2[nd] CENTRAL Summer School on *Analysis and Numerics of PDEs*

Humboldt-Universität zu Berlin, 29. August – 2. September 2016

Tutorial exercises for Mielke's lectures

Exercise 1: Gradient structures.
(a) Consider the diffusion equation $\dot{u} = \mathrm{div}\big(A(x, u(x))\nabla u\big)$ with Neumann boundary conditions, where $u(t, x) \geq 0$ is a density. Take any convex functions $\phi : [0, \infty[\to \mathbb{R}$ (with $\phi''(u) > 0$) and define the energy $\mathcal{E}(u) = \int_\Omega \phi(u(x))\,\mathrm{d}x$. Show that there exists an Osager operator $\mathbb{K}(u)$ such that the above diffusion equation is generated by $\mathcal{E}$ and $\mathbb{K}$.

(b) Consider the Reaction-Diffusion equation $\dot{u} = m\Delta u + k(4-u^2)$ for positive densities $u(t, x) > 0$. Construct $\mathbb{K}_{\mathrm{RD}}(u)$ such that the equation is induced by $\mathbb{K}_{\mathrm{RD}}$ and $\mathcal{E}(u) = \int_\Omega \big(u \log(u/w) - u\big)\,\mathrm{d}x$ for a suitable $w > 0$.

(c) Consider the PDE system for concentration $c(t, x) \geq 0$ and internal energy $e(t, x) \in R$ given by

$$\begin{aligned} \dot{c} &= m\Delta c + k(e-c^2) && \text{reaction-diffusion equation} \\ \dot{e} &= m\Delta e && \text{heat/energy equation} \end{aligned}$$

completed by Neumann boundary conditions for c and e. Consider the entropy functional

$$\mathcal{S}(c, e) = \int_\Omega S(c(x), e(x))\,\mathrm{d}x \quad \text{with } S(c, e) = \sigma(e) - c\log(c/\sqrt{e}) + c$$

with $\sigma''(e) \leq 0$. Show that S is a concave function and the $\mathcal{S}$ increases along solutions. of thePDE system.

Exercise 2: Homogenization. Set $\mathcal{J}_\varepsilon(u) = \int_0^\ell \frac{1}{2}u(x) \cdot G(x/\varepsilon)u(x)\,\mathrm{d}x$ where G satisfies $G \in \mathrm{C}^0_{\mathrm{per}}(\mathbb{R}; \mathbb{R}^{d\times d}_{\mathrm{spd}})$. Proof that the weak and strong Γ-limit in $\mathrm{L}^2(0, \ell)$ is given by the harmonic and arithmetic mean, respectively.

Exercise 3: Γ-convergence and Mosco convergence. Consider two separable and reflexive Banach spaces $\mathbf{X}$ and $\mathbf{Z}$ such that $\mathbf{Z}$ is compactly embedded in $\mathbf{X}$. Assume that

$$\text{(i) } (\mathcal{E}_\varepsilon)_{\varepsilon\in[0,1]} \text{ is equicoercive in } \mathbf{Z} \quad \text{and} \quad \text{(ii) } \mathcal{E}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{E}_0 \text{ in } \mathbf{Z}.$$

Show that $\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}$ in $\mathbf{X}$.

Exercise 4: Sina Reichelt's lemma. Assume $(u_\varepsilon)_{\varepsilon\in[0,1]}$ is bounded in $\mathrm{W}^{1,p}(0, T; \mathbf{X})$ and that $u_\varepsilon(t) \to u_0(t)$ for all $t \in [0, T]$. For suitable dissipation potentials Ψ_ε show that $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_0$ implies the liminf estimate

$$\int_0^T \Psi_0(\dot{u}_0(t))\,\mathrm{d}t \leq \liminf_{\varepsilon\to 0} \int_0^T \Psi_\varepsilon(\dot{u}_\varepsilon(t))\,\mathrm{d}t.$$

Hint: Approximate $\dot{u}_\varepsilon$ by piecewise affine functions.