u}(t)\| \leqq Ce^{-\alpha(l-|t|)}$, where C and α are independent of l, and $\tilde{u}$ is a solution on the center manifold. These results are applied to Saint-Venant's principle for the static deformation of nonlinearly elastic prismatic bodies. The use of the center manifold permits the effective treatment of the general case of non-zero resultant forces and moments acting on each cross-section.

1. Introduction

The theory of normal hyperbolicity of center manifolds (for ordinary differential equations) and the theory of Saint-Venant's principle in elastostatics (or more generally for nonlinear elliptic problems) have each attracted much attention in their respective research fields. One aim of this paper is to unite these two fields, showing that normal hyperbolicity and Saint-Venant's principle are only two different aspects of the same mathematical concept.

The main idea for this connection is the reformulation of elliptic problems in cylindrical domains as a formal evolution, the axial variable playing the role of the time. This approach was introduced by KIRCHGÄSSNER [1982], where a center manifold was constructed to prove the existence of solitary waves for inviscid fluid flows. This idea was extended to general semilinear elliptic equations by FISCHER [1984] and finally to quasilinear systems by MIELKE [1988a]. (It should also be possible to generalize some of the other concepts of dynamical systems to elliptic problems, such as stable, unstable, and inertial manifolds.)

Consider a system of differential equations

$$\begin{aligned} \frac{d}{dt} u_1 - L_1 u_1 &= f_1(t, u_1, u_2), \\ \frac{d}{dt} u_2 - L_2 u_2 &= f_2(t, u_1, u_2) \end{aligned} \tag{1.1}$$

for $u = (u_1, u_2)$ lying in some Banach space $X = X_1 \times X_2$. The norms on X, X_1, and X_2 are each denoted $\|\cdot\|$; the distinction will be evident from the context. Let $\text{spectrum}(L_1) \subset i\mathbb{R}$, $\alpha_L := \inf\{|\mathrm{Re}\, z| \quad |z \in \text{spectrum}(L_2)\} > 0$, and $\|f_j(t, u_1, u_2)\| \leqq C\,\|(u_1, u_2)\|^2$. Then u_1 is called the center part and u_2 the hyperbolic part of $u = (u_1, u_2)$. A center manifold of (1.1) is an invariant manifold of the form $CM = \{(u_1, h(t, u_1)) \in X \mid \|u_1\| < \varepsilon\}$. The normal hyperbolicity of center manifolds for ordinary differential equations (for which $\dim X < \infty$) was first proved by PALMER [1975] by using a suitable homeomorphism to linearize the flow normal to CM. This means that this flow behaves like that for the equation $\frac{d}{dt} v_2 - L_2 v_2 = 0$, where $v_2 = 0$ is a hyperbolic fixed point.

We do not prove the existence of such a homeomorphism, nor do we prove the normal hyperbolicity of our center manifold in the sense of HIRSCH *et al.* [1977]; we rather concentrate on the corresponding exponential decay estimates as developed by CHOW & HALE [1982, Ch. 9].

We introduce relevant results from center manifold theory in Section 2. In Section 3, we derive results of the following type:

Let $p \in (1, \infty]$ and $\alpha \in (0, \alpha_L)$ be fixed. Then there exist positive constants C and δ_0 such that for every $l > 0$ and every solution $u: (-l, l) \to X$ of (1.1), with $\|u(t)\| \leqq \delta \leqq \delta_0$ for all $|t| < l$, the estimate

$$\left(\int_{-\lambda}^{\lambda} \|u(t) - \tilde{u}(t)\|^p \, dt \right)^{1/p} \leqq C\, \delta e^{-\alpha(l-\lambda)} \tag{1.2}$$

holds for all $\lambda \in (0, l)$. Here $\tilde{u}$ is a suitably chosen solution on the center manifold. For solutions $u: (0, \infty) \to X$, with $\|u(t)\| \leqq \delta \leqq \delta_0$ for all $t > 0$, we obtain

$$\left(\int_{\lambda}^{\infty} \|u(t) - \tilde{u}(t)\|^p \, dt \right)^{1/p} \leqq C\, \delta e^{-\alpha\lambda} \tag{1.3}$$

for all $\lambda > 0$.

Note that only the existence of u is assumed, while that of $\tilde{u}$ follows from the center manifold theory.

Decay estimates of the latter type are widely established in the theory of dynamical systems (*cf.* CHOW & HALE [1982]) and also in the theory of Saint-Venant's principle (KNOPS & PAYNE [1983], OLEINIK & YOSIFIAN [1982]). However, the less common estimate (1.2) seems to be more useful for practical problems, since the semi-infinite cylinder usually studied is only a mathematical model for treating decay in just one direction.

Although SAINT-VENANT [1856] formulated his principle only in the context of linear elasticity, the notion of an *abstract Saint-Venant's principle* now refers to a certain decay property (like the Phragmén-Lindelöf principle) for solutions of (nonlinear) partial differential equations (*cf.* HORGAN & KNOWLES [1983], KIRCHGÄSSNER & SCHEURLE [1986]). While the corresponding linear theory has been developed quite extensively (*cf.* TOUPIN [1965], OLEINIK & YOSIFIAN [1982], HORGAN & KNOWLES [1983]), the results for nonlinear problems are less complete (*cf.* ROSEMAN [1967], BREUER & ROSEMAN [1977], KNOPS & PAYNE [1983]). In particular, up to now all work was restricted to the study of decay towards a hyperbolic t-independent solution $\tilde{u} = \text{const.}$ (so that $\dim X_1 = 0$ in (1.1)).

The essential new feature of the present paper is the treatment of problems having a nontrivial center manifold. We show that the hyperbolic fixed point can now be replaced by the *normally hyperbolic* center manifold, which, in general, contains a nontrivial flow. This means that the solutions $\tilde{u}$ in (1.2) or (1.3) can have a complicated t-dependence, *e.g.* periodic, quasiperiodic, or even chaotic.

While the center manifold theory (see references given above) allows only the construction of a finite-dimensional subclass of solutions, the normal hyperbolicity concept enables us to characterize *all* solutions inside of a full neighborhood of the center manifold. Loosely speaking, the estimates (1.2) and (1.3) say that the longer a solution remains in this neighborhood, the more it has to approach the center manifold. In this sense we obtain a more general abstract Saint-Venant principle.

Moreover, the methods developed here are applicable to general elliptic problems, whereas previous work on the abstract Saint-Venant principle relies heavily on the divergence structure of the equations under consideration. This restriction is circumvented here by extensive use of the machinery of functional analysis (see (H1)–(H4) in Section 2 and MIELKE [1988a]). However, this machinery also allows us to choose the decay rate α arbitrarily close to the best possible value α_L (but $\delta_0 = \delta_0(\alpha) \to 0$ for $\alpha \to \alpha_L$).

In Section 4 we apply these ideas to a simple model problem, namely,

$$\psi_{xx} + \psi_{yy} + \mu\psi + g(\psi, \psi_x, \psi_y) = 0 \quad \text{for } (x, y) \in \mathbb{R} \times (0, \pi),$$
$$\psi(x, 0) = \psi(x, \pi) = 0 \quad \text{for } x \in \mathbb{R}.$$

In Section 5 we treat the original Saint-Venant principle for three-dimensional long prismatic bodies in the context of fully nonlinear elasticity: Let $\Omega_l = \Sigma \times (-l, l)$ denote the undeformed body with Σ being the cross-section. A deformation $\varphi : \Omega_l \to \mathbb{R}^3$ gives rise to a stored energy density $\mathscr{W} = \mathscr{W}(\nabla\varphi)$. The equations of equilibrium, when there are no body forces and no tractions on the cylindrical surface $\partial\Sigma \times (-l, l)$, have the form

$$\operatorname{div}\left(\frac{\partial \mathscr{W}}{\partial F}(\nabla\varphi)\right) = 0 \quad \text{in } \Omega_l, \quad \frac{\partial \mathscr{W}}{\partial F}(\nabla\varphi)\, n = 0 \quad \text{on } \partial\Sigma \times (-l, l). \tag{1.4}$$

Then the resultant force $\mathscr{F}$ and resultant moment $\mathscr{M}$ per cross-section are independent of the axial variable $s \in (-l, l)$. We do not specify boundary conditions at the terminal surfaces $\Sigma \times \{\pm l\}$, since we are interested in the qualitative behavior of all solutions of (1.4) having sufficiently small strains along the whole body, *i.e.*, solutions φ for which

$$\|\nabla\varphi^T(\cdot, s)\,\nabla\varphi(\cdot, s) - I\|_{W^{1,p}(\Sigma)} \leqq \delta \leqq \delta_0 \quad \text{for all } |s| \leqq l. \tag{1.5}$$

It is the main result of MIELKE [1988b] that (1.4) has a twelve-dimensional center manifold. Moreover, the solutions on the center manifold are exactly in one-to-one correspondence with the solutions of the classical rod-equations of KIRCHHOFF [1859]; in particular, the solutions are uniquely characterized by the mean translation r and the mean rotation R of the cross-section $\Sigma\times\{0\}$ and the resultants $\mathcal{F}$ and $\mathcal{M}$. As the solutions of the rod equations exist for all $s \in \mathbb{R}$, the solutions on the center manifold correspond to deformations of an infinite prism.

With the methods developed here we are able to answer the question how the deformations of a finite prism or a semi-infinite prism are related to those of the infinite prism. ERICKSEN [1983] had conjectured that equilibrium states of very long prisms are rod-like, except for boundary layers at the terminal surfaces. We prove this conjecture for small strains by showing that:

For every solution $\varphi \in W^{2,p}(\Omega_l)$, $p > 2$, of (1.4) satisfying (1.5) there is a unique rod-like solution $\tilde{\varphi} \in W^{2,\infty}(\Sigma\times\mathbb{R})$ (on the center manifold) with the same resultants $\mathcal{F}$, $\mathcal{M}$ such that

$$\|\varphi(\cdot) - \tilde{\varphi}(\cdot)\|_{W^{2,p}(\Omega_\lambda)} \leqq C\,\delta e^{-\alpha(l-\lambda)} \quad \textit{for all } \lambda \in (0, l) \tag{1.6}$$

where α and C are positive constants depending solely on the shape of Σ and on the constitutive function $\mathcal{W}$.

Of course, estimate (1.6) gives nontrivial results only if l is sufficiently large. We remark that (1.6) implies exponential decay towards rod-like solutions which, for instance, can be buckled, twisted, sheared or can have helical shape. It is important to emphasize that $\mathcal{F}$ and $\mathcal{M}$ are regarded as prescribed in our analysis. ANTMAN & KENNEY [1981] have noted that otherwise the properties of solutions of the rod equations are *very sensitive to seemingly slight distinctions between boundary conditions.*

Moreover, estimate (1.6) allows us to assert, in the spirit of ERICKSEN [1983], that *Saint-Venant's principle is an important mathematical link between the theory of three-dimensional prismatic bodies and the theory of rods.*

2. Abstract formulation

We consider a differential equation in a real Banach space X

$$\frac{d}{dt}u - Lu = f(t, \mu, u), \tag{2.1}$$

where $\mu \in \mathbb{R}^n$ denotes a set of parameters. The whole analysis will be local with respect to some t-independent solution pair (μ_0, u_0) of (2.1). We may assume that

$(\mu_0, u_0) = (0, 0)$. The linear operator $L: D(L) \to X$ will be, in general, unbounded and $f = f(t, \mu, u) = \mathcal{O}(|\mu| + \|u\|^2)$ contains the nonlinear terms. (Here $\mathcal{O}$ is the Landau symbol.)

Our aim is to study the qualitative behavior of all solutions of (2.1) staying close to $u \equiv 0$ on a long t-interval, in particular, in the presence of a center manifold. To this end we assume that (2.1) splits into the following linearly decoupled system:

$$\frac{d}{dt} u_j - L_j u_j = f_j(t, \mu, u_1, u_2), \quad j = 1, 2, \tag{2.2}$$

where $u = (u_1, u_2) \in X = X_1 \times X_2$ and $Lu = (L_1 u_1, L_2 u_2)$. Here X_1 and X_2 denote the center part and the hyperbolic part respectively. Thus, spectrum $(L_1) \subset i \cdot \mathbb{R}$ and spectrum $(L_2) \subset \{z \in \mathbb{C} : |\mathrm{Re}\, z| \geqq \alpha_L\}$ for some positive α_L.

Since we intend to apply our results to elliptic problems, like Saint-Venant's principle in elasticity, we do not assume that the spectrum of L_2 is contained in a half plane $\{z \in \mathbb{C} : \mathrm{Re}\, z < \gamma\}$ for some real γ. This would be typically the case for parabolic and hyperbolic equations, but not for elliptic problems in cylindrical domains, t being the axial variable (see Sections 4 and 5).

We introduce a family of Banach spaces as follows. For each Banach space Y with norm $\|\cdot\|$, each number α, each interval I, and each $p \in [1, \infty]$ we define

$$Y(I, p, \alpha) = \left\{ f : I \to Y \,|\, f \text{ is measurable, } |f|_{I,p,\alpha} := \left(\int_I \|e^{-\alpha|t|} f(t)\|^p \, dt \right)^{1/p} < \infty \right\}$$

for $p < \infty$,

$$Y(I, \infty, \alpha) = \left\{ f : I \to Y \,|\, f \text{ is continuous, } |f|_{I,\infty,\alpha} := \sup_{t \in I} \|e^{-\alpha|t|} f(t)\| < \infty \right\}.$$

Note that $Y(I, p, 0)$ is more commonly denoted by $L_p(I; Y)$ when $p < \infty$ and $Y(I, \infty, 0)$ is not $L_\infty(I; Y)$, but rather $C_b^0(I; Y)$. However choosing the notation as above simplifies the formulation of the results considerably, since both cases can be stated at once. (Actually, all results remain true with minor changes if $Y(I, \infty, 0)$ is taken to be $L_\infty(I; Y)$.)

Let $C_{b,u}^2$ be the class of functions f that together with their derivatives up to second order are bounded and uniformly continuous. We impose the following hypotheses on (2.2):

(H1) For $j = 1, 2$ there exist Banach spaces Z_j with $D(L_j) = D(L) \cap X_j \subset Z_j \subset X_j$, there exist neighborhoods U_j of zero in Z_j, and there exist closed subspaces V_j of X_j such that

$$f_j = f_j(t, \mu, u_1, u_2) \in C_{b,u}^2(\mathbb{R} \times \mathbb{R}^n \times U_1 \times U_2, V_j), \quad j = 1, 2.$$

Moreover, $f_j(t, 0, 0, 0) = 0$, $D_{u_k} f_j(t, 0, 0, 0) = 0$ for $j, k = 1, 2$ and $t \in \mathbb{R}$.

(H2) Let $V = V_1 \times V_2$, $Z = Z_1 \times Z_2$. There is a $z \in \mathbb{C}$ such that the resolvents $(L_j - z)^{-1} : X_j \to D(L_j)$, $j = 1, 2$, exist. There is a positive α_L and a

$p \in [1, \infty]$ such that for each $\alpha \in (0, \alpha_L)$ and for each $(\xi, g_1, g_2) \in Z_1 \times V(\mathbb{R}, p, \alpha)$ the linear system

$$\begin{aligned} &\frac{d}{dt} u_1 - L_1 u_1 = g_1(t), \qquad u_1(0) = \xi, \\ &\frac{d}{dt} u_2 - L_2 u_2 = g_2(t) \end{aligned} \tag{2.3}$$

has a unique p-mild solution $(u_1, u_2) = (K_1(\xi, g_1), K_2 g_2) \in Z(\mathbb{R}, p, \alpha)$. (A p-mild solution is defined below.)

(H3) If X_1 is not finite-dimensional, then $Z_1 = X_1$ and $p = \infty$ in (H1) and (H2).

(H4) For $j = 1, 2$ cut-off functions $\chi_j \in C^2_{b,u}(Z_j, \mathbb{R})$ exist with $\chi_j(u_j) = 1$ for $\|u_j\|_{Z_j} \leqq 1$ and $\chi_j(u_j) = 0$ for $\|u_j\|_{Z_j} \geqq 2$.

In (H1) the space $Z = Z_1 \times Z_2$ with $D(L) \subset Z \subset X$ is needed to characterize the loss of regularity in the nonlinear function f. For semilinear problems Z is typically $D(L^\alpha)$ with $\alpha \in [0, 1)$ and for quasilinear problems $Z = D(L)$ (see Sections 4 and 5).

In (H2) we can choose $p = \infty$ in the semilinear case (*cf.* MIELKE [1986]), while in the quasilinear case we have to use $p \in (1, \infty)$ (*cf.* MIELKE [1988a]). The notation "p-mild solution of (2.3)" follows that of HALE & SCHEURLE [1985]; A triple (μ, u, I), with $I \subset \mathbb{R}$ being an interval, is called a p-mild solution of (2.3) if $u_j \in Z_j(J, p, 0)$ for every finite interval $J \subset I$, and if the equation $\frac{d}{dt}((L_j - z)^{-1} u_j) - (L_j - z)^{-1} L_j u_j = (L_j - z)^{-1} g_j$ is satisfied in $X_j(J, p, 0)$ (Here d/dt is to be taken in the distributional sense if $p < \infty$). Note that $(L_j - z)^{-1} L_j u_j = u_j + z(L_j - z)^{-1} u_j \in X_j(J, p, 0)$ since $Z_j \subset X_j$.

Usually the hypotheses (H1) and (H2) are supplemented by the general assumption that X_1 is finite-dimensional. Although this is true in the applications given below there are examples of infinite-dimensional center manifolds (MIELKE [1989]). In this case we have to impose the requirement that $Z_1 = X_1$, since we cannot expect smoothing properties in the u_1-equation in (2.3). To make the condition $u_1(0) = \xi$ meaningful we additionally require $p = \infty$. In particular, (H2) and (H3) imply the existence of the strongly continuous group $(e^{L_1 t})_{t \in \mathbb{R}}$ with $\|e^{L_1 t}\| \leqq c(\alpha)\, e^{\alpha |t|}$ for $\alpha > 0$, $t \in \mathbb{R}$.

The cut-off functions in (H4) allow us to localize equation (2.2) by changing the nonlinearities outside of a suitable neighborhood of $(\mu, u) = (0, 0)$. Hence the results will be restricted to small solutions. To be precise, let

$$f_j^\varepsilon(t, \mu, u_1, u_2) = f_j(t, \mu, u_1, u_2)\, \chi_0(\mu/\varepsilon)\, \chi_1(u_1/\varepsilon)\, \chi_2(u_2/\varepsilon),$$

where $\chi_0 \in C^2(\mathbb{R}^n, \mathbb{R})$ is a similar cut-off function for $\mu \in \mathbb{R}^n$. Then there is an $\varepsilon_0 > 0$ such that $f_j^\varepsilon \in C^2_{b,u}(\mathbb{R}^{n+1} \times Z_1 \times Z_2, V_j)$ for all $\varepsilon \in (0, \varepsilon_0)$. Moreover,

$$\sigma(\varepsilon) = \sup \{\|D_{u_k} f_j^\varepsilon(t, \mu, u)\|_{Z_k \to V_j} \mid j, k = 1, 2, \ (t, \mu, u) \in \mathbb{R}^{n+1} \times Z\} = \mathcal{O}(\varepsilon)$$

for $\varepsilon \to 0$. Whenever we replace f_j in (2.2) by f_j^ε we use the notation $(2.2)^\varepsilon$. We take ε as fixed but sufficiently small. (To make the following theory work, we have to restrict ε successively by a finite number of conditions of the form $c_k \sigma(\varepsilon) < 1$, $k = 1, \ldots, n$. Obviously there is an $\varepsilon \in (0, \varepsilon_0)$ satisfying all these relations).

According to (H2) all p-mild solutions of $(2.2)^\varepsilon$ with $z_2 \in Z_2(\mathbb{R}, p, \alpha)$ satisfy the integral equation

$$u = S(\xi, \mu, u),$$
$$S = (S_1, S_2) = (K_1(\xi, f_1^\varepsilon(\cdot, \mu, u)), K_2(f_2^\varepsilon(\cdot, \mu, u))), \tag{2.4}$$

where $\xi = u_1(0)$. For $\alpha \in (0, \alpha_L)$, $S(\xi, \mu, \cdot)$ may be considered as a mapping from $Z(\mathbb{R}, p, \alpha)$ into itself, being uniformly Lipschitz-continuous with constant $\|K\|_{Z_1 \times X(\mathbb{R},p,\alpha) \to Z(\mathbb{R},p,\alpha)} \cdot \sigma(\varepsilon)$. Thus, we have

Lemma 2.1. *Let* (H1) *to* (H4) *be satisfied and let* $\alpha \in (0, \alpha_L)$ *be fixed. Then, for sufficiently small* ε *and for all* $(\mu, \xi) \in \mathbb{R}^n \times Z_1$ *equation* (2.4) *has a unique solution* $u = U(\mu, \xi) \in Z(\mathbb{R}, p, \alpha)$, *where* $U_1(\mu, \xi)(0) = \xi$ *and* $\|U(\mu, \xi)\|_{Z(\mathbb{R},p,\alpha)} \leqq C(|\mu| + \|\xi\|_{Z_1})$ *for some constant* C.

Of course, whenever one of the solutions $u = U(\mu, \xi)$ satisfies $|\mu| \leqq \varepsilon$ and $\|u_j(t)\|_{Z_j} \leqq \varepsilon$ on some interval J, then $(\mu, (u_1, u_2), J)$ is a p-mild solution of the original system (2.2) without cut-off.

In fact we have more information on the solutions $u = U$:

Lemma 2.2. *Let* (H1) *to* (H4) *be satisfied. Then, for sufficiently small* ε, *there is a function* $h = h^\varepsilon(t, \mu, u_1) \in C^1_{b,u}(\mathbb{R} \times \mathbb{R}^n \times Z_1, Z_2)$ *with* $h(t, \mu, u_1) = \mathcal{O}(|\mu| + \|u_1\|^2_{Z_1})$ *such that every solution has the form* $u_2 = U_2(\mu, \xi)(t) = h(t, \mu, U_1(\mu, \xi)(t))$, $t \in \mathbb{R}$.

The existence of this reduction function h^ε is proved by Mielke [1986] for the case $p = \infty$ and by Mielke [1988a] for $p \in (1, \infty)$.

Hence all solutions $u \in Z(\mathbb{R}, p, \alpha)$ of (2.4) with $\alpha > 0$ have to lie on the manifold

$$CM = \{(t, \mu, u_1, h^\varepsilon(t, \mu, u_1)) \in \mathbb{R}^{n+1} \times Z \mid (t, \mu, u_1) \in \mathbb{R}^{n+1} \times Z\},$$

which is called the *center manifold* of $(2.2)^\varepsilon$ (and of (2.4)). Since u_2 is determined by (t, μ, u_1), it suffices to consider the reduced equation

$$\frac{d}{dt} u_1 - L_1 u_1 = f_1^\varepsilon(t, \mu, u_1, h(t, \mu, u_1)), \tag{2.5}$$

which, according to Lemma 2.1, has a unique solution $u_1 = U_1(\mu, \xi) \in Z_1(\mathbb{R}, p, \alpha)$ with $u_1(0) = \xi$ for each $\xi \in Z_1$.

3. Normal hyperbolicity

To motivate the method we first consider the case of a finite-dimensional X. Then it is known (CHOW & HALE [1982], PALMER [1975]) that, for sufficiently small ε, there is a homeomorphism H^ε with $(v_1, v_2) = H^\varepsilon(u_1, u_2) = (u_1, \tilde{H}^\varepsilon(u_1, u_2))$ such that $(2.2)^\varepsilon$ is equivalent to

$$\frac{d}{dt} v_1 - L_1 v_1 = f_1(t, \mu, v_1, h^\varepsilon(t, \mu, v_1)),$$

$$\frac{d}{dt} v_2 - L_2 v_2 = 0. \tag{3.1}$$

The transformed center manifold is given by $v_2 \equiv 0$ and the flow *normal* to it is linear with $v_2 = 0$ being a *hyperbolic* fixed point of $\frac{d}{dt} v_2 - L_2 v_2 = 0$ (*i.e.*, 0 is not contained in Re (spectrum (L_2))). Hence we call the center manifold *normally hyperbolic* (*cf.* HIRSCH, PUGH & SHUB [1977]).

Obviously for every solution $v: (-l, l) \to X$ of (3.1) with $\|v_j(t)\| \leqq \delta$ for $|t| \leqq l$ the estimate $\|v_2(t)\| \leqq C(\alpha)\, \delta e^{-\alpha(l-|t|)}$, $|t| < l$ with $\alpha \in (0, \alpha_L)$ holds. Let $\Delta = (\Delta_1, \Delta_2) = (0, v_2)$ be the difference between the given solution (v_1, v_2) and a reference solution $\tilde{v} = (v_1, 0)$ lying on the center manifold. Hence

$$\|\Delta\|_{X((-l,l),\infty,\alpha)} = \|v - \tilde{v}\|_{X((-l,l),\infty,\alpha)} \leqq C(\alpha)\, \delta e^{-\alpha l} \tag{3.2}$$

where the constant $C(\alpha)$ is independent of l and δ. This shows that the longer the solution v stays small the closer it must approach a reference solution $\tilde{v}$ on the center manifold. If v even remains small on $I = (0, \infty)$, then it converges to $\tilde{v}$ exponentially for $t \to \infty$.

Similar results, in particular, estimates like (3.2), can be proved directly for $(2.2)^\varepsilon$ in the infinite-dimensional case, without the use of the homeomorphism H^ε. The idea of the proof is to extend a given solution $u: (-l, l) \to Z$ of $(2.2)^\varepsilon$ to a function $\bar{u}: \mathbb{R} \to Z$ satisfying $(2.2)^\varepsilon$ up to some error $r = (r_1, r_2)$ with support close to $t = \pm l$. Then, $\bar{u}$ satisfies the integral equation (2.4) up to the error $R = K(0, r_1, r_2)$. Defining $\tilde{u}$ to be the solution on the center manifold with $\tilde{u}_1(0) = u_1(0)$ we are able to estimate the difference $\Delta = \bar{u} - \tilde{u}$ in the $Z(\mathbb{R}, p, \alpha)$-norm by use of the integral equation (2.4).

Theorem 3.1. *Let* (H1) *to* (H4) *be satisfied and* $\alpha \in (0, \alpha_l)$ *be fixed. Then for sufficiently small* ε *there is a constant* $C(\alpha)$ *such that for every* δ *the following assertions hold*:

a) *Every p-mild solution* $(\mu, u, (-l, l))$ *of* $(2.2)^\varepsilon$, *with* $l > 1$ *and* $\|u_j(t)\| \leqq \delta$ *for* $|t| \leqq l$, *satisfies the estimate*

$$\|u - \tilde{u}\|_{Z((-l+1,l-1),p,\alpha)} \leqq C(\alpha)\, \delta e^{-\alpha l} \tag{3.3}$$

where $\tilde{u}$ *is the unique solution on the center manifold with* $\tilde{u}_1(0) = u_1(0)$ (i.e., $\tilde{u} = U(\mu, u_1(0))$ *of Lemma 2.1*).

b) *For every p-mild solution* $(\mu, u, (0, \infty))$ *of* $(2.2)^\varepsilon$, *with* $\|u_j(t)\|_{Z_j} \leqq \delta$ *for* $t > 0$, *there is a unique solution* $\tilde{u}$ *on the center manifold such that*

$$\|\exp(\alpha \cdot)(u(\cdot) - \tilde{u}(\cdot))\|_{Z((1,\infty),p,0)} \leqq C(\alpha)\,\delta. \tag{3.4}$$

Before giving a proof of this theorem we point out how these global results for equation $(2.2)^\varepsilon$ (including cut-off) reduce to results for the original equation (2.2) (without cut-off). For u of Theorem 3.1 to be a solution of (2.2) we obviously have to impose the requirement that $\delta \leqq \varepsilon$. But this does not imply that $\|\tilde{u}_j(t)\| \leqq \varepsilon$ in general. However, if $p = \infty$, the estimates (3.3) and (3.4), respectively, imply that $\|\tilde{u}_j(t)\| \leqq \|u_j(t)\| + \|u_j(t) - \tilde{u}_j(t)\| \leqq (1 + C(\alpha))\,\delta$. Hence taking $\delta \leqq \varepsilon/(1 + C(\alpha))$ guarantees that u and $\tilde{u}$ are solutions of (2.2). For $p < \infty$ this method fails and a more careful analysis is neeeded. For instance, in Section 5 we use the fact that the solutions $\tilde{u}$ of (2.5) are always bounded and satisfy $\|\tilde{u}_1(t)\| \leqq g(\|\tilde{u}_1(0)\|)$, $t \in \mathbb{R}$, where $g: [0,\infty) \to [0,\infty)$ is continuous with $g(0) = 0$. Since $\|\tilde{u}_1(0)\| = \|u_1(0)\| \leqq \delta$ and $\tilde{u}_2(t) = h(t, \mu, u_1(t)) = \mathcal{O}(|\mu| + \|u_1(t)\|) = \mathcal{O}(\delta)$, we have $\|\tilde{u}_j(t)\| \leqq \varepsilon$ for small enough positive δ.

Moreover, we give estimates for the L^p-norm of $u - \tilde{u}$ on subintervals $(-\lambda, \lambda)$ and (λ, ∞), which are more common in the theory of Saint-Venant's principle (*cf.* HORGAN & KNOWLES [1983], TOUPIN [1965]).

Corollary 3.2. *Under the assumptions of Theorem 3.1 the following estimates are valid*:

a) *For a finite interval*:

$$\begin{aligned}\left(\int_{-\lambda}^{\lambda} \|u(t) - \tilde{u}(t)\|_Z^p\,dt\right)^{1/p} &\leqq C(\alpha)\,\delta e^{-\alpha(l-\lambda)} && \textit{for } 0 < \lambda \leqq l - 1, \quad \textit{if } p \geqq 1,\\ \|u(t) - \tilde{u}(t)\|_Z &\leqq C(\alpha)\,\delta e^{-\alpha(l-|t|)} && \textit{for } |t| \leqq l - 1, \quad \textit{if } p = \infty.\end{aligned} \tag{3.5}$$

b) *For the semi-infinite interval*:

$$\begin{aligned}\left(\int_{\lambda}^{\infty} \|u(t) - \tilde{u}(t)\|_Z^p\,dt\right) &\leqq C(\alpha)\,\delta e^{-\alpha\lambda} && \textit{for } \lambda \geqq 1, \quad \textit{if } p \geqq 1,\\ \|u(t) - \tilde{u}(t)\|_Z &\leqq C(\alpha)\,\delta e^{-\alpha t} && \textit{for } t \geqq 1, \quad \textit{if } p = \infty.\end{aligned} \tag{3.6}$$

Proof. a) Since $\|u - \tilde{u}\|_{Z((-\lambda,\lambda),p,0)} \leqq e^{\alpha\lambda}\|u - \tilde{u}\|_{Z((-\lambda,\lambda),p,\alpha)} \leqq e^{\alpha\lambda}\|u - \tilde{u}\|_{Z((-l+1,l-1),p,\alpha)}$ we use (3.3) to obtain (3.5).

b) Similarly, we estimate $\|u - \tilde{u}\|_{Z((-\lambda,\infty),p,-\alpha)} \leqq e^{-\alpha\lambda}\|u - \tilde{u}\|_{Z((\lambda,\infty),p,-\alpha)} \leqq e^{-\alpha\lambda}\|u - \tilde{u}\|_{Z((1,\infty),p,-\alpha)}$, and now use (3.4) to obtain (3.6). QED.

Proof of Theorem 3.1 a. Let $(\mu, u, (-l, l))$ be an arbitrary p-mild solution of $(2.2)^\varepsilon$ with $l > 1$. According to (H3), $u_1(0) \in Z_1$ is well-defined and hence we let $\tilde{u} = U(\mu, u_1(0))$ be the unique solution on the center manifold with $\tilde{u}_1(0) = u_1(0)$ (cf. Lemma 2.1). We extend u to a function $\bar{u}: \mathbb{R} \to Z$ through $\bar{u}(t) = \theta(l - |t|)\,u(t)$ for $|t| \leqq l$ and $\bar{u}(t) = 0$ elsewhere, where $\theta \in C^2(\mathbb{R}, [0, 1])$, $\theta(s) = 1$ for $s \geqq 1$, and $\theta(s) = 0$ for $s \leqq 0$. We have

$$\frac{d}{dt}\bar{u}_j - L_j\bar{u}_j = f_j^\varepsilon(t, \mu, \bar{u}) + r_j, \quad j = 1, 2, \tag{3.7a}$$

where the error r_j is identically zero except for $l-1<|t|<l$:

$$r_j(t)=\theta f_j^\varepsilon(t,\mu,u(t))-f_j^\varepsilon(t,\mu,\theta u)+\operatorname{sign}(t)\frac{d\theta}{ds}u(t). \tag{3.7b}$$

(Here θ and $d\theta/ds$ are evaluated at $l-|t|$.)

Using (H2) we find that $\overline{u}=S(\mu,u_1(0),\overline{u})+K(0,r)$. We set $\Delta=u-\tilde{u}$ where $\tilde{u}=S(\mu,u_1(0),\tilde{u})$, to obtain

$$\Delta=K(0,f^\varepsilon(\cdot,\mu,\overline{u})-f^\varepsilon(\cdot,\mu,\overline{u}-\Delta)+r). \tag{3.8}$$

Let us consider r and $\overline{u}$ as fixed and let ε be sufficiently small. Then the contraction mapping principle ensures that (3.8) has a unique solution Δ with

$$\|\Delta\|_{Z(\mathbb{R},p,\alpha)}\leqq c(\alpha)\,\|K(0,r)\|_{Z(\mathbb{R},p,\alpha)}\leqq\tilde{c}(\alpha)\,\|r\|_{X(\mathbb{R},p,\alpha)}. \tag{3.9}$$

Now the inequality $\|f_j^\varepsilon(t,\mu,u)\|_{X_j}\leqq C(|\mu|+\|u\|_Z)\leqq\tilde{c}\,\delta$ implies that $\|r(t)\|\leqq c\,\delta$ for $l-1\leqq|t|\leqq l$. Hence

$$\|r\|_{X(\mathbb{R},p,\alpha)}\leqq\left(2\int_{l-1}^{l}(e^{-\alpha t}\,c\,\delta)^p\,dt\right)^{1/p}=\overline{c}(\alpha)\,\delta e^{-\alpha l}.$$

Since $\overline{u}(t)=u(t)$ for $|t|\leqq l-1$ we deduce that

$$\|u-\tilde{u}\|_{Z((-l+1,l-1),p,\alpha)}\leqq\|\overline{u}-\tilde{u}\|_{Z((-l,l),p,\alpha)}\leqq\|\Delta\|_{Z(\mathbb{R},p,\alpha)}\leqq C(\alpha)\,\delta e^{-\alpha l},$$

which is the desired result. QED.

To derive the decay estimate (3.4) we introduce another space

$$Y^0(I,p,\alpha)=\{y\in Y(I,p,\alpha)\mid e^{-\alpha(\cdot)}y(\cdot)\in Y(I,p,0)\},$$

$$\|y\|_{Y^0(I,p,\alpha)}=\|e^{-\alpha(\cdot)}y(\cdot)\|_{Y(I,p,0)}.$$

Lemma 3.3. *Let* (H2) *and* (H3) *be satisfied. Then for every* $\alpha\in(0,\alpha_L)$ *the linear system*

$$\frac{d}{dt}u_j-L_ju_j=g_j,\quad j=1,2, \tag{3.10}$$

has a unique p-mild solution $u_j=K_j^0(g_j)\in Z_j^0(\mathbb{R},p,\alpha)$ *for each* $g\in X^0(\mathbb{R},p,\alpha)$.

Proof. The remarks accompanying (H3) imply that u_1 takes the form $u_1(t)=(K_1^0g_1)(t)=\int_t^\infty e^{A(t-s)}g_1(s)\,ds$. Introducing $u_1^0=e^{\alpha t}u_1$, $g_1^0(t)=e^{\alpha t}g_1(t)$ we obtain $u_1^0(t)=\int_t^\infty e^{(A-\alpha)(t-s)}g(s)\,ds$. Young's inequality (*cf.* STEIN [1970]) implies

$$\begin{aligned}\|u_1\|_{Z^0(\mathbb{R},p,\alpha)}=\|u_1^0\|_{Z(\mathbb{R},p,0)}&\leqq\int_0^\infty\|e^{(A-\alpha)(\tau)}\|_{X_1\to Z_1}\,d\tau\,\|g_1^0\|_{X(\mathbb{R},p,0)}\\&\leqq C(\alpha)\,\|g_1\|_{X^0(\mathbb{R},p,\alpha)}.\end{aligned}$$

For u_2 we proceed as follows. Since $X_2^0(\mathbb{R}, p, \alpha) \subset X_2(\mathbb{R}, p, \alpha)$, we immediately obtain a unique solution $u_2 = K_2 g_2 \in Z_2(\mathbb{R}, p, \alpha)$. We only have to show that $u_2 \in Z_2^0(\mathbb{R}, p, \alpha)$. To this end we establish the relation

$$\|y\|_{Y^0(\mathbb{R},p,\alpha)} = \|y\|_{\sup} \overset{\text{def}}{=} \sup_{s>0} e^{\alpha s} \|y(\cdot + s)\|_{Y(\mathbb{R},p,\alpha)}. \tag{3.11}$$

Since $\|e^{\alpha(t-s)} y(t+s)\| \geqq e^{\alpha s} \|e^{-\alpha|t|} y(t+s)\|$ we have $\|y\|_{Y^0(\mathbb{R},p,\alpha)} \geqq \|y\|_{\sup}$. On the other hand, for fixed y and $\varepsilon > 0$ there is a $s = s(\varepsilon, y)$ such that $\|y\|_{Y^0(\mathbb{R},p,\alpha)} \leqq \varepsilon + \|y\|_{Y^0((-\infty,s),p,\alpha)}$. Moreover, $e^{\alpha t} = e^{\alpha s} e^{-\alpha|t-s|}$ for $t \leqq s$. Thus

$$\begin{aligned}\|y\|_{Y^0(\mathbb{R},p,\alpha)} &\leqq \varepsilon + e^{\alpha s} \|e^{-\alpha|\cdot - s|} y\,(\cdot)\|_{Y((-\infty,s),p,0)} \\ &= \varepsilon + e^{\alpha s} \|y(\cdot + s)\|_{Y((-\infty,0)p,\alpha)} \leqq \varepsilon + \|y\|_{\sup}.\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, (3.11) is proved.

Now the autonomy of (3.10) implies that $(K_2 g_2(\cdot + s))(t) = (K_2 g_2(\cdot))(t+s)$ for $s \in \mathbb{R}$ and hence that

$$\|u_2(\cdot + s)\|_{Z_2(\mathbb{R},p,\alpha)} \leqq \|K_2\| \cdot \|g_2(\cdot + s)\|_{X_2(\mathbb{R},p,\alpha)}.$$

Thus $\|u_2\|_{\sup} \leqq \|K_2\| \|g_2\|_{\sup} \leqq \|K_2\| \|g_2\|_{X_0^2(\mathbb{R},p,\alpha)}$. This shows that $u_2 \in Z_2^0(\mathbb{R}, p, \alpha)$. QED.

Proof of Theorem 3.1b. We again extend $u: (0, \infty) \to Z$ to $\bar{u}: \mathbb{R} \to Z$ by $\bar{u}(t) = \theta(t) u(t)$ for $t \geqq 0$ and $\bar{u}(t) = 0$ otherwise. Equation (3.7) is still valid where θ and $d\theta/ds$ is evaluated at t. We are looking for a p-mild solution $\tilde{u}$ of $(2.2)^\varepsilon$ such that $\Delta = u - \tilde{u} \in Z^0(\mathbb{R}, p, \alpha)$. Hence Δ has to satisfy the equation

$$\frac{d}{dt} \Delta_j - L_j \Delta_j = f_j^\varepsilon(t, \mu, \bar{u}) - f_j^\varepsilon(t, \mu, \bar{u} - \Delta) + r_j. \tag{3.12}$$

According to Lemma 3.3 a solution Δ of (3.12) with $\Delta \in Z^0(\mathbb{R}, p, \alpha)$ satisfies

$$\Delta = K^0(f^\varepsilon(\cdot, \mu, \bar{u}) - f_j^\varepsilon(t, \mu, \bar{u} \in \Delta) + r). \tag{3.13}$$

Note that r and $f^\varepsilon(\cdot, \mu, \bar{u}) - f^\varepsilon(\cdot, \mu, \bar{u} - \Delta)$ are in $X^0(\mathbb{R}, p, \alpha)$ since $\|f^\varepsilon(t, \mu, \bar{u}) - f^\varepsilon(t, \mu, \bar{u} - \Delta)\|_X \leqq \sigma(\varepsilon) \|\Delta\|_Z$. Again, for sufficiently small ε, the contraction mapping principle delivers a unique solution Δ with $\|\Delta\|_{Z^0(\mathbb{R},p,\alpha)} \leqq C(\alpha) \|r\|_{X^0(\mathbb{R},p,\alpha)}$. As in case a) the assertion follows. QED.

Often it is desirable to estimate $\|u - \tilde{u}\|_{Z((-\lambda,\lambda),p,\alpha)}$ in terms of $\|u - \tilde{u}\|_{Z((-l,l),p,\alpha)}$ instead of $C(\alpha)\,\delta$ (*cf.* Oleinik & Yosifian [1982], Toupin [1965]). This can be done by a slightly more careful analysis. We obtain

Theorem 3.4. *Let* (H1)–(H4) *hold. Let* $0 < \beta < \beta + \alpha < \alpha_L$. *Then, for sufficiently small* ε, *there is a* $C(\alpha, \beta)$ *such that*:

a) *For every p-mild solution* $(\mu, u, (-l, l))$ *of* $(2.2)^\varepsilon$ *the estimate*

$$\|u - \tilde{u}\|_{Z((-\lambda,\lambda),p,\beta)} \leqq C(\alpha, \beta)\, e^{-\alpha(l-\lambda)} \|u - \tilde{u}\|_{Z((-l,-l+1)\cup(l-1,l),p,\beta)}, \tag{3.14}$$

holds for $0 < \lambda < l - 1$. *Here* $(\mu, \tilde{u}, \mathbb{R})$ *is the unique solution on the center manifold with* $\tilde{u}_1(0) = u_1(0)$.

b) *For every p-mild solution* $(\mu, u, (0, \infty))$ *of* $(2.2)^\varepsilon$ *with* $u \in Z((0,\infty), p, \alpha + \beta)$ *there is a unique solution* $(\mu, \tilde{u}, \mathbb{R})$ *on the center manifold with*

$$\|u - \tilde{u}\|_{Z((\lambda,\infty),p,\beta)} \leqq C(\alpha, \beta)\, e^{-\alpha\lambda}\, \|u - \tilde{u}\|_{Z((0,1),p,\beta)} \quad \textit{for } \lambda \geqq 1. \tag{3.15}$$

Remark 3.5. In the case without center part (so that $\dim X_1 = 0$) it is possible to prove these estimates also for $\beta = 0$. These results would match those cited above for linear elasticity. According to our proof $C(\alpha, \beta)$ blows up as $\beta \to 0$ if $\dim X_1 > 0$. As yet it is not clear whether this is only a deficiency of our method or whether it is really due to the center part.

Proof of Theorem 3.4. We follow the lines of the proof of Theorem 3.1, but instead of $\bar{u}$ we use the extension $\hat{u}$ given by $\hat{u}(t) = \tilde{u}(t) + \theta(l - |t|)(u(t) - \tilde{u}(t))$ for $|t| < l$ and $\hat{u}(t) = \tilde{u}(t)$ elsewhere, where $\tilde{u}$ is defined as above. Letting $\hat{\Delta} = \hat{u} - \tilde{u}$, we deduce that

$$\hat{\Delta} = K(0, f^\varepsilon(\cdot, \mu, \hat{u}) - f^\varepsilon(\cdot, \mu, \hat{u} - \hat{\Delta}) + \hat{r})$$

with $\hat{r} = \theta[f^\varepsilon(\cdot, \mu, u) - f^\varepsilon(\cdot, \mu, \tilde{u})] + f^\varepsilon(\cdot, \mu, \tilde{u}) - f^\varepsilon(\cdot, \mu, \hat{u}) + \operatorname{sign}(t)\dfrac{d\theta}{ds}(u - \tilde{u})$. Hence $r(t) = 0$ except for $l - 1 < |t| < l$. Thus we have

$$\begin{aligned}
\|\hat{r}\|_{X(\mathbb{R},p,\alpha+\beta)} &\leqq e^{-\alpha(l-1)}\, \|\hat{r}\|_{X(\mathbb{R},p,\beta)} \\
&\leqq C(\alpha)\, e^{-\alpha l}\Bigg(\int\limits_{l-1<|t|<l} (e^{-\beta t}(2 + 2\sigma(\varepsilon))\, \|u(t) - \tilde{u}(t)\|_Z)^p\, dt\Bigg)^{1/p} \\
&\leqq \tilde{c}(\alpha)\, e^{-\alpha l}\, \|u - \tilde{u}\|_{Z((-l,-l+1)\cup(l-1,l),p,\beta)}.
\end{aligned} \tag{3.16}$$

Using (3.9) once again we obtain for $\lambda \leqq l - 1$

$$\begin{aligned}
\|u - \tilde{u}\|_{Z((-\lambda,\lambda),p,\beta)} &= \|\hat{\Delta}\|_{Z((-\lambda,\lambda),p,\beta)} \leqq e^{\alpha\lambda}\, \|\hat{\Delta}\|_{Z((-\lambda,\lambda),p\alpha+\beta)} \\
&\leqq e^{\alpha\lambda} \|\hat{\Delta}\|_{Z(\mathbb{R},p,\alpha+\beta)} \leqq C(\alpha + \beta)\, e^{\alpha\lambda}\, \|\hat{r}\|_{X(\mathbb{R},p,\alpha+\beta)}.
\end{aligned}$$

The last two inequalities yield the estimate (3.14).

For the case b) we define $\hat{u}$, $\hat{\Delta}$, and $\hat{r}$ as above, except that the argument of θ is replaced by t. Then the support of $\hat{r}$ is contained in $[0, 1]$ and thus

$$\begin{aligned}
\|\hat{r}\|_{X^0(\mathbb{R},p,\alpha+\beta)} &= \|\hat{r}\|_{X((0,1),p,\alpha+\beta)} \leqq e^{\alpha}\, \|\hat{r}\|_{X((0,1),p,\beta)} \\
&\leqq c(\alpha)\, \|u - \tilde{u}\|_{Z((0,1),p,\beta)}.
\end{aligned} \tag{3.17}$$

For $\lambda \geqq 1$ we derive

$$\begin{aligned}
\|u - \tilde{u}\|_{Z((\lambda,\infty),p,\beta)} &= \|\hat{\Delta}\|_{Z((\lambda,\infty),p,\beta)} \leqq e^{-\alpha\lambda}\, \|\hat{\Delta}\|_{Z((\lambda,\infty),p,\alpha+\beta)} \\
&\leqq e^{-\alpha\lambda}\, \|\hat{\Delta}\|_{Z^0(\mathbb{R},p,\alpha+\beta)} \leqq C(\alpha + \beta)\, e^{-\alpha\lambda}\, \|\hat{r}\|_{X^0(\mathbb{R},p,\alpha+\beta)}.
\end{aligned}$$

Statement (b) then follows from these two estimates. QED.

4. A first example

To see how the theory just developed applies to elliptic problems we first treat the following example.

$$\psi_{xx} + \psi_{yy} + \lambda\psi + g(\psi, \psi_x, \psi_y) = 0 \quad \text{for } (x, y) \in \mathbb{R}\times(0, \pi),$$
$$\psi(x, 0) = \psi(x, \pi) = 0 \quad \text{for } x \in \mathbb{R}. \tag{4.1}$$

This equation may be viewed as a model problem for some inviscid channel flows, ψ being the perturbation of the stream function with respect to the parallel flow (*cf.* KIRCHGÄSSNER [1982], MIELKE [1986]).

For the abstract formulation let $\lambda = \bar{\mu} + \mu$, $u = (\psi, \varphi)$, and $X = \mathring{H}_1(0, \pi)\times L_2(0, \pi)$. Then (4.1) takes the form

$$\frac{d}{dx} u - L(\bar{\mu})\, u = f(\mu, u), \tag{4.2}$$

where $D(L(\bar{\mu})) = (H_2(0, \pi) \cap \mathring{H}_1(0, \pi))$, $L(\mu)(\psi, \varphi) = (\varphi, -\psi_{xx} - \bar{\mu}\psi)$, and $f(\mu, u) = (0, -\mu\psi - g(\psi, \psi_y, \varphi))$. The spectrum of $L(\bar{\mu})$ consists of eigenvalues only, given by $z = \pm z_n = \pm(n^2 - \bar{\mu})^{1/2}$. Hence $L(\bar{\mu})$ has for $\bar{\mu} < 1$ no eigenvalue and for $\bar{\mu} \in [m^2, (m+1)^2)$, $m \in \mathbb{N}$, exactly $2m$ eigenvalues on the imaginary axis.

Using the projection $P_{\bar{\mu}} : X \to X$, $(\psi, \varphi) = \sum_{k=1}^{\infty} (\psi_k, \varphi_k) \sin ky \mapsto P_{\bar{\mu}}(\psi, \varphi) = \sum_{k=1}^{m} (\psi_k, \varphi_k) \sin ky$ for $\bar{\mu} \in [m^2, (m+1)^2)$, $m \in \mathbb{N}$, and $P_{\bar{\mu}} = 0$ for $\bar{\mu} < 1$, we split equation (4.2), using $u_1 = P_{\bar{\mu}} u \in X_1 = P_{\bar{\mu}} X$, $u_2 = u - u_1 \in X_2 = (I - P_{\bar{\mu}})X$, into

$$\begin{aligned} \frac{d}{dx} u_1 - L_1(\bar{\mu})\, u_1 &= P_{\bar{\mu}} f(\mu, u_1 + u_2), \\ \frac{d}{dx} u_2 - L_2(\bar{\mu})\, u_2 &= (I - P_{\bar{\mu}}) f(\mu, u_1 + u_2). \end{aligned} \tag{4.3}$$

To verify the hypotheses (H1)–(H4), let $Z = (H_{s+1}(0, \pi) \cap \mathring{H}_1(0, \pi)) \times \mathring{H}_s(0, \pi)$ and $Z_j = Z \cap X_j$ for some $s \in (\frac{1}{2}, 1)$. (If $g(\psi, \psi_y, \psi_x) = g(\psi)$ it suffices to take $s = 0$, so that $Z_j = X_j$.) Since $H_s(0, \pi)$ is continuously embedded in $C([0, \pi], \mathbb{R})$ we see that $f \in C^2(\mathbb{R}\times Z, V)$ with $V = \{0\}\times L_2(0, \pi) \subset X$. If, moreover, $g(\psi, \psi_y, \psi_x) = \mathcal{O}(\psi^2 + \psi_x^2 + \psi_y^2)$ for $(\psi, \psi_y, \psi_x) \to 0$, then (H1) holds.

From the explicit form of $L = L_1 \otimes L_2$ the hypothesis (H2) follows with $p = \infty$ and $\alpha_L = z_{m+1} = ((m+1)^2 - \bar{\mu})^{1/2} > 0$. Since $\dim X_1 = 2m < \infty$, the properties of $u_1 = K_1(\xi, g_1)$ are trivially satisfied. Moreover, since $g_2 = (0, \tilde{g}) \in V_2(\mathbb{R}, \infty, \alpha)$, the operator K_2 is given by $K_2 g_2(\cdot) = \int_{-\infty}^{\infty} M_{\bar{\mu}}(\cdot - \tau)\, \tilde{g}(\tau)\, d\tau$, where

$$M_{\bar{\mu}}(t)\, \tilde{g} = \sum_{k=m+1}^{\infty} \left(-\frac{1}{z_k}, \operatorname{sign}(t)\right) e^{-z_k|t|} \frac{1}{\pi} \int_0^{\pi} \tilde{g}(\eta) \sin k\eta \, d\eta \, \sin ky.$$

Using $\|\psi\|^2_{\mathring{H}_s(0,\pi)} = \frac{\pi}{2} \sum_{k=1}^{\infty} (k^s \psi_k)^2$ we obtain

$$\|M_{\bar\mu}^-(t)\|_{V\to Z} \leqq c \sup_{k\geqq m+1} k^s \left(\frac{k}{z_k} + 1\right) e^{-z_k|t|}$$
$$\leqq C(\bar\mu) \max\{1, |t|^{-s}\} e^{-z_{m+1}|t|}.$$

For $\alpha \in (0, z_{m+1})$ it then follows that

$$\|K_2 g_2\|_{Z(\mathbb{R},\infty,\alpha)} \leqq \sup_{t\in\mathbb{R}} e^{-\alpha|t|} \int_{-\infty}^{\infty} \|M_{\bar\mu}^-(t-\tau)\|_{V\to Z}\, e^{\alpha|\tau|} \|\tilde g\|_{V(\mathbb{R},\infty,\alpha)}\, d\tau$$
$$\leqq \int_{-\infty}^{\infty} c(\bar\mu) \max\{1, |\xi|^{-s}\} e^{-(z_{m+1}-\alpha)|\xi|}\, d\xi\, \|\tilde g\|_{V(\mathbb{R},\infty,\alpha)}$$
$$= C(\bar\mu, \alpha, s)\, \|\tilde g\|_{V(\mathbb{R},\infty,\alpha)}.$$

Thus (H2) holds. By taking $\chi_j(u_j) = \tilde\chi(\|u_j\|^2_{Z_j})$ for a suitable $\tilde\chi: \mathbb{R} \to [0, 1]$ we easily see that (H3) and (H4) are also satisfied.

Thus the local behavior of (4.1) is described by Theorems 3.1 and 3.4. For example, in the case of a reversible equation, for which $g(\psi, \psi_y, -\psi_x) = g(\psi, \psi_y, \psi_x)$, it is proved in FISCHER [1984] that all solutions on the (two-dimensional) center manifold are periodic with period close to $2\pi/z_1(\bar\mu)$, for $\bar\mu \in (1, 4)$. Hence any solution (μ, ψ) of (4.1) with $\psi: (-l, l)\times(0, \pi) \to \mathbb{R}$ and $|\mu| + \|\psi(x,\cdot)\|_{H_{s+1}(0,\pi)} + \|\psi_x(x,\cdot)\|_{H_s(0,\pi)} \leqq \varepsilon$ for sufficiently small $\varepsilon = \varepsilon(\bar\mu)$, is essentially periodic in $x \in (-l, l)$, *i.e.*, periodic up to perturbations of the size $C(\bar\mu)\, \varepsilon(\bar\mu)\, e^{-z_2(\bar\mu)(l-|x|)/2}$.

5. The classical Saint-Venant principle

We briefly introduce the notations of nonlinear elasticity and report on the results on Saint-Venant's problem obtained in MIELKE [1988b]. Then we show that the abstract methods developed above yield a natural formulation of Saint-Venant's principle for nonlinearly elastic prisms. In particular, there is no restriction to zero cross-sectional resultants or to a special constitutive assumption as in BREUER & ROSEMAN [1977] and KNOPS & PAYNE [1983]. We only have to assume that the strains are pointwise small along the whole body, although the displacements may be arbitrarily large.

We restrict our attention to the case of finite prisms and leave the case of semi-infinite prisms to the reader. Let $\Omega = \Omega_l = \Sigma\times(-l, l)$ be the undeformed prismatic body, where the cross-section Σ with coordinates $x = (x_1, x_2)$ is a bounded region in $\mathbb{R}^2$ with C^2-boundary $\partial\Sigma$ and where $(-l, l)$ is the domain of the axial coordinate s.

For a deformation $\varphi: \Omega_l \to \mathbb{R}^3$ the symmetric strain tensor $E = E(\varphi)$ is given by $\frac{1}{2}(\nabla\varphi^T \nabla\varphi - I)$. Here the superscript T denotes transpose. The stress-strain relation is given in terms of the symmetric second Piola-Kirchhoff stress tensor $S = S(x, E) = C(x)[E] + \mathcal{O}(\|E\|^2)$, where $C(x)$ is the fourth-order tensor of linear elasticity with components $c_{ijkl}(x)$. C is assumed to be independent of s

and positive-definite: $c_{ijkl}(x)\,E_{ij}E_{kl} \geqq c\,\|E\|^2$ with $c > 0$. For hyperelastic materials there is a stored-energy function $\mathscr{W} = \mathscr{W}(x, E(\varphi))$ such that $S = \partial\mathscr{W}/\partial E$, *i.e.*, $S_{ij} = \partial\mathscr{W}/\partial E_{ij}$. In the following we suppress the explicit x-dependence of S and $\mathscr{W}$ for simplicity of notation.

We consider deformations φ satisfying the field equations

$$\begin{aligned} \operatorname{div}\,[\nabla\varphi\, S(E(\varphi))] &= 0 \quad \text{in } \Omega_l, \\ S(E(\varphi))\, n &= 0 \quad \text{on } \partial\Sigma\times(-l, l). \end{aligned} \tag{5.1}$$

In particular, there are neither body forces (in Ω_l) nor tractions on the cylindrical surface. Note that, in keeping with the approach in Section 3, we do not specify boundary conditions on the terminal surfaces $\Sigma\times\{\pm l\}$.

The cross-sectional resultants of forces and moments,

$$\mathscr{F} = \int\limits_{\Sigma\times\{s\}} \nabla\varphi\, S(E(\varphi))\, e_3\, dx, \qquad \mathscr{M} = \int\limits_{\Sigma\times\{s\}} \varphi\times[\nabla\varphi\, S(E(\varphi))\, e_3]\, dx, \tag{5.2}$$

are s-independent vectors for each solution of (5.1). Moreover for hyperelastic materials the scalar

$$\mathscr{E} = \int\limits_{\Sigma\times\{s\}} \left(\frac{\partial\varphi}{\partial s}\cdot[\nabla\varphi\, S(E(\varphi))\, e_3] - \mathscr{W}(E(\varphi))\right) dx \tag{5.3}$$

is also independent of s.

Observe that (5.1) is invariant under rigid transformations, that is, for every $(r, R) \in \mathbb{R}^3\times SO(3)$ the deformation $\tilde\varphi = r + R\varphi$ is a solution of (5.1) whenever φ is. Moreover, $\tilde\varphi$ has resultants $\tilde{\mathscr{F}} = R\mathscr{F}$, $\tilde{\mathscr{M}} = r\times\tilde{\mathscr{F}} + R\mathscr{M}$, and $\tilde{\mathscr{E}} = \mathscr{E}$. These observations suggest that we express φ by

$$\varphi(x, s) = r(s) + R(s)\begin{pmatrix} x_1 + w_1(x, s) \\ x_2 + w_2(x, s) \\ w_3(x, s) \end{pmatrix}, \quad \frac{\partial\varphi}{\partial s}(x, s) = R(s)\begin{pmatrix} w_4(x, s) \\ w_5(x, s) \\ 1 + w_6(x, s) \end{pmatrix}, \tag{5.4}$$

where $\int\limits_\Sigma w_i\, dx = \int\limits_\Sigma x_j w_3\, dx = \int\limits_\Sigma (x_1 w_2 - x_2 w_1)\, dx = 0$ for $i = 1, 2, 3$ and $j = 1, 2$. Hence $r(s)$ and $R(s)$ denote the mean displacement and mean orientation of the cross-section $\Sigma\times\{s\}$. The strains can be expressed in terms of $w(\cdot, s) = (w_1(s), \ldots, w_6(\cdot, s))$.

In the variables r, R and w, (5.1) assumes the form

$$\frac{d}{ds} r = R(e_3 + \gamma(w)), \quad \frac{d}{ds} R = R\Gamma(w), \quad (\Gamma = -\Gamma^T), \tag{5.5}$$

$$\begin{aligned} \frac{d}{ds} w - Lw &= F(w) \quad \text{in } \Omega_l, \\ Bw &= G(w) \quad \text{on } \partial\Sigma\times(-l, l). \end{aligned} \tag{5.6}$$

with $\gamma(w), \Gamma(w) = \mathcal{O}(\|w\|_{D(L)})$ and $F(w), G(w) = \mathcal{O}(\|w\|^2_{D(L)})$ for $w \to 0$. Here we consider $w(\cdot, s)$ as an element of the Banach space $X = \{w \in W^{1+\theta,q}(\Sigma)^3 \times W^{\theta,q}(\Sigma)^3 \mid \int w_i \, dx = \ldots = 0\}$ or of $D(L) = (W^{2+\theta,q}(\Sigma)^3 \times W^{1+\theta,q}(\Sigma)^3) \cap X$, where θ and q are such that $q \in (1, \infty)$, $\theta \in [0, 1/q)$, and $\theta > 2/q - 1$. Although the methods in MIELKE [1988b] are developed only for $q = 2$, they generalize to $W^{\theta,q}(\Sigma)$ according Remark 5.6 therein. For simplicity we henceforth assume that $\theta = 0$, $q > 2$.

An additional nonlinear mapping $w = W(z) = z + \mathcal{O}(\|z\|^2_{D(L)})$ transforms the boundary conditions on $\partial\Sigma \times (-l, l)$ into the linear condition $Bz = 0$. Letting $\mathcal{L}z = Lz$ and $D(\mathcal{L}) = \{z \in D(L) \mid Bz = 0\}$, we can write (5.6) in the form

$$\frac{d}{ds} z - \mathcal{L} z = f(z)$$

where $f = f(z) = \mathcal{O}(\|z\|_{D(\mathcal{L})}^2)$ is a smooth mapping from $D(\mathcal{L})$ into X, since $W^{1,q}(\Sigma)$ is continuously embedded in $C^0(\overline{\Sigma})$. Since the only eigenvalue of $\mathcal{L}$ on the imaginary axis is 0 with a six-dimensional generalized eigenspace X_1, this equation splits into

$$\frac{d}{ds} z_j - \mathcal{L}_j z_j = f_j(z_1, z_2), \quad j = 1, 2.$$

Moreover, defining the local contact force and couple n and m:

$$\mathcal{F} = Rn, \quad \mathcal{M} = Rm + r \times \mathcal{F}, \tag{5.7}$$

we find that $z_1 = Z_1(n, m, z_2) = A(n, m) + \mathcal{O}(|n|^2 + |m|^2 + \|z_2\|^2)$ with $A : \mathbb{R}^6 \to X_1$ invertible. Problem (5.1) is thus locally equivalent to the system

$$\frac{d}{ds} r = R(e_3 + \gamma(n, m, z_2)), \quad \frac{d}{ds} R = R\Gamma(n, m, z_2), \tag{5.8a}$$

$$\frac{d}{ds} n = -\Gamma(\cdots)\, n, \quad \frac{d}{ds} m = -\Gamma(\cdots)\, m + n \times (e_3 + \gamma(\cdots)), \tag{5.8b}$$

$$\frac{d}{ds} z_2 - \mathcal{L} z_2 = f_2(n, m, z_2). \tag{5.8c}$$

(To derive (5.8b) one uses $\frac{d}{ds}\mathcal{F} = \frac{d}{ds}\mathcal{M} = 0$.) We recall that r and R are decoupled from (5.8b, c) due to the symmetry of (5.1) under rigid-body transformations. As is shown in MIELKE [1988a and b], the system (5.8b,c) satisfies the Hypotheses (H1)–(H3) of Section 2, if we let $V_j = X_j$, $Z_j = D(\mathcal{L}_j)$ and $p = q$. Furthermore, (H4) follows from the work of BONIC & FRAMPTON [1966], since $\theta = 0$ and $q > 2$ allows us to take $\chi_j(z_j) = \chi(\|z_j\|_{D(\mathcal{L}_j)})$ for a suitable $\chi : \mathbb{R} \to \mathbb{R}$.

Hence there is a local center manifold for (5.8b and c) with $z_2 = h(n, m) = \mathcal{O}(|n|^2 + |m|^2)$. Substituting $z_2 = h(n, m)$ into (5.8a and b) gives a twelve-dimensional ordinary differential equation having exactly the form of Kirchhoff's rod equations [1859]. See also ANTMAN & KENNEY [1981] for the case of general nonlinear constitutive laws. Here the constitutive relation $(n, m) \leftrightarrow (\tilde{\gamma}, \tilde{\Gamma}) =$

$(\gamma(n, m, h(n, m)), \Gamma(n, m, h(n, m)))$ is obtained in a mathematically rigorous way from the three-dimensional material behavior given by $S = S(x, E)$, $x \in \Sigma$. These rod-like solutions on the center manifold are called the nonlinear *Saint-Venant solutions* of (5.1), as they generalize the well-known Saint-Venant solutions of linear elasticity.

Moreover, the reduced equation

$$\frac{d}{ds} n = -\tilde{\Gamma}(n, m)\, n, \qquad \frac{d}{ds} m = -\tilde{\Gamma}(n, m)\, m + n \times (e_3 + \tilde{\gamma}(n, m)) \tag{5.9}$$

has the first integrals $I_1 = |n|^2$ and $I_2 = n \cdot m$. Hyperelastic materials admit the additional first integral $I_3 = I_3(n, m) = \mathscr{E}(W(z(n, m))) = \frac{1}{2} \sum_{i,j=1}^{3} a_{ij} m_i m_j + n_3 + \mathcal{O}(|n|^2 + |n| \cdot |m| + |m|^3)$ with (a_{ij}) positive definite. Rotationally symmetric prisms have $I_4 = m_3$ for a first integral (*cf.* Holmes & Mielke [1988]). In both cases all solutions on the center manifold are bounded in s and satisfy $|n(s)|^2 + |m(s)|^2 \leqq C(|n(0)| + |m(0)|)$ for $s \in \mathbb{R}$ and a fixed positive C. (If the prism is not rotationally symmetric, then (5.9) exhibits *chaos* in the sense of existence of *Smale horseshoes*.)

So far these are the results of Mielke [1988b] concerning the existence of the center manifold and the structure of the reduced equation. Now we consider arbitrary solutions of (5.1) with small strains and large l. We show that each solution is close to a corresponding Saint-Venant solution, having the same resultants $\mathscr{F}$ and $\mathscr{M}$. The difference between these two solutions decays exponentially with the distance from the terminal surfaces.

To obtain these results we apply Theorem 3.1 to the system (5.8b, c) to obtain an estimate relating the given solution (n, m, z_2) to the center manifold solution $(\tilde{n}, \tilde{m}, \tilde{z}_2)$ uniquely defined by $(\tilde{n}(0), \tilde{m}(0)) = (n(0), m(0))$. The smoothness of the transformation $w \to (n, m, z_2)$ and of its inverse imply estimates relating the given solution $w: (-l, l) \to D(L)$ of (5.5), (5.6) and the reference solution $\tilde{w} = W(z(\tilde{n}, \tilde{m}))$, where $(\tilde{n}, \tilde{m})$ satisfies (5.9).

The general assumption of smallness on w is

$$\|w(\cdot, s)\|_{W^{2,p}(\Sigma)^3 \times W^{1,p}(\Sigma)^3} \leqq \delta \leqq \delta_0 \tag{5.10}$$

for all $s \in (-l, l)$. From (5.4), (5.5), we deduce that

$$\nabla\varphi = R(s)\left[I + \begin{pmatrix} w_{1,1} & w_{1,2} & w_4 \\ w_{2,1} & w_{2,2} & w_5 \\ w_{3,1} & w_{3,2} & w_6 \end{pmatrix} + \left\{\gamma(w) + \Gamma(w)\begin{pmatrix} x_1 + w_1 \\ x_2 + w_2 \\ w_3 \end{pmatrix}\right\} \otimes e_3\right]$$

where $w_{i,j} = \partial w_i / \partial x_j$. Thus, (5.10) implies that

$$\|\nabla\varphi^T(\cdot, s)\, \nabla\varphi(\cdot, s) - I\|_{W^{1,p}(\Sigma)} \leqq C\,\delta \leqq C\,\delta_0. \tag{5.11}$$

However, using Korn's inequality for L^p-spaces (*cf.* Orazov [1989]), we can show that (5.11) also implies (5.10), with $\tilde{C}\delta$ substituted for δ.

To derive estimates for the full deformation $\varphi = r + R(x_1 + w_1, x_2 + w_2, w_3)^T$ we still have to estimate the corresponding translations r and rotations R. To obtain $\tilde{\varphi} = \tilde{r} + \tilde{R}(x_1 + \tilde{w}_1, x_2 + \tilde{w}_2, \tilde{w}_3)^T$ we integrate (5.5) with $w = \tilde{w} = W(z(\tilde{n}, \tilde{m}))$ and initial condition $(r(0), \tilde{R}(0)) = r((0), R(0))$. Let $M(s) = R^T(s)\,\tilde{R}(s) - I$. Then

$$\frac{d}{ds} M = M\Gamma(\tilde{w}) - \Gamma(w)\,M + \Gamma(\tilde{w}) - \Gamma(w), \qquad M(0) = 0.$$

Since $|\Gamma(\tilde{w})|, |\Gamma(w)| \leqq \varrho(\delta) = \mathcal{O}(\delta)$ for $\delta \to 0$, we obtain $|M(s)| \leqq \left| \int\limits_0^s e^{2(s-\sigma)\varrho} \cdot |\Gamma(\tilde{w}[(\sigma)) - \Gamma(w(\sigma))|\, d\sigma \right|$. Hence for small enough δ the estimate

$$\|M\|_{\mathbb{R}^{3\times 3}((-l,l),p,\alpha)} \leqq C\,\|\tilde{w} - w\|_{D(L)((-l,l),p,\alpha)} \tag{5.12a}$$

holds. Similarly, the relation $r(s) - \tilde{r}(s) = \int\limits_0^s \{\tilde{R}(\gamma(w) - \gamma(\tilde{w})) - RM(e_3 + \gamma(w))\}\, d\sigma$ leads to

$$\|r - r\|_{\mathbb{R}^3((-l,l),p,\alpha)} \leqq C\,\|\tilde{w} - w\|_{D(L)((-l,l),p,\alpha)}\,. \tag{5.12b}$$

Note that (5.7) and the special choice of initial conditions imply $\tilde{\mathscr{F}} = \mathscr{F}$, $\tilde{\mathscr{M}} = \mathscr{M}$.

Theorem 5.1. *Consider a prismatic body with cross-section Σ and constitutive law $S = S(x, E)$. Then there is a positive $\alpha_L = \alpha_L(\Sigma, S)$ with the property that for all $\alpha \in (0, \alpha_L)$ and $p > 2$ there are positive constants C and δ_0 such that the following assertion holds:*

For all $l > 0$ and all solutions $\varphi \in W^{2,p}(\Sigma\times(-l, l))^3$ of (5.1) satisfying (5.11) with $\delta \leqq \delta_0$ there is a Saint-Venant solution $\tilde{\varphi}$ of (5.1) with $\tilde{\mathscr{F}} = \mathscr{F}$, $\tilde{\mathscr{M}} = \mathscr{M}$ such that

$$\|\varphi(\cdot,\cdot) - \tilde{\varphi}(\cdot,\cdot)\|_{W^{2,p}(\Sigma\times(-\lambda,\lambda))} \leqq C\,\delta e^{-\alpha(l-\lambda)} \quad \textit{for all } \lambda \in (0, l). \tag{5.13}$$

Proof. By applying Theorem 3.1a to (5.8b, c), we find that $\|w - \tilde{w}\|_{D(L)((-l,l),p,\alpha)} \leqq C\,\delta e^{-\alpha l}$. From (5.12) we then deduce that $\|r - \tilde{r}\|_{\mathbb{R}^3((-l,l),p,\alpha)} + \|R^T\tilde{R} - I\|_{\mathbb{R}^{3\times 3}((-l,l),p,\alpha)} \leqq C\,\delta e^{-\alpha l}$. Using the relations

$$\varphi - \tilde{\varphi} = r - \tilde{r} + R\begin{pmatrix} w_1 - \tilde{w}_1 \\ w_2 - \tilde{w}_2 \\ w_3 - \tilde{w}_3 \end{pmatrix} - R(R^T\tilde{R} - I)\begin{pmatrix} x_1 + \tilde{w}_1 \\ x_2 + \tilde{w}_2 \\ \tilde{w}_3 \end{pmatrix},$$

$$\frac{\partial\varphi}{\partial s} - \frac{\partial\tilde{\varphi}}{\partial s} = R\begin{pmatrix} w_4 - \tilde{w}_4 \\ w_5 - \tilde{w}_5 \\ w_6 - \tilde{w}_6 \end{pmatrix} - R(R^T\tilde{R} - I)\begin{pmatrix} \tilde{w}_4 \\ \tilde{w}_5 \\ 1 + \tilde{w}_6 \end{pmatrix}$$

we obtain $\|\varphi - \tilde{\varphi}\|_{W^{2,p}(\Sigma)((-l,l),p,\alpha)} + \|\partial(\varphi - \tilde{\varphi})/\partial s\|_{W^{1,p}(\Sigma)((-l,l),p,\alpha)} \leqq C\,\delta e^{-\alpha l}$. Moreover, $\partial^2\varphi/(\partial s)^2$ can be calculated by (5.1) in the form $\partial^2\varphi/(\partial s)^2 = g(\varphi, \partial\varphi/ds)$, g being a smooth function from some open set in $W^{2,p}(\Sigma)^3 \times W^{1,p}(\Sigma)^3$ into $L^p(\Sigma)^3$.

Hence $\|\partial^2(\varphi - \tilde{\varphi})/(\partial s)^2\|_{L^p(\Sigma)((-l,l),p,\alpha)} \leqq C\,\delta e^{-\alpha l}$ follows from the estimates for $\varphi - \tilde{\varphi}$ and $(\partial\varphi - \tilde{\varphi})/\partial s$. Now the desired estimate (5.13) follows by noting that

$$\|\varphi - \tilde{\varphi}\|_{W^{2,p}(\Omega_\lambda)} = \|\varphi - \tilde{\varphi}\|_{W^{2,p}(\Sigma)((-\lambda,\lambda),p,0)} + \left\|\frac{\partial\varphi}{\partial s} - \frac{\partial\tilde{\varphi}}{\partial s}\right\|_{W^{1,p}(\Sigma)((-\lambda,\lambda),p,0)} + \left\|\frac{\partial^2\varphi}{\partial s^2} - \frac{\partial^2\tilde{\varphi}}{\partial s^2}\right\|_{L^p(\Sigma)((-\lambda,\lambda),p,0)}$$

and by using the same technique as in the proof of Corollary 3.2a. QED.

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