Spatially Complex Equilibria of Buckled Rods

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1. Introduction

In this paper we study the spatial aspects of equilibrium states exhibited by infinitely or arbitrarily long rods, buckled by loads applied at their ends. Our methods exploit the Hamiltonian structure of the equilibrium equations and use regular perturbation methods based on completely integrable cases (MELNIKOV theory). We obtain a qualitative description of classes of solutions close to such limiting cases, which correspond to geometrical symmetries and the vanishing of certain stress components. Our results imply that there exist spatially irregular or chaotic equilibrium states for rods under the appropriate load conditions.

KIRCHHOFF [1859] was apparently the first to remark the analogy between the equilibrium equations of a rod loaded at its end and the equations of motion of a heavy rigid body pivoted at a fixed point. In this analogy the arclength along the axis of the rod plays the rôle of a time-like coordinate. LARMOR [1884] subsequently extended the analogy to rods with initial curvature and twist; *cf.* LOVE [1927, §§ 259–264]. The discussions in KIRCHHOFF and LOVE assume linear constitutive relations and the equations thus obtained are precisely analogous to the rigid body equations. However, as we show in the next section, the analogy extends to general nonlinear hyperelastic materials. The (non-canonical) Hamiltonian structure of the rigid body equations is preserved, while the quadratic Hamiltonian is replaced by a general function. This structure, and the existence of certain integrals, derive from underlying symmetries and group structures in the problem. While the derivation of the rod equations is relatively well known (*cf.* ANTMAN & KENNEY [1981], ANTMAN [1984], MIELKE [1987]), the Hamiltonian structure is not normally emphasized and so we outline it in Section 2. Here we mean the Hamiltonian structure of the static problem with the arclength as time-like variable, in contrast to the dynamic problem which is a partial differential equation with time and arclength as independent variables. See KRISHNAPRASAD, MARSDEN, & SIMO [1986] for the Hamiltonian structure in that case.

While there is an elegant non-canonical formulation for the three degree of freedom rigid body equations (*cf.* HOLMES & MARSDEN [1983]) we find it more convenient here to work with a reduced two degree of freedom system in canonical coordinates. In general this system is expected to be non integrable; in fact our main results prove this to be true near certain limiting cases. However, in the two cases of zero resultant force and of circular symmetry, an additional integral can be found and the equations solved completely. The former case corresponds to the absence of gravitational forces (moments) and the latter to the well known 'Lagrange' top (GOLDSTEIN [1980, Chapter 5]). In both cases smooth manifolds of heteroclinic or homoclinic orbits exist, and using the perturbative techniques of MELNIKOV [1963] in the Hamiltonian context of HOLMES & MARSDEN [1982, 1983], we are able to prove that these manifolds break to give transverse homoclinic orbits in the presence of small resultant forces or asymmetries. Then, by arguments familiar in dynamical systems (SMALE [1963, 1967], GUCKENHEIMER & HOLMES [1988, Ch. 4–5]), it follows that spatially chaotic equilibrium states occur.

This paper is organized as follows. In Section 2 we outline the derivation of the equilibrium equations and discuss the Hamiltonian structure. In Section 3 we perform our first reduction, obtaining a canonical two degree of freedom Hamiltonian system. The main results are given in Section 4, together with a discussion of their physical implications, including a rough description of the spatial shapes exhibited by rods in such 'chaotic' states. The remainder of the paper is devoted to proofs of the two main theorems of Section 4. Section 5 contains a brief outline of the second reduction and application of MELNIKOV'S method to the resulting periodically perturbed single degree of freedom system. A detailed treatment of this material is contained in HOLMES & MARSDEN [1982, 1983]. The two main theorems are then proved in Section 6 and 7 by computation of Melnikov functions for appropriate limiting cases. Computational details are relegated to the Appendix. In order to make explicit computations we restrict ourselves to stresses which are sufficiently small in magnitude, so that linear elasticity dominates. The geometric nonlinearities are, however, unrestricted.

2. The Rod Equations in Hamiltonian Form

The model of a rod treated in this paper takes the following form (*cf.* ANTMAN [1984], KRISHNAPRASAD, MARSDEN, & SIMO [1986]). We consider a prismatic, elastic body with reference configuration $\Omega = \mathbb{R}\times\Sigma \in \mathbb{R}^3$ or $\bar{\Omega} = I\times\Sigma \in \mathbb{R}^3$, where $t \in \mathbb{R}$ or $t \in I \subset \mathbb{R}$ denotes the axial variable and $x = (x_1, x_2) \in \Sigma$, the cross-section Σ being a bounded domain in $\mathbb{R}^2$. We assume the deformations of the rod have the approximate form $\varphi\colon \Omega, \bar{\Omega} \to \mathbb{R}^3$, with

$$\varphi(t, x) = \boldsymbol{r}(t) + \boldsymbol{R}(t)\begin{pmatrix} x \\ 0 \end{pmatrix}. \tag{2.1}$$

Here the $\boldsymbol{r}(t) \in \mathbb{R}^3$ denotes the position of the deformed axis and $\boldsymbol{R}(t) \in \mathrm{SO}(3)$ specifies the position of the rigidly transformed cross-section. See Figure 1. Note that $\boldsymbol{d}_3 = \boldsymbol{R}(t)\,\boldsymbol{e}_3$, the local coordinate axis perpendicular to the transformed cross-section, need not be tangent to $\boldsymbol{r}'(t)$. Thus we allow for shear deformations.

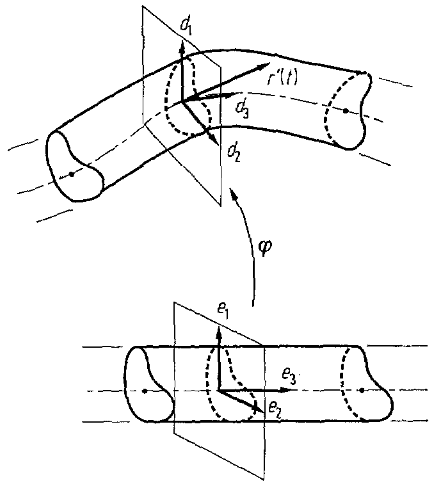

Fig. 1. The rod model.

However, the local coordinate system $\{d_1, d_2, d_3\}$, $d_i = R(t)\, e_i$, is still orthogonal so that $d_3(t) = d_1(t) \times d_2(t)$, as in ANTMAN'S formulation.

There are differing conventions for definition of strains. We adopt the following. We set

$$v = R^T r' - e_3, \qquad \Omega = R^T R' \tag{2.2a, b}$$

so that $\Omega = -\Omega^T \in \mathrm{so}(3)$, and finally define u by

$$u \times a = \Omega a \tag{2.2c}$$

for arbitrary vectors $a \in \mathbb{R}^3$. Since u and Ω are related by (2.2c), we sometimes write $\Omega(u)$ subsequently. Thus $u = u(t)$, $v = v(t)$ are both 3-vectors; they are the strains in body coordinates. Specifically, u_1 and u_2 represent bending in the d_2, d_3 and d_1, d_3 planes respectively and u_3 is torsion ($=$ bending in d_1, d_2); v_1 and v_2 are shears in the d_1, d_2 directions and v_3 is extension in the d_3 direction; *cf.* ANTMAN & KENNEY [1981, § 2].

The elastic properties of the rod are described by a strain energy function $W(u, v): \mathbb{R}^6 \to \mathbb{R}$, which is assumed to have a nondegenerate minimum at $(u, v) = (0, 0)$, corresponding to the undeformed state $r(t) = te_3$, $R(t) = I$. The stresses in body coordinates are then given by

$$m = \frac{\partial W}{\partial u}, \qquad n = \frac{\partial W}{\partial v}, \tag{2.3}$$

the components of m being bending and torsion moments and of n, shear and extension forces.

The equilibrium equations for a rod loaded only at its ends are easily expressed in terms of the force and moment vectors F, M in spatial coordinates (e_1, e_2, e_3). They simply express the fact that F and M are constants. Thus, via (2.1), in body coordinates, we have

$$F = Rn = \text{const}, \tag{2.4a}$$

$$M = Rm + r \times Rn = \text{const}. \tag{2.4b}$$

Differentiating (2.4a), we have $R'n + Rn' = 0$ or

$$n' = -R^T R' n = -\Omega n = n \times u. \tag{2.5}$$

Differentiating (2.4b) and applying R^T from the left yields $R^T R' m + m' + R^T(r' \times Rn + r \times (Rn)') = 0$ or, in view of (2.4a),

$$R^T R' m + m' + R^T r' \times n = 0. \tag{2.6}$$

Using (2.2) equation (2.6) becomes

$$m' = -\Omega m + n \times (e_3 + v) = m \times u + n \times (e_3 + v). \tag{2.7}$$

Equations (2.5) and (2.7) involve the stresses and the strains, but, as W has a nondegenerate minimum at $u = v = 0$, it is locally convex and hence the relations (2.3) are locally invertible. We may therefore write the inverses

$$u = u(m, n), \qquad v = v(m, n) \tag{2.8}$$

and there furthermore exists a real valued function $H(m, n)\colon \mathbb{R}^6 \to \mathbb{R}$ such that

$$u = \frac{\partial H}{\partial m}, \qquad e_3 + v = \frac{\partial H}{\partial n}. \tag{2.9}$$

H and W are related via a Legendre transform; in the dynamical analogy, W plays the rôle of a Lagrangian and H of the Hamiltonian (*cf.* ARNOLD [1978]).

We can now write the equilibrium equations (2.5), (2.7) entirely in terms of the stresses as a non-canonical Hamiltonian system

$$\begin{pmatrix} m \\ n \end{pmatrix}' = J(m, n)\, \nabla H(m, n), \tag{2.10}$$

where

$$J = -J^T = \begin{pmatrix} \Omega(m) & \Omega(n) \\ \Omega(n) & 0 \end{pmatrix}, \qquad \Omega(a)\, b = a \times b. \tag{2.11}$$

Alternatively (2.10) can be written in terms of the Lie-Poisson bracket defined in HOLMES & MARSDEN [1983, eqns. (3.17–18)] for the heavy rigid body. In that case the Hamiltonian is simply $H = \frac{1}{2}\left(\frac{m_i^2}{I_i}\right) + \text{Mgl}\, n_3$, but we note that the symplectic structure permits formulation of equations for general Hamiltonian functions.

Regardless of the precise form of H, the group structure implies that the two functions $I_1 = |n|^2$ and $I_2 = m \cdot n$ are constants of motion for (2.10), along with the Hamiltonian H itself, as one can readily verify. These are the only quantities derivable from $F = \text{const}$ and $M = \text{const}$, that are invariant under motions in $\text{SO}(3) \times \mathbb{R}^3$; *i.e.*, invariant under changes in R and r. (Note, $F \to RF$,

$M \to RM + r \times RF$). In the rigid body analogy I_1 corresponds to conservation of magnitude of the gravity vector and I_2 to conservation of the component of angular momentum in the direction of the gravity vector.

Remark. In the above discussion it is not clear how this special Hamiltonian structure in (2.11) arises. Here we point out briefly how it can be deduced using the methods for Hamiltonian systems on Lie groups as discussed in ABRAHAM & MARSDEN [1978, Ch. 4].

In our case the Lie group is the Euclidean group $SO(3) \times \mathbb{R}^3$, *i.e.* the group of rigid transformations with multiplication $(R_2, r_2)(R_1, r_1) = (R_2 R_1, R_2 r_1 + r_2)$ The Lagrangian L is given by $L(R, r, R', r') = W(R^T R, R^T r' - e_3)$. Since W is invariant under the action of G, it is more convenient to use the "body coordinates" (basis (d_1, d_2, d_3)) than the "spatial coordinates" (with basis (e_1, e_2, e_3)), *i.e.* u and $v + e_3$ are the strains in body coordinates. Now L can be written as a function $\tilde{L}: G \times g \to \mathbb{R}$; $(R, r, \Omega(u), v + e_3) \to W(u, v)$, where $g = T_e G$ is the Lie algebra of G with Lie bracket $[(\Omega_1, v_2), (\Omega_2, v_2)] = (\Omega_1 \Omega_2 - \Omega_2 \Omega_1, \Omega_1 v_2 - \Omega_2 v_1)$.

Similarly the corresponding Hamiltonian H is defined on $G \times g^*$ (g^* = dual of g) rather than on the cotangent bundle T^*G. The variables in g^*, being conjugate to the variables in G, are exactly the stresses m and n in body coordinates. Of course $H = H(m, n)$ is given as in (2.9).

The price for working in body coordinates must be paid when the Hamiltonian structure of $G \times g^*$ is calculated, since "inertial effects" appear. Using Theorem 4.4.1 of ABRAHAM & MARSDEN [1978], we obtain the symplectic form

$$\begin{aligned}
&\omega_{(R,r,\Gamma(m),n)}((R_1', r_1', \Omega_1, n_1), (R_2', r_2', \Omega_2, n_2)) \\
&= R^T R_1' : \Omega_2 - R^T R_2' : \Omega_1 + R^T r_1' \cdot n_2 - R^T r_2' \cdot n_1 \\
&+ \Gamma(m) : (R^T R_1' R^T R_2' - R^T R_2' R^T R_1') + n \cdot (R^T R_1' R^T r_2' - R^T R_2' R^T r_1')
\end{aligned}$$

where $\Omega_1 : \Omega_2 = \operatorname{tr}(\Omega_1^T \Omega_2)$ and $\Gamma(m)\, b = m \times b$.

The associated vector field X_H is defined by

$$\omega_{(R,r,\Omega,u)}(X_H, (R_2', r_2', \Omega_2, n_2)) = \frac{\partial H}{\partial R} : R_2' + \frac{\partial H}{\partial r} r_2' + \frac{\partial H}{\partial \Gamma} : \Omega_2 + \frac{\partial H}{\partial n} \cdot n_2.$$

A direct calculation results in

$$\begin{aligned}
R' &= R \frac{\partial H}{\partial \Gamma}, \; r' = R \frac{\partial H}{\partial n}, \; n' = \frac{\partial H}{\partial \Gamma} n, \\
\Gamma' &= \Gamma \frac{\partial H}{\partial \Gamma} - \frac{\partial H}{\partial \Gamma} \Gamma + \tfrac{1}{2}\left(n \otimes \frac{\partial H}{\partial n} - \frac{\partial H}{\partial n} \otimes n\right),
\end{aligned} \tag{2.12}$$

which are exactly the equations established above, when Γ and m are identified. In addition, we remark that the invariants $F = Rn$ and $M = Rm + r \times Rn$ are exactly the momentum mappings obtained from the invariance of L under the action of G (ABRAHAM & MARSDEN [1978, Theorem 2.12]).

In the analysis below we find it convenient to fix $I_1 = a^2$, $I_2 = ab$ and work with the reduced Hamiltonian system restricted to this (family of) four manifold(s). In Section 3 we show that a symplectic structure exists such that this restriction is a canonical two degree of freedom system. However, the limiting cases which we study below are best described initially in terms of the non-canonical coordinates $(\boldsymbol{m}, \boldsymbol{n})$.

The set $\boldsymbol{n} = \boldsymbol{0}$ is clearly an invariant manifold for (2.10); restricted to this manifold the equation becomes

$$\boldsymbol{m}' = \boldsymbol{m} \times \boldsymbol{u}(\boldsymbol{m}, \boldsymbol{0}), \tag{2.13}$$

which has the additional integral $I_3 = |\boldsymbol{m}|^2$. From MIELKE [1987] we know that, without loss of generality, we can take $\boldsymbol{u}$ to have the form

$$\boldsymbol{u}(\boldsymbol{m}, \boldsymbol{0}) = (\alpha_1 m_1, \alpha_2 m_2, \alpha_3 m_3) + \mathcal{O}(|\boldsymbol{m}|^2) \tag{2.14}$$

where $0 < \alpha_1 \leqq \alpha_2$ and $\alpha_3 > \frac{1}{2}(\alpha_1 + \alpha_2)$. Hence there are two generic cases: $\alpha_1 < \alpha_2 < \alpha_3$ and $\alpha_1 < \alpha_3 < \alpha_2$. We will also consider the special case $\alpha_1 = \alpha_2 < \alpha_3$, which occurs in the case of certain cross-sectional symmetries. The phase portraits on spheres $I_3 =$ constant, sufficiently small, are then qualitatively identical to those for the Euler equations for the gravity free rigid body (GOLDSTEIN [1980, Ch. 5]). Figure 2 shows the case $\alpha_1 < \alpha_2 < \alpha_3$. Note the heteroclinic orbits connecting the hyperbolic fixed points at $m_2 = \pm I_2$, $m_1 = m_3 = 0$. Solutions can be written down explicitly in terms of elliptic functions (WHITTAKER [1937, § 69]) and the heteroclinic orbits are hyperbolic functions (*cf.* HOLMES & MARSDEN [1983]).

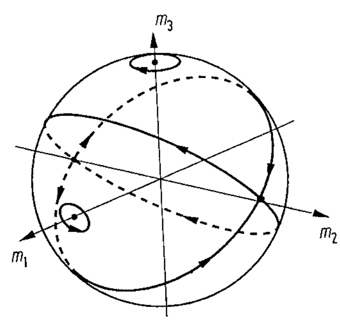

Fig. 2. Heteroclinic cycles for the pure bending case.

This limit corresponds to a rod in pure bending and torsion with zero shear and extension. The fixed points $m_2 = \pm I_2$ correspond to circularly coiled rods: one of the classic planar equilibria found by EULER [1744].

In the case $\alpha_1 < \alpha_3 < \alpha_2$ the phase portraits are similar to Figure 2, but the heteroclinic orbits now connect the fixed points $\boldsymbol{m} = (0, 0, \pm m_3)$. When $\alpha_1 = \alpha_2 < \alpha_3$ the portrait degenerates to a family of circles parallel to the equatorial circle $m_1^2 + m_2^2 =$ const., $m_3 = 0$ and m_3 is an additional constant of motion (see below).

The existence of these integrable limits suggests that we consider a rod with small shear and extension. Letting

$$n = \varepsilon \bar{n}, \quad |\bar{n}| = 1 \tag{2.15}$$

and dropping the bars, we put (2.11) into the form

$$\begin{aligned} m' &= m \times u(m, \varepsilon n) + \varepsilon n \times (e_3 + \mathcal{O}(\varepsilon)), \\ n' &= n \times u(m, \varepsilon n), \end{aligned} \tag{2.16}$$

An analysis of this system lies behind our first main result, Theorem 4.1.

A second important case is provided by rods whose cross-sections are symmetric with respect to the dihedral group D_N with $N \geqq 3$. (Thus, Σ is invariant under rotation by $2\pi/N$). In that case the two constants α_1 and α_2, which appear at leading order in the function $u(m, 0)$ of (2.14) are necessarily equal. If we truncate the Hamiltonian at quadratic terms, then equation (2.16) admits the additional integral m_3, which results from the rotational symmetry of the rod at that order. This corresponds to the integrable Lagrange top, m_3 being the component of angular momentum along the axis of symmetry, and $1/\alpha_1$, $1/\alpha_2$ being the (equal) moments of inertia. Our second main result, Theorem 4.2, concerns this case. Here the perturbation parameter is the scaling parameter δ, with $|n| = \delta^2$ and $m = \delta \hat{m}$. As $\delta \to 0$ we approach the rotationally symmetric Hamiltonian $\hat{H}(\hat{m}, \hat{n}) = \frac{1}{2}(\alpha_i \hat{m}_i^2) + \hat{n}_3$ with $\alpha_1 = \alpha_2 (= \alpha)$. The term of lowest order that breaks circular symmetry while respecting D_N-symmetry is then $\delta^{N-2} \operatorname{Re} (\hat{m}_1 + i\hat{m}_2)^N$.

The fact that H contains terms of higher order in the nonlinear elastic case makes the present analysis more subtle than that of the dynamical analogue carried out by Holmes & Marsden; for example delicate scaling arguments, carried out in Section 3, are necessary to prove Theorems 4.1 and 4.2.

Solution of the Hamiltonian system (2.10) does not specify the spatial state of the rod. Equipped with the stresses $m(t)$ and $n(t)$ as functions of arclength $t \in \mathbb{R}$ (respectively I) we must then integrate the relations (2.2a, b)

$$r' = R(v(m, n) + e_3) \tag{2.17a}$$

$$R' = R\Omega(u(m, n)), \tag{2.17b}$$

using the functions $u = \dfrac{\partial H}{\partial m}$ and $v = \dfrac{\partial H}{\partial n} - e_3$ from (2.9). This yields $r = r(t)$, $R = R(t)$ and hence, via (2.1), the deformation $\varphi(t, x)$.

3. First Reduction: A Symplectic Transformation

The Hamiltonian system (2.10) has a degenerate symplectic structure and is, in fact, a parameterized family of two degree of freedom systems. To reveal this explicitly and make elementary calculations possible, we wish to reduce via the momentum mapping which defines the invariants $I_1 = |n|^2$ and $I_2 = m \cdot n$ (Abraham & Marsden [1978, § 4.3]).

Let the reduced phase space be denoted

$$P_{a,b} = \{(\boldsymbol{m}, \boldsymbol{n}) \in \mathbb{R}^6 \mid |\boldsymbol{n}|^2 = a^2, \boldsymbol{m} \cdot \boldsymbol{n} = ab\} \tag{3.0}$$

for real constants $a \geqq 0$, $b \in \mathbb{R}$. We are especially concerned with limiting cases in which additional integrals $|\boldsymbol{m}|$ and m_3 arise. This suggests that we pick a coordinate system in $P_{a,b}$ which has $|\boldsymbol{m}|$ and m_3 as two of the new variables. We define these variables (r, s, σ, ϱ) by

$$\boldsymbol{m} = \begin{pmatrix} \tilde{r} \cos \sigma \\ \tilde{r} \sin \sigma \\ s \end{pmatrix}, \tag{3.1a}$$

$$\boldsymbol{n} = \frac{ab}{r^2} \begin{pmatrix} \tilde{r} \cos \sigma \\ \tilde{r} \sin \sigma \\ s \end{pmatrix} + a \frac{\bar{r}}{r^2} \begin{pmatrix} s \cos \sigma \\ s \sin \sigma \\ -\tilde{r} \end{pmatrix} \cos \varrho + a \frac{\bar{r}}{r} \begin{pmatrix} -\sin \sigma \\ \cos \sigma \\ 0 \end{pmatrix} \sin \varrho, \tag{3.1b}$$

where $\tilde{r} = \sqrt{r^2 - s^2}$ and $\bar{r} = \sqrt{r^2 - b^2}$. Thus $m_3 = s$, $|\boldsymbol{m}| = r$ and it can be verified that $|\boldsymbol{n}|^2 = a^2$ and $\boldsymbol{m} \cdot \boldsymbol{n} = ab$, so that the point $(\boldsymbol{m}, \boldsymbol{n}) = (\boldsymbol{m}(r, s, \sigma), \boldsymbol{n}(r, s, \sigma, \varrho))$ lies in $P_{a,b}$ as required.

To obtain the symplectic form $\bar{\omega}$ on the reduced space $P_{a,b}$ in (r, s, σ, ϱ) coordinates, we need the transformation matrix

$$G = \frac{\partial(r, s, \sigma, \varrho)}{\partial(\boldsymbol{m}, \boldsymbol{n})}, \tag{3.2}$$

where elements are computed by inversion of equations (3.1 a, b). The variables r, s and σ are easily obtained in terms of $\boldsymbol{m}, \boldsymbol{n}$:

$$r = \sqrt{m_1^2 + m_2^2 + m_3^2}, \quad s = m_3, \quad \sigma = \arctan (m_2/m_1) \tag{3.3a}$$

and ϱ is implicitly defined via

$$r^2 n_3 = abs - a\bar{r}\tilde{r} \cos \varrho \Rightarrow \varrho = \bar{\varrho}(r, s, n_3). \tag{3.3b}$$

Thus we have

$$G = \begin{bmatrix} \frac{\tilde{r}}{r} \cos \sigma & \frac{\tilde{r}}{r} \sin \sigma & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{\tilde{r}} \sin \sigma & \frac{1}{\tilde{r}} \cos \sigma & 0 & 0 & 0 & 0 \\ \frac{\tilde{r}}{r} \varrho_1 \cos \sigma & \frac{\tilde{r}}{r} \varrho_1 \sin \sigma & \frac{s}{r} \varrho_1 + \varrho_2 & 0 & 0 & \varrho_3 \end{bmatrix}, \tag{3.4}$$

where

$$\varrho_1 = \frac{\partial \bar{\varrho}}{\partial r}, \varrho_2 = \frac{\partial \bar{\varrho}}{\partial s} \quad \text{and} \quad \varrho_3 = \frac{\partial \bar{\varrho}}{\partial n_3} = \frac{r^2}{a\bar{r}\tilde{r} \sin \varrho} \tag{3.5}$$

are obtained from (3.3b). We shall not require the explicit expressions for ϱ_1 and ϱ_2.

Writing the matrix J of (2.12) explicitly in $\boldsymbol{m}, \boldsymbol{n}$ coordinates, we have

$$J = 0 \begin{bmatrix} 0 & -m_3 & m_2 & 0 & -n_3 & n_2 \\ m_3 & 0 & -m_1 & n_3 & 0 & -n_1 \\ -m_2 & m_1 & 0 & -n_2 & n_1 & 0 \\ 0 & -n_3 & n_2 & & & \\ n_3 & 0 & -n_1 & & \mathbf{0} & \\ -n_2 & n_1 & 0 & & & \end{bmatrix}. \tag{3.5}$$

The new symplectic form $\bar{\omega}$ is then given by the matrix

$$J_{ab} = GJG^T, \tag{3.7}$$

and an elementary calculation using (3.4)–(3.6) and (3.1) yields

$$J_{a,b} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{3.8}$$

Thus the reduced Hamiltonian system can be written

$$\dot{\boldsymbol{x}} = J_{a,b} \nabla_{\boldsymbol{x}} H_{a,b}(\boldsymbol{x}) \tag{3.9}$$

where $\boldsymbol{x} = (r, s, \sigma, \varrho)^T$ and $H_{a,b}(r, s, \sigma, \varrho) = H(\boldsymbol{m}(r, s, \sigma), \boldsymbol{n}(r, s, \sigma, \varrho))$. The explicit form of $J_{a,b}$ implies that (3.9) is canonical with (s, σ) and (r, ϱ) as conjugate pairs of variables.

To appreciate that this reduction process simplifies the problem, we consider the following special cases. For a rotationally symmetric rod we know that $H(\boldsymbol{m}, \boldsymbol{n})$ is of the form

$$H(m, n) = \mathscr{H}(m_1^2 + m_2^2, m_3, n_1^2 + n_2^2, n_3, m_1 n_1 + m_2 n_2, m_1 n_2 - m_2 n_1) \tag{3.10}$$

(*cf.* Antman & Kenney [1981]). Now $H_{a,b}$ is given by

$$H_{a,b}(r, s, \delta, \varrho) = \mathscr{H}\left(r^2 - s^2, s, a^2\left(1 - \frac{(bs - \bar{r}\tilde{r}\cos\varrho)^2}{r^4}\right), a\frac{bs - \bar{r}\tilde{r}\cos\varrho}{r^2}, \right.$$
$$\left. \frac{a}{r^2}(b\tilde{r}^2 + \bar{r}^2 s\cos\varrho), \frac{a\bar{r}\tilde{r}}{r}\sin\varrho\right) \tag{3.11}$$

and is therefore independent of σ. Thus σ is a cyclic variable and s a constant of the motion.

On the other hand the case $a = 0$ $(\boldsymbol{n} = \boldsymbol{0})$ leads to

$$H_{0,b}(r, s, \sigma, \varrho) = H(\tilde{r}\cos\sigma, \tilde{r}\cos\sigma, s, 0, 0, 0), \tag{3.12}$$

and here ϱ is cyclic and r a constant of the motion.

To perform explicit calculations we restrict our analysis to very small $\boldsymbol{m}$ and $\boldsymbol{n}$. Then only the lowest order terms of $H(m, n) = \frac{1}{2}\alpha_i m_i^2 + n_3 + \mathcal{O}(|\boldsymbol{m}|^3, |\boldsymbol{n}|\,|\boldsymbol{m}|, |\boldsymbol{n}|^2)$ are relevant. We use the scaling $\boldsymbol{m} = \delta\hat{\boldsymbol{m}}$, $\boldsymbol{n} = \varepsilon\,\delta^2\hat{\boldsymbol{n}}$ with $|\hat{\boldsymbol{n}}| = 1$ and define

$$\hat{H}(\hat{m}, \hat{n}, \varepsilon, \delta) = \frac{1}{\delta^2} H(\delta\hat{m}, \varepsilon\,\delta^2\hat{n}) = \tfrac{1}{2}\alpha_i\hat{m}_i^2 + \varepsilon\hat{n}_3 + \mathcal{O}(\delta). \tag{3.13}$$

The corresponding $(a, b, r, s, \sigma, \varrho)$-scaling is

$$a = \varepsilon\,\delta^2\hat{a}, \quad b = \varepsilon\,\delta^3\hat{b}, \quad r = \delta\hat{r}, \quad s = \delta\hat{s}, \quad (\sigma = \hat{\sigma}, \varrho = \hat{\varrho}), \tag{3.14}$$

where $\hat{a} = 1$. Henceforth we need the (r, s, σ, ϱ)-coordinates in the scaled version only. Thus we drop the hats on (r, s, σ, ϱ) and b but retain them on $(\hat{\boldsymbol{m}}, \hat{\boldsymbol{n}})$. We define

$$\begin{aligned}\tilde{H}_b(r, s, \sigma, \varrho, \varepsilon, \delta) &= \frac{1}{\delta^2} H_{\varepsilon\delta^2,\varepsilon\delta^3 b}(\delta r, \delta s, \sigma, \varrho)\\ &= \tfrac{1}{2}(\alpha_1\tilde{r}^2\cos^2\sigma + \alpha_2\tilde{r}^2\sin^2\sigma + \alpha_3 s^2)\\ &\quad + \frac{\varepsilon}{r^2}(bs - \bar{r}\tilde{r}\cos\varrho) + \mathcal{O}(\delta).\end{aligned} \tag{3.15}$$

Observe that the limit $\delta = 0$ is identical to the heavy rigid body if $\varepsilon \neq 0$ and note that ε is not necessarily small; this is important in our analysis of the D_N-symmetric rod.

In both cases the canonical structure of the reduced system explicitly reveals the symmetries implicit in the original non-canonical structure.

The case $\alpha_1 < \alpha_3 < \alpha_2$ requires a slightly different coordinate system. In that case the unperturbed $(\varepsilon = 0)$ behavior is essentially the identical to that of $\alpha_1 < \alpha_2 < \alpha_3$ with m_2, m_3 and n_2, n_3 interchanged. Instead of (3.1 a, b), we define an analogous coordinate system by interchanging m_2 and m_3 in definition (3.1 a) and n_2 and n_3 in (3.1 b). Everything goes through in the same manner as previously, except that the Hamiltonian (3.15) is replaced by

$$\begin{aligned}\tilde{H}_b &= \tfrac{1}{2}(\alpha_1\tilde{r}^2\cos^2\sigma + \alpha_3\tilde{r}^2\sin^2\sigma + \alpha_2 s^2)\\ &\quad + \frac{\varepsilon a}{r^2}((b\tilde{r} + \bar{r}s\cos\varrho)\sin\sigma + r\bar{r}\cos\sigma\sin\varrho) + \mathcal{O}(\delta).\end{aligned} \tag{3.16}$$

In the rotationally symmetric case we use the same scaling but with $\varepsilon = 1$. Here, since $\alpha_1 = \alpha_2$, for $\varepsilon = 0$ the invariant spheres $|\boldsymbol{m}| = \delta$ are filled with periodic orbits lying in the planes $m_3 = \text{const}$. To get a homoclinic solution we fix $\varepsilon = 1$ and treat the limit $\delta = 0$, which is exactly the case treated in HOLMES & MARSDEN [1983].

4. The Main Results: Equilibrium States of Buckled Rods

We state our main result in terms of the two-dimensional systems on the reduced phase spaces $P_{a,b} = \{(\boldsymbol{m}, \boldsymbol{n}) \mid |\boldsymbol{n}| = a, \boldsymbol{m}\cdot\boldsymbol{n} = ab\}$:

Theorem 4.1. *Let* $H(\boldsymbol{m}, \boldsymbol{0}) = \frac{1}{2}\alpha_i m_i^2 + \mathcal{O}(|m_i|^3)$. *We then have the following cases:*

I. $0 < \alpha_1 < \alpha_2 < \alpha_3$. *For all sufficiently small* $\varepsilon,\ \delta > 0$ *and all* $b \in (-1, 1)$ *there exists a pair of periodic solutions* $(\boldsymbol{m}, \boldsymbol{n}) = (0, \pm\delta, 0; \boldsymbol{0}) + \mathcal{O}(\delta^2)$ *on* $P_{\varepsilon\delta^2, \varepsilon\delta^3 b} \cap \{H = \frac{1}{2}\alpha_2\,\delta^2\}$ *which are connected by transverse heteroclinic cycles.*

II. $0 < \alpha_1 < \alpha_3 < \alpha_2$. *For all sufficiently small* $\varepsilon,\ \delta > 0$ *and all b satisfying*

$$b^2 < \frac{\pi^2\alpha_3^2}{\pi^2\alpha_3^2 + 4(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)\cosh^2\left(\dfrac{\pi\alpha_3}{2\sqrt{(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)}}\right)} \tag{4.1}$$

there exists a pair of periodic solutions $(\boldsymbol{m}, \boldsymbol{n}) = (0, 0, \pm\delta; \boldsymbol{0}) + \mathcal{O}(\delta^2)$ *on* $P_{\varepsilon\delta^2, \varepsilon\delta^3 b} \cap \{H = \frac{1}{2}\alpha_3\,\delta^2\}$ *which are connected by transverse heteroclinic cycles.*

The second result concerns rods which are symmetric with respect to rotations through the angle $2\pi/N$ about their axes in the reference configuration. In the (r, s, σ, ϱ)-coordinates this corresponds exactly to the transformation $\sigma \to \sigma + 2\pi/N$. Moreover we assume that the cross-section has also a reflectional symmetry; and without loss of generality let the $\boldsymbol{d}_2$-axis be the symmetry axis. Then the Hamiltonian $H = H(\boldsymbol{m}, \boldsymbol{n})$ is invariant under the reflections $S_1 : (\boldsymbol{m}, \boldsymbol{n}) \to (m_1, -m_2, -m_3, -n_1, n_2, n_3)$ and $S_3(\boldsymbol{m}, \boldsymbol{n}) \to (m_1, m_2, -m_3, -n_1, -n_2, n_3)$ which correspond to reflection of material points in the rod with respect to the $\boldsymbol{d}_2, \boldsymbol{d}_3$-plane and $\boldsymbol{d}_1, \boldsymbol{d}_2$-plane respectively. Observe that only the composition $S = S_1 S_3$ maps $P_{a,b}$ onto itself if $b \neq 0$. Since $S : (r, s, \sigma, \varrho) \to (r, s, -\sigma, -\varrho)$ the Hamiltonian $\tilde{H}_b$ satisfies

$$\tilde{H}_b(r, s, \sigma, \varrho, \varepsilon, \delta) = \tilde{H}_b(r, s, \sigma + 2\pi/N, \varrho, \varepsilon, \delta) = \tilde{H}_b(r, s, -\sigma, -\varrho, \varepsilon, \delta). \tag{4.2}$$

To find the lowest order at which the first nontrivial D_N-symmetric term can occur we have to appeal to the methods in invariant theory (*cf.* Buzano, Geymonat, & Poston [1985]). In our case the corresponding polynomials, being D_N-invariant but not rotationally symmetric, are of the form $\operatorname{Re}(m_1 + im_2)^k (n_1 + in_2)^{N-k}$ where $k = 0, 1, \ldots, N$. However, since our scaling is of the form $\boldsymbol{m} = \delta\hat{\boldsymbol{m}},\ \boldsymbol{n} = \delta^2\hat{\boldsymbol{n}}$, (recall $\varepsilon = 1$ here) the lowest order term is $\operatorname{Re}(m_1 + im_2)^N$ with order $\mathcal{O}(\delta^N)$.

For $N \geqq 3$ and under the assumption that $H = H(\boldsymbol{m}, \boldsymbol{n})$ is $N + 1$ times continuously differentiable, we conclude that the scaled Hamiltonian has the form

$$\hat{H}_b(r, s, \sigma, \varrho, \varepsilon, \delta) = \hat{H}_b^*(r, s, \varrho, \varepsilon, \delta) + c\,\delta^{N-2}(r^2 - s^2)^{N/2}\cos N\sigma + \mathcal{O}(\delta^{N-1}). \tag{4.3}$$

Observe that the quadratic terms are σ-independent, *i.e.* $H(\boldsymbol{m}, \boldsymbol{n}) = \frac{1}{2}(\alpha(m_1^2 + m_2^2) + \alpha_3 m_3^2) + \mathcal{O}(|\boldsymbol{m}|^3)$ and thus $\alpha_1 = \alpha_2 = \alpha < \alpha_3$. Without loss of generality we restrict our attention to the case $\varepsilon = 1$.

We can now state the second main result:

Theorem 4.2. *Let the Hamiltonian satisfy* (4.2) *with* $N \geqq 3$ *and assume that c in* (4.3) *is nonzero. Then, for all sufficiently small* $\delta > 0$ *and all b with* $0 < b^2 < 4/\alpha$

there exists a periodic solution $m = \delta b e_3 + \mathcal{O}(\delta^2)$, $n = \delta^2 e_3 + \mathcal{O}(\delta^3)$ *on* $P_{\delta^2,\delta^3 b} \cap \left\{H = \delta^2 \left(1 + \frac{\alpha_3}{2} b^2\right)\right\}$ *which possesses transverse homoclinic orbits.*

To appreciate the physical implications of these results, we must anticipate some of the material outlined in Section 5. The theorems are proved by perturbation arguments involving solutions lying close to homoclinic or heteroclinic orbits to hyperbolic saddle points of the unperturbed systems. The existence of transverse homoclinic or heteroclinic points then implies, via the Smale-Birkhoff homoclinic theorem (GUCKENHEIMER & HOLMES [1983, § 5.3]), that there exist solutions which remain in a neighborhood of the unperturbed homoclinic or heteroclinic orbits and which are chaotic in the following sense. There are two (or more) disjoint closed sets in the phase space which the solutions pass through in any prescribed sequence. The sets can be chosen to lie in any neighborhood of the saddle point(s); in the event of a set of multiple homoclinic orbits or heteroclinic cycles, this implies that the perturbed solution passes near different members of the set in any order. The consequence is that nonperiodic orbits near the transverse homoclinic or heteroclinic orbits are, to first order, quasi-random superpositions of single, unperturbed homoclinic or heteroclinic orbits. Thus, to understand typical global structures of perturbed orbits, and the spatial equilibrium states to which they correspond, we must first consider the spatial states corresponding to unperturbed orbits.

As we observed at the end of Section 2, to obtain equilibrium shapes from the stresses $(m(t), n(t))$, we must integrate equations (2.17a, b) and recall the original spatial description of the deformed rod of equation (2.1). We will concentrate on the implications for the vector $r(t)$ describing the position of the axis of the deformed rod. To arrange the discussion clearly we will only deal with the scaling limit $\delta = 0$. Thus we scale $r(t)$ and $R(t)$ in the following way

$$r(t) = \frac{1}{\delta}\hat{r}(\delta t), \qquad R(t) = \hat{R}(\delta t). \tag{4.4}$$

For $\delta = 0$ we are left with

$$\hat{r}' = \hat{R} e_3, \qquad \hat{R}' = \hat{R} \begin{bmatrix} 0 & -\alpha_3 \hat{m}_3 & \alpha_2 \hat{m}_2 \\ \alpha_3 \hat{m}_3 & 0 & -\alpha_1 \hat{m}_1 \\ -\alpha_2 \hat{m}_2 & \alpha_1 \hat{m}_1 & 0 \end{bmatrix} \tag{4.5a, b}$$

if $\hat{m}(t)$ is already known.

In the case $0 < \alpha_1 < \alpha_2 < \alpha_3$ the four heteroclinic solutions on the manifold $|\hat{m}| = 1$, $|\hat{n}| = 0$ are given by

$$\begin{bmatrix} \hat{m}_1(t) \\ \hat{m}_2(t) \\ \hat{m}_3(t) \end{bmatrix} = \begin{bmatrix} K_1 \sqrt{-a_1/a_2}\,\operatorname{sech}(\sqrt{a_1 a_3}\, t) \\ -K_1 K_2 \tanh(\sqrt{a_1 a_3}\, t) \\ K_2 \sqrt{-a_3/a_2}\,\operatorname{sech}(\sqrt{a_1 a_3}\, t) \end{bmatrix} \tag{4.6}$$

where $K_1, K_2 \in \{-1, 1\}$, $a_1 = \alpha_3 - \alpha_2 > 0$, $a_2 = \alpha_1 - \alpha_3 < 0$, and $a_3 = \alpha_2 - \alpha_1 > 0$. It will be convenient to express the rotation matrix $\hat{R}$ in terms of the classical Euler angles ψ, θ, φ:

$$\hat{\boldsymbol{R}} = \boldsymbol{D}_\psi \overline{\boldsymbol{D}}_\theta \boldsymbol{D}_\varphi, \tag{4.7a}$$

where

$$\boldsymbol{D}_\psi = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \overline{\boldsymbol{D}}_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}. \tag{4.7b}$$

Since $\boldsymbol{F} = \hat{\boldsymbol{R}}\hat{\boldsymbol{n}} = \boldsymbol{0}$ for the heteroclinic orbits, we have $\boldsymbol{M} = \hat{\boldsymbol{R}}\hat{\boldsymbol{m}} = \text{const.}$ We are free to orient our spatial coordinates and, therefore, we pick $\boldsymbol{M} = \boldsymbol{e}_3$ since this simplifies the calculations. Use of the definition (4.7) yields then

$$\hat{\boldsymbol{m}} = \hat{\boldsymbol{R}}^T \boldsymbol{e}_3 = \begin{bmatrix} \sin\theta \quad \sin\varphi \\ \sin\theta \quad \cos\varphi \\ \cos\theta \end{bmatrix}. \tag{4.8}$$

Comparing this expression with (4.6), we find

$$\cos\theta = \hat{m}_3, \qquad \varphi = \arctan(\hat{m}_1/\hat{m}_2). \tag{4.9}$$

We will also require ψ. From (4.5b) and (4.7) we derive, after some calculation, the relation

$$\psi' \sin\theta = \alpha_1 \hat{m}_1 \sin\varphi + \alpha_2 \hat{m}_2 \cos\varphi, \tag{4.10}$$

or, with (4.8) and (4.9),

$$\psi' = \alpha_1 \sin^2\varphi + \alpha_2 \cos^2\varphi = \frac{\alpha_1 \hat{m}_1^2 + \alpha_2 \hat{m}_2^2}{\hat{m}_1^2 + \hat{m}_2^2}. \tag{4.11}$$

Assuming without loss of generality that $\psi(0) = 0$, by direct integration we obtain

$$\psi(t) = \alpha_2 t - \arctan(\sqrt{a_3/a_1} \tanh(\sqrt{a_1 a_3}\, t)). \tag{4.12}$$

We are now in a position to compute the equilibrium shape in terms of the displacement vector $\hat{\boldsymbol{r}}(t)$ by integrating (4.5a). Using (4.7) we obtain

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}(0) + \int_0^t \begin{bmatrix} \sin\theta \quad \sin\psi \\ -\sin\theta \quad \cos\psi \\ \cos\theta \end{bmatrix} dt. \tag{4.13}$$

Realizing that $s = \hat{m}_3 = \cos\theta$, $\sin\theta = \sqrt{1 - s^2}$, and $\psi(t) = -\varrho(t)$ (*cf.* (6.2)), we use the relations (A.5a, b) to obtain

$$\begin{aligned} \hat{r}_1(t) &= \hat{r}_1(0) - \frac{1}{\alpha_2} \sin\theta \cos\psi + \frac{\sqrt{-a_1 a_2}}{\alpha_2} \int_0^t \cos^2\theta \sin\alpha_2 t \, dt, \\ \hat{r}_2(t) &= \hat{r}_2(0) + \frac{1}{\alpha_2} \sin\theta \sin\psi + \frac{\sqrt{-a_1 a_2}}{\alpha_2} \int_0^t \cos^2\theta \cos\alpha_2 t \, dt, \\ \hat{r}_3(t) &= \hat{r}_3(0) + K_2 \frac{2}{\sqrt{-a_1 a_2}} \arctan(\tanh(\sqrt{a_1 a_3}\, t/2)). \end{aligned} \tag{4.14}$$

To understand the geometric shape of these solutions we first look at the limits $t \to \pm\infty$. Since $\cos\theta(t) \to 0$ and $\psi(t) - \alpha_2 t \to \pm A_1$ with $A_1 = \arctan(\sqrt{a_3/a_1})$, we have for $t \to \pm\infty$

$$\hat{r}(t) = \hat{r}(0) + \begin{bmatrix} A_2 \\ \pm A_3 \\ \pm K_2 A_4 \end{bmatrix} + \frac{1}{\alpha_2}\begin{bmatrix} \cos(\alpha_2 t \pm A_1) \\ \sin(\alpha_2 t \pm A_1) \\ 0 \end{bmatrix} + \mathcal{O}(e^{-\sqrt{a_1 a_3}|t|}), \quad (4.15)$$

where A_2 is some constant and A_3 and A_4 are given by

$$A_3 = \frac{\pi}{2\sqrt{-a_1 a_2}} \operatorname{cosech}\left(\frac{\pi\alpha_2}{2\sqrt{a_1 a_3}}\right) < \frac{1}{\alpha_2}, \qquad A_4 = \pi/2\sqrt{-a_1 a_2}\,. \quad (4.16)$$

The oscillatory behavior for large t corresponds to the fact that the saddle points $\hat{m} = (0_1, \pm 1, 0)$, $\hat{n} = \mathbf{0}$, represent rods in pure bending, coiled onto themselves in one plane, here parallel to the 1, 2-plane. The full rod configuration, obtained by numerical integration of (4.14) from $t = -10$ to 10, is sketched in Figure 3.

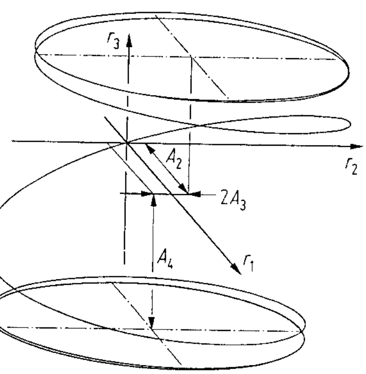

Fig. 3. Sketch of the rod configuration corresponding to the unperturbed heteroclinic orbits for $0 < \alpha_1 = 2 - 1/\sqrt{3} < \alpha_2 = 2 < \alpha_3 = 2 + \sqrt{3}$.

As Figure 2 indicates, on the sphere $|\hat{m}| = 1$ there are four heteroclinic solutions. Yet we obtain only two different geometric shapes ($K_2 = \pm 1$). The difference between the cases $K_1 = +1$ and $K_1 = -1$ in (4.6) is obtained by changing φ to $\varphi + \pi$, *i.e.* the centerline stays the same but the rod is first rotated by π around its centerline and then bent into the same configuration.

To obtain an idea of what the chaotic equilibrium states of such a rod may look like, we remind the reader that such solutions always remain near one of the perturbed heteroclinic solutions. These are themselves close ($\mathcal{O}(\varepsilon)$) to the unperturbed solutions. Hence we may think of the chaotic states as fairly arbitrary combinations of "elements", each of which is, up to $\mathcal{O}(\varepsilon)$, an unperturbed solution taken over a large but finite arclength. Of course we have to satisfy two conditions. First there is a "dynamical" condition that near a saddle point there are only three choices for the continuing solution: either it remains there or leaves in the neighborhood of one of the two branches of the unstable manifold. Second, there is the overall compatibility of the equilibrium to consider. We must arrange each "elementary" configuration in $\mathbb{R}^3$ in such a way that the resultants $\boldsymbol{F}$ and $\boldsymbol{M}$ are the same as for the other elements.

The two other cases, $0 < \alpha_1 < \alpha_3 < \alpha_2$ and $0 < \alpha_1 = \alpha_2 < \alpha_3$, lead in a similar fashion to the corresponding shapes of heteroclinic or homoclinic solutions in the unperturbed case. For $0 < \alpha_1 < \alpha_3 < \alpha_2$ the four heteroclinic solutions are

$$\hat{m}(t) = \begin{bmatrix} K_1 \sqrt{-a_1/a_3} \operatorname{sech}(\sqrt{a_1 a_2}\, t) \\ K_2 \sqrt{-a_2/a_3} \operatorname{sech}(\sqrt{a_1 a_2}\, t) \\ K_1 K_2 \tanh(\sqrt{a_1 a_2}\, t) \end{bmatrix}. \tag{4.17}$$

From (4.8) we obtain $\cos\theta = K_1 K_2 \tanh(\sqrt{a_1 a_2}\, t)$ and $\varphi = K_1 K_2 \arctan \sqrt{a_1/a_2}$. Hence, (4.11) yields $\psi(t) = \alpha_3 t$ and (4.13) gives

$$\hat{r}(t) = \hat{r}(0) + \int_0^t \begin{bmatrix} \operatorname{sech}(\sqrt{a_1 a_2}\, t) \sin \alpha_3 t \\ -\operatorname{sech}(\sqrt{a_1 a_2}\, t) \cos \alpha_3 t \\ K_1 K_2 \tanh(\sqrt{a_1 a_2}\, t) \end{bmatrix} dt.$$

In the limit $t \to \pm\infty$ $\hat{r}(t)$ behaves as follows

$$\hat{r}(t) = \hat{r}(0) + \begin{bmatrix} B_1 \\ \pm B_2 \\ \pm B_3 \end{bmatrix} + K_1 K_2 \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} + \mathcal{O}(e^{-\sqrt{a_1 a_2}|t|}) \tag{4.18}$$

where $B_2 = \dfrac{\pi}{2\sqrt{a_1 a_2}} \operatorname{sech}\left(\dfrac{\pi \alpha_3}{2\sqrt{a_1 a_2}}\right)$.

In the case $0 < \alpha_1 = \alpha_2 = \alpha < \alpha_3$ the unperturbed solution is a homoclinic solution defined by

$$\begin{aligned} r(t) &= \sqrt{b^2 + \gamma^2 \operatorname{sech}^2\left(\frac{\alpha\gamma}{2} t\right)}, \qquad s = b, \\ \sigma(t) &= \left(\frac{\alpha}{2} - \alpha_3\right) bt, \qquad \cos\varrho = \frac{\alpha}{2} r^2 - 1, \end{aligned} \tag{4.19}$$

where $\gamma^2 = \dfrac{4}{\alpha} - b^2$. Now $\boldsymbol{F} = R\hat{n}$ is different from zero, and we may assume $\boldsymbol{F} = \boldsymbol{e}_3$. In a manner similar to (4.8) we obtain by using (3.1)

$$\hat{n} = \begin{bmatrix} \dfrac{b\bar{r}}{r^2} \cos\sigma\,(1 + \cos\varrho) - \dfrac{\bar{r}}{r} \sin\sigma \sin\varrho \\ \dfrac{b\bar{r}}{r^2} \sin\sigma(1 + \cos\varrho) + \dfrac{\bar{r}}{r} \cos\sigma \sin\varrho \\ \dfrac{b^2}{r^2} - \dfrac{\bar{r}^2}{r^2} \cos\varrho \end{bmatrix} = \begin{bmatrix} \sin\theta \quad \sin\varphi \\ \sin\theta \quad \cos\varphi \\ \cos\theta \end{bmatrix}.$$

Hence, (4.19) gives us

$$\cos\theta = 1 - \frac{\alpha\gamma^2}{2}\operatorname{sech}^2\left(\frac{\alpha\gamma}{2}t\right).$$

Using (4.13) again, we find that the foregoing implies for $\hat{r}(t)$ the limit behavior

$$\hat{r}(t) = \begin{bmatrix} \pm C_1 \\ C_2 \\ \pm C_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} + \mathcal{O}\left(e^{-\frac{\alpha\gamma}{2}|t|}\right) \tag{4.20}$$

for $t \to \pm\infty$. In these two cases the asymptotic states (corresponding to the saddle points in $(\boldsymbol{m}, \boldsymbol{n})$-space) are straight rods; in (4.18) in pure torsion and in (4.20) in mixed torsion and tension.

5. Second Reduction: Melnikov's Method

In this section we outline the method of MELNIKOV [1963], as generalized and adapted to the analysis of Hamiltonian systems having two degrees of freedom by HOLMES & MARSDEN [1982, 1983]. We first outline the reduction procedure (*cf.* BIRKHOFF [1927, Ch. 8], WHITTAKER [1937, Ch. 12], ARNOLD [1978, § 45B]).

We start with a Hamiltonian of the form

$$H_\varepsilon = H_0(q, p, I) + \varepsilon H_1(q, p, \theta, I, \varepsilon), \tag{5.1}$$

where (q, p) are conjugate variables and (I, θ) are conjugate variables in action-angle form, so that H_1 is 2π-periodic in θ. For $\varepsilon = 0$ θ is a cyclic variable and Hamilton's equations are completely integrable. We suppose $H_1(q, p, \theta, I, 0)$ is bounded on bounded sets and that H_0 and H_1 are sufficiently differentiable for the power series manipulations which follow (C^3 will suffice).

Our specific assumptions on the unperturbed Hamiltonian $H_0(q, p, I)$ are that

(1) The system

$$\dot{q} = \partial H_0/\partial p, \quad \dot{p} = -\partial H_0/\partial q \tag{5.2}$$

possesses a homoclinic orbit $\bar{x}_h = (\bar{q}(t - t_0; h), \bar{p}(t - t_0; h))$ to a hyperbolic fixed point $x_0 = (q_0, p_0)$ for each total energy $H_0 = h$ in some interval $J \subset \mathbb{R}$. Note that $\bar{x}_h$ depends on h via the action $I = I_h$ corresponding to the homoclinic orbit and total energy: $H_0(\bar{x}_h, I_h) = h$.

(2) For $h \in J$ and $(q, p) = \bar{x}_h(t - t_0)$, the frequency

$$\Omega_0 = \frac{\partial H_0}{\partial I} = \Omega_0(\bar{x}_h(t - t_0), I_h) \tag{5.3}$$

of the unperturbed system $H_0(\bar{x}_h, I_h) = h$ satisfies $|\Omega_0| \geqq \delta > 0$.

Under these hypotheses, we may invert the equation

$$H_\varepsilon(q, p, \theta, I) = h \in J \tag{5.4}$$

in a neighborhood of the unperturbed homoclinic orbit and solve for the action I_h as a function of $(q, p, \theta; h)$:

$$I_h = \mathscr{I}_\varepsilon(q, p, \theta; h) = \mathscr{I}_0 + \varepsilon\mathscr{I}_1 + \mathcal{O}(\varepsilon^2). \tag{5.5}$$

Moreover, hypothesis (2) implies that the equation $d\theta/dt = \Omega_\varepsilon = \Omega_0 + \varepsilon\frac{\partial H_1}{\partial I}$ can be inverted for small ε, and hence that "real" time t can be replaced by the angle θ to give

$$\frac{dq}{d\theta} = \frac{dq}{dt}\Big/\frac{d\theta}{dt} = \Omega_\varepsilon^{-1}\frac{\partial H_\varepsilon}{\partial p}, \quad \frac{dp}{d\theta} = -\Omega_\varepsilon^{-1}\frac{\partial H_\varepsilon}{\partial q}. \tag{5.6}$$

Implicit differentiation of $H_\varepsilon = h$ yields

$$\frac{\partial H_\varepsilon}{\partial q} + \Omega_\varepsilon\frac{\partial \mathscr{I}_\varepsilon}{\partial q} = 0 = \frac{\partial H_\varepsilon}{\partial p} + \Omega_\varepsilon\frac{\partial \mathscr{I}_\varepsilon}{\partial p}, \tag{5.7}$$

so that, from (5.6)–(5.7), we obtain the reduced equation

$$q' = -\frac{\partial \mathscr{I}_\varepsilon}{\partial p}(q, p, \theta; h), \quad p' = \frac{\partial \mathscr{I}_\varepsilon}{\partial p}(q, p, \theta; h) \tag{5.8}$$

on each energy surface $H_\varepsilon = h \in J$. Here $(\,)'$ denotes $\frac{d}{d\theta}(\,)$, differentiation with respect to the angle variable, which plays the rôle of the new time.

The series expansion (5.5) for $\mathscr{I}_\varepsilon$ can be computed directly by expanding $H_0 + \varepsilon H_1 = h$ with I replaced by $\mathscr{I}_0 + \varepsilon\mathscr{I}_1 + \ldots$ One obtains

$$\mathscr{I}_0 = \mathscr{I}_0(q, p; h) = H_0(q, p)^{-1}(h), \tag{5.9a}$$

$$\mathscr{I}_1 = \mathscr{I}_1(q, p, \theta; h) = \frac{-H_1(q, p, \theta, H_0^{-1}(q, p)(h))}{\Omega_0(q, p, H_0^{-1}(q, p)(h))}, \tag{5.9b}$$

where $H_0(q, p)^{-1}(h)$ denotes inversion of H_0 with respect to the variable I. Thus (5.8) takes the form of a periodic perturbation of an integrable Hamiltonian system: specifically, from (5.7) and the expansion (5.5), (5.9), we have

$$\left.\begin{aligned} q' &= -\frac{\partial \mathscr{I}_0}{\partial p} - \varepsilon\frac{\partial \mathscr{I}_1}{\partial p} \\ p' &= \frac{\partial \mathscr{I}_0}{\partial q} + \varepsilon\frac{\partial \mathscr{I}_1}{\partial q} \end{aligned}\right\} + \mathcal{O}(\varepsilon^2), \tag{5.10}$$

and the unperturbed vector field $\left(-\frac{\partial \mathscr{I}_0}{\partial p}, \frac{\partial \mathscr{I}_0}{\partial q}\right)$ is simply $\Omega_0^{-1}\left(\frac{\partial H_0}{\partial p}, -\frac{\partial H_0}{\partial q}\right)$, a scaled version of the unperturbed field of the original problem, restricted to (q, p) space. Thus hypothesis (1) implies that, for $\varepsilon = 0$, (5.10) has a homoclinic orbit to a hyperbolic fixed point.

The standard Melnikov method as developed in Guckenheimer & Holmes [1953, § 4.5] can be applied directly to (5.10). We have

Proposition 5.1. *Let* $\bar{x}_h = (\bar{q}_h(\theta - \theta_0), \bar{p}_h(\theta - \theta_0))$ *denote the homoclinic orbit to the fixed point* p_0 *of the unperturbed Hamiltonian system* $\mathscr{I}_0(q, p; h)$ *in the energy surface* $H_\varepsilon = h$ *and define the Melnikov function*

$$M_h(\theta_0) = \int_{-\infty}^{\infty} \{\mathscr{I}_0, \mathscr{I}_1\} (\bar{q}_h(\theta), \bar{p}_h(\theta), \theta + \theta_0)\, d\theta. \tag{5.11}$$

Then for $\varepsilon \neq 0$ *sufficiently small, if* $M_h(\theta_0)$ *has simple zeros, the stable and unstable manifolds of the perturbed fixed point* $p_\varepsilon = p_0 + \mathcal{O}(\varepsilon)$ *of the Poincaré map corresponding to* (5.8) *intersect transversely for the perturbed system* $\mathscr{I}_\varepsilon$. *If* $M_h(\theta_0)$ *is bounded away from zero, then the manifolds do not intersect.*

Proof. See GREENSPAN & HOLMES [1982] or GUCKENHEIMER & HOLMES [1983]. The main ideas and original proof are due to MELNIKOV [1963] (*cf.* ARNOLD [1964]).

It is unnecessary to compute $\mathscr{I}_0$ and $\mathscr{I}_1$ explicitly, for we have

Lemma 5.2. (HOLMES & MARSDEN [1983])

$$M_h(\theta_0) = \int_{-\infty}^{\infty} \left\{H_0, \frac{H_1}{\Omega_0}\right\}_{(q,p)} (\bar{q}_h(t), \bar{p}_h(t), I_h, \bar{\theta}(t) + \theta_0)\, dt, \tag{5.12}$$

where $\{\cdot, \cdot\}_{(q,p)}$ *denotes that only the variables* (q, p) *are used in the bracket evaluation.* I_h *is the (constant) action given by* $H_0(\bar{q}, \bar{p}, I_h) = h$ *and* $\bar{\theta}(t) = \int_0^t \Omega(\bar{q}_h(s), \bar{p}_h(s), I_h)\, ds$.

Proof. Consider the equation

$$H_0(q, p, \mathscr{I}_0 + \varepsilon\mathscr{I}_1) + \varepsilon H_1(q, p, \theta, \mathscr{I}_0 + \varepsilon\mathscr{I}_1) = h + \mathcal{O}(\varepsilon^2);$$

this implies that

$$\frac{\partial H_0}{\partial q} = -\Omega_0 \frac{\partial \mathscr{I}_0}{\partial q}\; \frac{\partial H_0}{\partial p} = -\Omega_0 \frac{\partial \mathscr{I}_0}{\partial p} \tag{5.13}$$

as well as $\mathscr{I}_1 = -H_1/\Omega_0$, as in (5.9b). Therefore

$$\begin{aligned}\{\mathscr{I}_0, \mathscr{I}_1\} &= \frac{\partial \mathscr{I}_0}{\partial q}\frac{\partial \mathscr{I}_1}{\partial p} - \frac{\partial \mathscr{I}_0}{\partial p}\frac{\partial \mathscr{I}_1}{\partial q} \\ &= -\frac{1}{\Omega_0}\frac{\partial H_0}{\partial q}\frac{\partial}{\partial p}(H_1/\Omega_0) + \frac{1}{\Omega_0}\frac{\partial H_0}{\partial p}\frac{\partial}{\partial q}(-H_1/\Omega_0) \\ &= \frac{1}{\Omega_0}\left\{H_0, \frac{H_1}{\Omega_0}\right\}_{(q,p)}.\end{aligned} \tag{5.14}$$

Since $d\theta = \Omega\, dt$, substitution of (5.14) in (5.11) yields (5.12). □

Proposition 5.1 and Lemma 5.2 together yield:

Theorem 5.3. *If $H_\varepsilon = H_0 + \varepsilon H_1$ satisfies hypotheses* (1) *and* (2) *and the Melnikov function $M_h(\theta_0)$ of* (5.12) *has simple zeros, then, for $\varepsilon \neq 0$ sufficiently small, there exist transverse homoclinic orbits to a hyperbolic periodic orbit on the energy surface $H_\varepsilon = h$.*

The Smale-Birkhoff homoclinic theorem (SMALE [1963], [1967], GUCKENHEIMER & HOLMES [1983]) then implies

Corollary 5.4. *The Poincaré map associated with H_ε on the level set $H_\varepsilon^{-1}(h)$ has a hyperbolic, non-wandering Cantor set Ω_h on which the map is conjugate to a subshift of finite type.*

As MOSER [1973] shows, this in turn implies

Corollary 5.5. *H_ε possesses no analytic integrals of motion independent of the total energy H_ε itself.*

In the first two situations treated in this paper, rather than a homoclinic orbit to a fixed point we have a cycle of four heteroclinic orbits connecting a pair of saddle points (*cf.* Figure 2). The transverse homoclinic orbits of Theorem 5.3 become transverse heteroclinic cycles, but otherwise it and the conclusions of the corollaries stand unchanged. See HOLMES & MARSDEN [1983, Figure 5] for an impression of the structure of such cycles.

6. Proof of Theorem 4.1

To prove the theorem we compute Melnikov functions for suitably scaled versions of the Hamiltonian. Specifically, for case I, $\alpha_1 < \alpha_2 < \alpha_3$, we take (3.15) and for case II, $\alpha_1 < \alpha_3 < \alpha_2$, (3.16). In particular we restrict the computations to the limit case $\delta = 0$; this suffices since the dependence on δ is continuous. The conclusions of the theorem then follow upon application of Theorem 5.3. Certain computational details are relegated to the Appendix.

We remark that the unperturbed Hamiltonians $H_0(r, s, \sigma)$ differ only in transposition of α_2 and α_3. It therefore suffices to consider the unperturbed heteroclinic solutions only the first case. The unperturbed Hamilton's equations may be written

$$\begin{aligned} \dot{s} &= \frac{\tilde{r}^2}{2}\beta'(\sigma), \\ \dot{\sigma} &= s(\beta(\sigma) - \alpha_3), \\ \dot{r} &= 0, \\ \dot{\varrho} &= -r\beta(\sigma) \stackrel{\text{def}}{=} \Omega(r, \sigma) \end{aligned} \tag{6.1}$$

where $\beta(\sigma) = \alpha_1 \cos^2 \sigma + \alpha_2 \sin^2 \sigma$ and we recall $\tilde{r}^2 = r^2 - s^2$. Without loss of generality we fix the unperturbed "momentum" $r = 1$. in which case one of the four heteroclinic orbits connecting the fixed points $(s, \sigma) = (0, \pm\pi/2)$ takes the form

$$\begin{aligned} s &= \sqrt{-a_3/a_2} \operatorname{sech}(-\sqrt{a_1 a_3}\, t), \\ \sigma &= \arctan(\sqrt{-a_2/a_1} \sinh(-\sqrt{a_1 a_3}\, t)), \\ \varrho &= -[\alpha_2 t + \arctan(\sqrt{a_3/a_1} \tanh(-\sqrt{a_1 a_3}\, t))], \end{aligned} \tag{6.2}$$

where $a_1 = \alpha_3 - \alpha_2 > 0$, $a_2 = \alpha_1 - \alpha_3 < 0$, $a_3 = \alpha_2 - \alpha_1 > 0$ and $a_1 + a_2 + a_3 = 0$. Moreover $\sigma(t) \to \pm\pi/2$ and $\varrho(t) \to \pm\infty$ as $t \to \pm\infty$ and the orbits lie on the level set $H_0 = \alpha_2/2$. The other heteroclinic orbits are obtained by appropriate sign changes (*cf.* Figure 2).

To apply Lemma 5.2, we must compute

$$M(\varrho_0) = \int_{-\infty}^{\infty} \left\{H_0, \frac{H_1}{\Omega}\right\}_{(s,\sigma)} (s_h(t), \sigma_h(t), r_h = 1, \varrho_h(t) + \varrho_0)\, dt.$$

The Poisson bracket is given by

$$\frac{\partial H_0}{\partial s}\frac{\partial(H_1/\Omega)}{\partial \sigma} - \frac{\partial H_0}{\partial \sigma}\frac{\partial(H_1/\Omega)}{\partial s}$$
$$= s(\alpha_3 - \beta(\sigma))\,(b_s - \bar{r}\tilde{r} \cos(\varrho + \varrho_0)) \frac{\partial\beta/\partial\sigma}{r^2\beta^2(\sigma)} + \frac{\tilde{r}^3\, \partial\beta/\partial\sigma}{2r^2\beta(\sigma)} \left(b + \frac{\bar{r}s}{\tilde{r}} \cos(\varrho + \varrho_0)\right). \tag{6.3}$$

Using (6.2), we see that the ϱ_0-independent part of (6.3) is odd in t and therefore vanishes in integration to yield

$$\begin{aligned} M(\varrho_0) &= \int_{-\infty}^{\infty} \frac{\bar{r}\tilde{r}\, \partial\beta/\partial\sigma}{2\beta^2(\sigma)} s(3\beta(\sigma) - 2\alpha_3) \cos(\varrho + \varrho_0)\, dt \\ &= \sqrt{1 - b^2} \left(\int_{-\infty}^{\infty} \frac{s\dot{s}}{\beta^2(\sigma)\sqrt{1 - s^2}} (2\alpha_3 - 3\beta(\sigma)) \sin \varrho\, dt \right) \sin \varrho_0, \end{aligned} \tag{6.4}$$

where we have again used the odd/even properties of (6.2) and the differential relation $\partial\beta/\partial\sigma = 2\dot{s}/\tilde{r}^2$, as well as setting $r = 1$. Equation (6.4) is evaluated in the Appendix to yield

$$M(\varrho_0) = \frac{\sqrt{1 - b^2}\, \pi}{\sqrt{(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)}} \operatorname{cosech}\left(\frac{\alpha_2 \pi}{2\sqrt{(\alpha_3 - \alpha_2)(\alpha_2 - \alpha_1)}} \right) \sin \varrho_0. \tag{6.5}$$

This function has simple zeros for all $\alpha_1 < \alpha_2 < \alpha_3$ and $|b| < 1$, and the proof in case I is complete.

In case II we obtain the Poisson bracket from (3.16). As before, the Melnikov function has a constant (ϱ_0-independent) part and a part periodic in ϱ_0. However, here the constant part does not vanish identically. After using the properties of the unperturbed solutions (6.2) and interchanging the rôles of α_2 and α_3, so that now $\beta(\sigma) = \alpha_1 \cos^2 \sigma + \alpha_3 \sin^2 \sigma$, we obtain

$$\begin{aligned} M(\varrho_0) = b \int_{-\infty}^{\infty} \frac{\tilde{r}s}{\beta(\sigma)} \left[(\alpha_2 - \beta) \left(\frac{\partial\beta/\partial\sigma}{\beta(\sigma)} \sin\sigma - \cos\sigma \right) - \frac{\partial\beta/\partial\sigma}{2\beta(\sigma)} \sin\sigma \right] dt \\ + \sqrt{1-b^2} \left(\int_{-\infty}^{\infty} \frac{s(\alpha_2 - \beta(\sigma))}{\beta(\sigma)} \left[\left(\frac{\partial\beta/\partial\sigma}{\beta(\sigma)} \sin\sigma - \cos\sigma \right) s \cos\varrho \right. \right. \\ \left. \left. + \left(\frac{\partial\beta/\partial\sigma}{\beta(\sigma)} \cos\sigma + \sin\sigma \right) \sin\varrho \right] dt + \int_{-\infty}^{\infty} \frac{\tilde{r}^2\, \partial\beta/\partial\sigma}{2\beta(\sigma)} \sin\sigma \cos\varrho \, dt \right) \cos\varrho_0 . \end{aligned} \tag{6.6}$$

These integrals are evaluated in the Appendix to give

$$M(\varrho_0) = -\frac{2b}{\alpha_3} - \frac{\sqrt{1-b^2}\,\pi}{\sqrt{(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)}} \operatorname{sech}\left(\frac{\alpha_3 \pi}{2\sqrt{(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)}} \right) \cos\varrho_0 . \tag{6.7}$$

We conclude that, if

$$b^2 < \frac{\alpha_3^2 \pi^2}{\alpha_3^2\pi^2 + 4(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)\cosh^2\left(\dfrac{\alpha_3\pi}{2\sqrt{(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)}} \right)}, \tag{6.8}$$

then $M(\varrho_0)$ has simple zeros and thus that transverse homoclinic orbits exist for ε, δ sufficiently small. The proof of Theorem 4.1 is complete. □

Remarks on Case II: $\alpha_1 < \alpha_3 < \alpha_2$.

In this case the Melnikov function has a constant part and, if $|b|$ is sufficiently close to 1, so that $|b| < 1$ but (6.8) is violated, then $M(\varrho_0)$ has no zeros. Thus, by

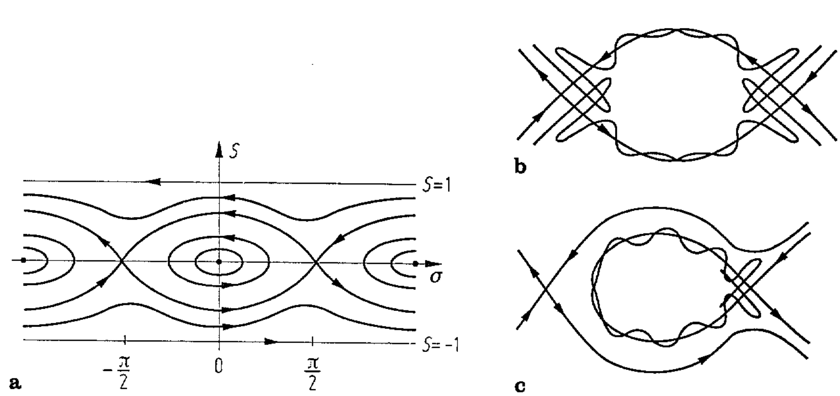

Fig. 4a–c. Hyperbolic fixed points and invariant manifolds in the (σ, s) cross section, $r = 1$. (a) Unperturbed problem, $\alpha_1 < \alpha_2 < \alpha_3$ and $\alpha_1 < \alpha_3 < \alpha_2$. (b) Perturbed problem, $\alpha_1 < \alpha_2 < \alpha_3$ and $\alpha_1 < \alpha_3 < \alpha_2$ if b^2 satisfies (6.8). (c) A possible situation if $\alpha_1 < \alpha_3 < \alpha_2$ and b^2 fails to satisfy (6.8) so that $M(\varrho_0)$ has no zero.

Proposition 5.1, the stable and unstable manifolds the two perturbed fixed points do not intersect. We observe that this does not necessarily imply that no transverse homoclinic orbits exist, since the stable and unstable manifolds of a single fixed point might still intersect, as indicated in Figure 4(c). However, the perturbation calculations give no information on this.

7. Proof of Theorem 4.2

As is shown in Section 4 the scaled Hamiltonian of a D_N-symmetric rod has the form

$$\begin{aligned}\tilde{H}_b(r, s, \sigma, \varrho, 1, \delta) = \tilde{H}_0(r, s, \varrho) + \delta \tilde{H}_s(r, s, \varrho, \delta) \\ + c\, \delta^{N-2} \tilde{r}^N \cos N\sigma + \mathcal{O}(\delta^{N-1}),\end{aligned} \tag{7.1}$$

where ε is chosen equal to 1 and

$$\tilde{H}_0(r, s, \varrho) = \tfrac{1}{2}(\alpha \tilde{r}^2 + \alpha_3 s^2) + \frac{1}{r^2}(bs - \bar{r}\tilde{r} \cos \varrho). \tag{7.2}$$

The rotationally symmetric part $\delta \tilde{H}_s$ is even in ϱ (*cf.* (4.2)) and allows for nonlinear terms of lower order $(\mathcal{O}(\delta^M),\ 1 \leqq M \leqq N-2)$ than the first D_N-symmetric term $\delta^{N-2} \tilde{r}^N \cos N\sigma$.

The unperturbed equations are

$$\begin{aligned}\dot{r} &= \frac{\bar{r}\tilde{r}}{r^2} \sin \varrho, \\ \dot{\varrho} &= -\alpha r + \frac{2bs}{r^3} + \frac{\cos \varrho}{r^3}\left(\frac{b^2 \tilde{r}}{\bar{r}} + \frac{s^2 \bar{r}}{\tilde{r}}\right), \\ \dot{s} &= 0, \\ \dot{\sigma} &= (\alpha - \alpha_3)\, s - \frac{b}{r^2} - \frac{s\bar{r}}{r^2 \tilde{r}} \cos \varrho \stackrel{\text{def}}{=} \Omega(r, s, \varrho).\end{aligned} \tag{7.3}$$

Here σ is the cyclic variable. We restrict our attention to the special case $s = b$ with $0 < b^2 < \frac{4}{\alpha}$, for which the phase portrait in the (r, ϱ)-plane appears as in Figure 5. The line $r = b$ is degenerate in the $(\boldsymbol{m}, \boldsymbol{n})$ coordinate system: for $\bar{r} = \sqrt{r^2 - b^2} = 0$, $(\boldsymbol{m}, \boldsymbol{n})$ is independent of ϱ (*cf.* (3.1 b)). Thus the heteroclinic solution lying on the level set

$$\tfrac{1}{2}(\alpha(r^2 - b^2) + \alpha_3 b^2) + \frac{b^2 - (r^2 - b^2)\cos \varrho}{r^2} = \frac{\alpha_3}{2} b^2 + 1 \tag{7.4}$$

connecting the fixed points $(r, \varrho) = \left(b, \arccos\left(\frac{\alpha b^2}{2} - 1\right)\right)$ is a homoclinic orbit to the saddle point $(\boldsymbol{m}, \boldsymbol{n}) = (0, 0, b; 0, 0, 1)$ in the original coordinates. The condition $b^2 < \frac{4}{\alpha}$, necessary for the saddle to exist, corresponds to the angular momentum condition for a "slow top" in the classical Lagrange analysis (*cf.* GOLDSTEIN [1980, Ch. 5]).

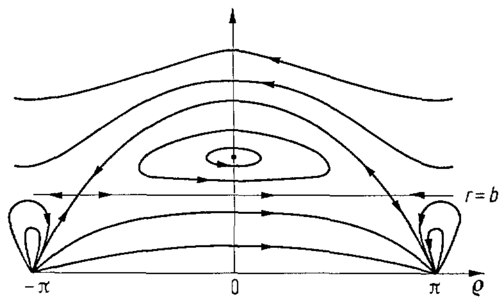

Fig. 5. Unperturbed (r, ϱ) phase plane for equation (7.3) with $s = b$.

We shall require an explicit expression for the homoclinic solution as a function of r, on the set $s = b$, as well as an implicit relation:

$$\tilde{r}(t) = \bar{r}(t) = \sqrt{r^2(t) - b^2} = \gamma \operatorname{sech}\left(\frac{\alpha\gamma}{2} t\right), \tag{7.5a}$$

$$r^2 = \frac{2}{\alpha}(1 + \cos\varrho), \tag{7.5b}$$

where $\gamma^2 = \frac{4}{\alpha} - b^2 > 0$.

To apply the Melnikov method we must compute the Poisson bracket

$$\left\{H_0, \frac{H_1}{\Omega}\right\} = \frac{\partial H_0}{\partial r}\left(\frac{1}{\Omega}\frac{\partial H_0}{\partial \varrho} - \frac{H_1}{\Omega^2}\frac{\partial \Omega}{\partial \varrho}\right) - \frac{\partial H_0}{\partial \varrho}\left(\frac{1}{\Omega}\frac{\partial H_0}{\partial r} - \frac{H_1}{\Omega^2}\frac{\partial \Omega}{\partial r}\right). \tag{7.6}$$

From (7.3) and (7.5b) we see that, on the homoclinic orbit, $\Omega(r, s = b, \varrho)$ is constant and takes the value

$$\Omega = b\left(\frac{\alpha}{2} - \alpha_3\right), \tag{7.7}$$

which is nonzero since $\alpha_3 > \alpha$.

The perturbation Hamiltonian H_1 divides into two parts: δH_s and $c\,\delta^{N-2}\,\tilde{r}^N \cos N\sigma$. Since H_0, H_s and Ω are even in ϱ, $\frac{\partial H_0}{\partial \varrho}$, $\frac{\partial H_0}{\partial \varrho}$ and $\frac{\partial \Omega}{\partial \varrho}$ are odd in ϱ and $\left\{H_0, \frac{\partial H_s}{\Omega}\right\}$ is also odd in ϱ. From this fact and the evenness of r, $\bar{r}$, and $\tilde{r}$ in t we deduce that this part of the Poisson bracket is odd in t and so does not contribute to the Melnikov integral. Thus we need only compute the part involving

$c\delta^{N-2}\bar{r}^N \cos N\sigma$. Using (7.6a, b), we obtain

$$\frac{\partial H_0}{\partial r} = \frac{\alpha \bar{r}^2}{r}, \qquad \frac{\partial H_0}{\partial \varrho} = \frac{\bar{r}^2}{r^2}\sin\varrho,$$

$$\frac{\partial(H_1/\Omega)}{\partial r} = \left(\frac{Nr\bar{r}^{N-2}}{\Omega} - \frac{\alpha b\bar{r}^N}{\Omega^2 r}\right)\cos N\sigma, \quad \frac{\partial(H_1/\Omega)}{\partial \varrho} = -\frac{cb\bar{r}^N}{\Omega^2 r^2}\sin\varrho\cos N\sigma. \tag{7.8}$$

The Melnikov integral therefore reduces to

$$M(\sigma_0) = \int_{-\infty}^{\infty}\left(-\frac{cN\bar{r}^N}{\Omega r}\sin\varrho\cos(N(\sigma+\sigma_0))\right)dt, \tag{7.9}$$

where $\sigma(t) = \Omega t$, since Ω is constant.

Using (7.8) and the fact that $r, \bar{r}$ are even in t while $\sin\varrho$ is odd and $\dot{r} = (\bar{r}^2/r^2)$ $\sin\varrho$, we may rewrite (7.9) as follows:

$$M(\varrho_0) = c\left(\int_{-\infty}^{\infty}\frac{\sin(N(\Omega t))}{\Omega}Nr\bar{r}^{N-2}\dot{r}\,dt\right)\sin(N\sigma_0).$$

Integrating by parts and substituting from (7.5), we obtain

$$\begin{aligned} M(\varrho_0) &= c\left(\int_{-\infty}^{\infty} -N\cos(N\Omega t)\,\bar{r}^N(t)\,dt\right)\sin(N\sigma_0) \\ &= -Nc\gamma^N\left(\int_{-\infty}^{\infty}\operatorname{sech}^N\left(\frac{\alpha\gamma}{2}t\right)\cos(N\Omega t)\,dt\right)\sin(N\sigma_0) \qquad (7.10) \\ &= \frac{-2Nc\gamma^{N-1}}{\alpha}\left(\int_{-\infty}^{\infty}\operatorname{sech}^N(\tau)\cos\left(\frac{2N\Omega}{\alpha\gamma}\tau\right)d\tau\right)\sin(N\sigma_0). \end{aligned}$$

Writing the integral of (7.10) as $I_N(\omega)$, $\omega = \dfrac{2N\Omega}{\alpha\gamma} = \dfrac{bN}{\gamma}\left(1 - \dfrac{2\alpha_3}{\alpha}\right)$, by integrating by parts twice we deduce the recurrence relation

$$I_{N+2}(\omega) = \frac{\omega^2 + N^2}{N(N+1)}I_N(\omega), \qquad N \geqq 0. \tag{7.11}$$

This relation, together with the evaluations of I_1 and I_2 by the calculus of residues, namely

$$\begin{aligned} I_1(\omega) &= \pi\operatorname{sech}\left(\frac{\pi\omega}{2}\right), \\ I_2(\omega) &= \pi\omega\operatorname{cosech}\left(\frac{\pi\omega}{2}\right), \end{aligned} \tag{7.12}$$

guarantees that $I_N(\omega) \neq 0$ for all $N \geqq 1$ and $\alpha, \gamma \neq 0$. We conclude that the Melnikov function (7.10) has simple zeroes under the hypotheses of the theorem. Applying Theorem 5.3 completes the proof. □

Remark. We have assumed $N \geqq 3$, so that the explicit σ-dependent term in the Hamiltonian appears at higher order. In the case $N = 2$ the relevant term is $c\bar{r}^2\cos 2\sigma$, which occurs at the same order as the rotationally symmetric part H_0.

In this case we have

$$H = \tfrac{1}{2}\left(\tilde{r}^2(\alpha + 2\,c\cos 2\sigma) + \alpha_3 s^2\right) + \frac{bs - \tilde{r}\tilde{r}\cos\varrho}{r^2}. \tag{7.14}$$

Here the problem reduces to the perturbed Lagrange top considered by HOLMES & MARSDEN [1983]. Using the fact that $\beta(\sigma) = \alpha_1 \cos^2\sigma + \alpha_2 \sin^2\alpha = \left(\frac{\alpha_1 + \alpha_2}{2}\right) + \left(\frac{\alpha_1 - \alpha_2}{2}\right)\cos 2\sigma$, we see explicitly that the condition $c \neq 0$ corresponds to inequality of the (inverse)-moments of inertia, $\alpha_1 \neq \alpha_2$.

Appendix. Computation of Melnikov Functions for Theorem 4.1

We give the necessary calculations to evaluate the first order approximations of the Melnikov functions in equations (6.4) and (6.6).

In the case $0 < \alpha_1 < \alpha_2 < \alpha_3$ the relevant parts of the Hamiltonian are

$$H_0 = \tfrac{1}{2}\left((r^2 - s^2)\,\beta(\sigma) + \alpha_3 s^2\right),$$

$$\frac{H_1}{\Omega} = \frac{\tilde{r}\tilde{r}}{r^3\beta(\sigma)}\cos\varrho - \frac{bs}{r^3\beta(\sigma)}. \tag{A.1}$$

From (6.2) we deduce that the heteroclinic solution with $r = 1$ satisfies

$$s(t) = \sqrt{-a_3/a_2}\,S(t), \qquad \beta(\sigma) = \frac{\alpha_2 - \alpha_3 s^2}{1 - s^2}, \tag{A.2a + b}$$

$$\sin\sigma(t) = -\frac{T(t)}{\sqrt{1 - s^2}}, \qquad \cos\sigma(t) = \sqrt{-a_1/a_2}\,\frac{S(t)}{\sqrt{1 - s^2}}, \tag{A.2c + d}$$

$$\sin\varrho(t) = \sqrt{-a_1/a_2}\,\frac{1}{\sqrt{1 - s^2}}\left(\sqrt{a_3/a_1}\,T(t)\cos\alpha_2 t - \sin\alpha_2 t\right) \tag{A.2e}$$

$$\cos\varrho(t) = \sqrt{-a_1/a_2}\,\frac{1}{\sqrt{1 - s^2}}\left(\cos\alpha_2 t + \sqrt{a_3/a_1}\,T(t)\sin\alpha_2 t\right) \tag{A.2f}$$

where $a_1 = \alpha_3 - \alpha_2$, $a_2 = \alpha_1 - \alpha_3$, $a_3 = \alpha_2 - \alpha_1$ and

$$S(t) = \operatorname{sech}\left(\sqrt{a_1 a_3}\,t\right), \qquad T(t) = \tanh\left(\sqrt{a_1 a_3}\,t\right).$$

We further need the relations

$$d\varrho = \dot{\varrho}\,dt = -\beta\,dt, \qquad d\sigma = \dot{\sigma}\,dt = -s(\alpha_3 - \beta)\,dt, \tag{A.3a + b}$$

$$\dot{s} = \frac{\partial H}{\partial\sigma} = \tfrac{1}{2}\,\tilde{r}^2\frac{\partial\beta}{\partial\sigma} = \tfrac{1}{2}(1 - s^2)\frac{\partial\beta}{\partial\sigma}, \tag{A.4}$$

$$\frac{d}{dt}\left(\sqrt{1 - s^2}\sin\varrho\right) = -\alpha_2\sqrt{1 - s^2}\cos\varrho + a_3\sqrt{-a_1/a_2}\,S^2(t)\cos\alpha_2 t, \tag{A.5a}$$

$$\frac{d}{dt}\left(\sqrt{1 - s^2}\cos\varrho\right) = \alpha_2\sqrt{1 - s^2}\sin\varrho + a_3\sqrt{-a_1/a_2}\,S^2(t)\sin\alpha_2 t. \tag{A.5b}$$

The Melnikov integral then yields

$$M(\varrho_0) = \int_{\mathbb{R}} \left(\frac{\partial H_0}{\partial s} \frac{\partial (H_1/\Omega)}{\partial \sigma} - \frac{\partial H_0}{\partial \sigma} \frac{\partial (H_1/\Omega)}{\partial s} \right) dt$$

$$= \int_{\mathbb{R}} \Big\{ s(\alpha_3 - \beta) \left[bs - \sqrt{1 - b^2} \sqrt{1 - s^2} \cos(\varrho + \varrho_0) \right]$$

$$+ \tfrac{1}{2}(1 - s^2) \beta \left[b + \frac{\sqrt{1 - b^2}}{\sqrt{1 - s^2}} s \cos(\varrho + \varrho_0) \right] \Big\} \frac{\frac{\partial \beta}{\partial \sigma}}{\beta^2} dt.$$

Since s, $\beta \cos \varrho$ are even in t and $\sin \varrho$, $\frac{\partial \tau}{\partial \sigma}$ are odd this integral simplifies, after use of (A.4), to

$$M(\varrho_0) = b \cdot 0 + \sqrt{1 - b^2}\, I^* \sin \varrho_0,$$

$$I^* = \int_{\mathbb{R}} \frac{s \dot{s}}{\sqrt{1 - s^2}\, \beta^2} (2\alpha_3 - 3\beta) \sin \varrho \, dt,$$

as in equation (6.4). Expressing β in terms of s via (A.2b), we have

$$I^* = \int_{\mathbb{R}} g(s)\, \dot{s} \sin \varrho \, dt, \tag{A.6}$$

where

$$g(s) = \frac{s\sqrt{1 - s^2}}{(\alpha_2 - \alpha_3 s^2)^2} (2\alpha_3(1 - s^2) - 3(\alpha_2 - \alpha_3 s^2))$$

and

$$G(s) = \int g(s)\, ds = \frac{(1 - s^2)^{3/2}}{\alpha_2 - \alpha_3 s^2} = \frac{\sqrt{1 - s^2}}{\beta}.$$

Hence, upon integration by parts, we have

$$\int g(s)\, \dot{s} \sin \varrho \, dt = G(s) \sin \varrho - \int G(s)\, \dot{\varrho} \cos \varrho \, dt$$

$$= \frac{\sqrt{1 - s^2}}{\beta} \sin \varrho + \int \sqrt{1 - s^2} \cos \varrho \, dt$$

$$= \frac{\sqrt{1 - s^2}}{\beta} \sin \varrho - \frac{\sqrt{1 - s^2}}{\alpha_2} \sin \varrho + \frac{a_3}{\alpha_2} \sqrt{\frac{a_1}{-a_2}} \int S^2(t) \cos \alpha_2 t \, dt.$$

The last equality is a consequence of (A.5a). Now

$$I^* = \left(\frac{(1 - s^2)}{\alpha_2 - \alpha_3 s^2} - \frac{1}{\alpha_2} \right) \sqrt{1 - s^2} \sin \varrho \Big|_{-\infty}^{\infty} + \frac{a_3}{\alpha_2} \sqrt{\frac{a_1}{-a_2}} \int_{\mathbb{R}} S^2(t) \cos \alpha_2 t \, dt,$$

so that the boundary terms vanish and, letting $\tau = \sqrt{a_1 a_3}\, t$, we are left whith

$$I^* = \frac{1}{\alpha_2}\sqrt{\frac{a_3}{-a_2}} \int_{\mathbb{R}} \operatorname{sech}^2(\tau) \cos\left(\frac{\alpha_2}{\sqrt{a_1 a_3}}\tau\right) d\tau$$

$$= \frac{\pi}{\sqrt{-a_1 a_2}} \operatorname{cosech}\left(\frac{\pi\alpha_2}{2\sqrt{a_1 a_3}}\right), \tag{A.7}$$

from which we obtain (6.5). The final integral is evaluated by the method of residues.

For the case $0 < \alpha_1 < \alpha_3 < \alpha_2$, as indicated in Section 4, we use the (r, s, σ, ϱ)-coordinates for the vectors $(m_1, m_3, m_2)^T$ and $(n_1, n_3, n_2)^T$ rather than for $\boldsymbol{m}$ and $\boldsymbol{n}$. Thus, the unperturbed solutions are given by the formulae (6.2), (A.2) and (A.3), as above, but with α_2 and α_3 interchanged and the heteroclinic orbits now connect the points $\boldsymbol{m} = (0, 0, \pm 1)$. The perturbation Hamiltonian, H_1, however is quite different. From (3.16) we have

$$H_0 = \tfrac{1}{2}\left((r^2 - s^2)\,\beta(\sigma) + \alpha_2 s^2\right),$$

$$\frac{H_1}{\Omega} = -\frac{b\tilde{r}\sin\sigma}{r^3\beta} - \frac{\bar{r}}{r^3\beta}(s\sin\sigma\cos\varrho + r\cos\sigma\sin\varrho)$$

where now $\beta(\sigma) = \alpha_1\cos^2\sigma + \alpha_3\sin^2\sigma$.

The Melnikov function is

$$M(\varrho_0) = \int_{-\infty}^{\infty}\left(\frac{\partial H_0}{\partial s}\frac{\partial(H_1/\Omega)}{\partial\sigma} - \frac{\partial H_0}{\partial\sigma}\frac{\partial(H_1/\Omega)}{\partial s}\right) dt.$$

With $r = 1$ and after omitting the odd terms in the integrand, we are left with

$$M(\varrho_0) = bI_1 + \sqrt{1 - b^2}\,(I_2 - I_3)\cos\varrho_0, \tag{A.8}$$

where I_1, I_2 and I_3 are the integrals defined implicitly by comparison of (A.8) and (6.6). For the first, we have

$$I_1 = \int_{-\infty}^{\infty}\sqrt{1 - s^2}\left[(\alpha_2 - \beta(\sigma))\, s\frac{d}{d\sigma}\left(-\frac{\sin\sigma}{\beta}\right) - \frac{s}{2\beta(\sigma)}\frac{\partial\beta}{\partial\sigma}\sin\sigma\right] dt. \tag{A.9}$$

Using $\dot{\sigma} = s(\beta - \alpha_2)$ from (6.1) (with $\alpha_3 \leftrightarrow \alpha_2$) and $\beta'(\sigma) = 2\dot{s}/\tilde{r}^2$, we may rewrite (A.9) and obtain

$$I_1 = \int_{-\pi/2}^{\pi/2}\sqrt{1 - s^2}\frac{d}{d\sigma}\left(\frac{-\sin\sigma}{\beta(\sigma)}\right) d\sigma - \int_{-\infty}^{\infty}\frac{s\sin\sigma}{\beta(\sigma)\sqrt{1 - s^2}}\,\dot{s}\, dt$$

$$= \sqrt{1 - s^2}\left(\frac{-\sin\sigma}{\beta(\sigma)}\right)\Bigg|_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2}\frac{s\sin\sigma}{\beta(\sigma)\sqrt{1 - s^2}}\frac{ds}{d\sigma}\, d\sigma + \int_{-\pi/2}^{\pi/2}\frac{s\sin\sigma}{\beta(\sigma)\sqrt{1 - s^2}}\frac{ds}{d\sigma}\, d\sigma$$

$$= -\frac{2}{\alpha_3}, \tag{A.10}$$

since the two last integrals cancel one another.

In the ϱ_0-dependent part of (A.8) we have

$$I_2 = \int_{-\infty}^{\infty} s(\alpha_2 - \beta(\sigma)) \left[s \cos \varrho \frac{d}{d\sigma}\left(-\frac{\sin \sigma}{\beta(\sigma)}\right) + \sin \varrho \frac{d}{d\sigma}\left(-\frac{\cos \sigma}{\beta(\sigma)}\right)\right] dt, \quad \text{(A.11 a)}$$

$$I_3 = \int_{-\infty}^{\infty} \frac{(1 - s^2)}{2\beta(\sigma)} \frac{\partial \beta}{\partial \sigma} \sin \sigma \cos \varrho \, dt. \quad \text{(A.11 b)}$$

Using $s(\beta - \alpha_2) = \dot{\sigma}$ again we have, in terms of $d\sigma$,

$$I_2 = \int_{-\pi/2}^{\pi/2} \left[s \cos \varrho \frac{d}{d\sigma}\left(-\frac{\sin \sigma}{\beta(\sigma)}\right) + \sin \varrho \frac{d}{d\sigma}\left(-\frac{\cos \sigma}{\beta(\sigma)}\right)\right] d\sigma,$$

or, after integration by parts and transformation back to dt:

$$I_2 = \int_{-\pi/2}^{\pi/2} \left[\sin \sigma \frac{d}{d\sigma}(s \cos \varrho) + \cos \sigma \frac{d}{d\sigma}(\sin \varrho)\right] \frac{d\sigma}{\beta(\sigma)}$$

$$= \int_{-\infty}^{\infty} \left[\sin \sigma \frac{d}{dt}(s \cos \varrho) + \cos \sigma \frac{d}{dt}(\sin \varrho)\right] \frac{dt}{\beta(\sigma)}. \quad \text{(A.12)}$$

Note that, in integration by parts, the boundary terms vanish since s and $\cos \sigma \to 0$ as $\sigma \to \pm\pi/2$. In I_3 we use (A.4) to obtain

$$I_3 = \int_{-\infty}^{\infty} \dot{s} \sin \sigma \cos \varrho \frac{dt}{\beta}. \quad \text{(A.13)}$$

Thus, from (A.12) and (A.13), we have, after cancellation and using $\dot{\varrho} = -\beta(\sigma)$:

$$I_2 - I_3 = \int_{-\infty}^{\infty} [s \sin \sigma \sin \varrho - \cos \sigma \cos \varrho] \, dt. \quad \text{(A.14)}$$

Finally, we use the relations (A.2) (with $\alpha_2 \leftrightarrow \alpha_3$, so that $a_1 \to -a_1$, $a_2 \to -a_3$, $a_3 \to -a_2$) and the fact that $S^2 + T^2 = 1$ to reduce (A.14) to the form

$$I_2 - I_3 = -\int_{-\infty}^{\infty} S(t) \cos \alpha_3 t \, dt$$

$$= -\frac{1}{\sqrt{a_1 a_2}} \int_{-\infty}^{\infty} \operatorname{sech} \tau \cos\left(\frac{\alpha_2 \tau}{\sqrt{a_1 a_2}}\right) d\tau$$

$$= -\frac{\pi}{\sqrt{a_1 a_2}} \operatorname{sech}\left(\frac{\alpha_2 \pi}{2\sqrt{a_1 a_2}}\right). \quad \text{(A.15)}$$

Thus from (A.8), (A.10) and (A.15), we have

$$M(\varrho_0) = -\frac{2b}{\alpha_3} - \frac{\sqrt{1 - b^2}\,\pi}{\sqrt{a_1 a_2}} \operatorname{sech}\left(\frac{\alpha_2 \pi}{2\sqrt{a_1 a_2}}\right) \cos \varrho_0, \quad \text{(A.16)}$$

which gives (6.7).

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