Höhere Analysis II: Partial Differential Equations
Summer Term 2011
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# *Exercise Sheet 12*

**Exercise 46. Quadratic membrane.** On the square $\Omega = ]0,\pi[ \times ]0,\pi[$ we consider the wave equation $u_{tt} = \Delta u = u_{x_1x_1} + u_{x_2x_2}$ with DIRICHLET boundary conditions $u(t,\cdot)|_{\partial\Omega} = 0$.

(a) Determine a complete orthonormal system $\{\, \Phi_j \mid j \in \mathbb{N} \,\}$ in $\mathrm{L}^2(\Omega)$, such that $u(t,x) = \cos(\omega_j t)\Phi_j(x)$ provides a solution to the wave equation for suitable $\omega_j$.

(b) Show that for $\omega^2 = 10$ there exist two different eigenfunctions $\Phi_j$, such that $\Phi_i(x_1,x_2) = \Phi_j(x_2,x_1) \neq \Phi_j(x_1,x_2)$ holds for $x \in \Omega$. Discuss the nodal lines ( = zero set) of the solution $u(t) = \cos(\sqrt{10}\,t)\big(\alpha\Phi_i + \beta\Phi_j\big)$. In particular study the cases $\alpha/\beta \in \{\infty, 1, -1, 0\}$.

**Exercise 47. The Schrödinger equation** for a free particle on the full space reads

$$\mathrm{i}\partial_t \Psi(t,x) = \Delta\Psi(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d,$$

where $\psi(t,x) \in \mathbb{C}$ is the wave function.

(a) Show that the particle number $E_1$ and the kinetic energy $E_2$ are constant along solutions (only formal argument for smooth and decaying solutions needed):

$$E_1(\psi) = \int_{\mathbb{R}^d} |\psi(x)|^2 \,\mathrm{d}x \quad \text{and} \quad E_2(\psi) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla\psi(x)|^2 \,\mathrm{d}x.$$

(b) Derive an abstract solution formula via spatial Fourier transform.

(c) Prove that for initial conditions $\psi_0 \in \mathrm{H}^{2k}(\mathbb{R}^d)$ the solution $\psi$ satisfies

$$\psi \in \mathrm{C}^0(\mathbb{R}, \mathrm{H}^{2k}(\mathbb{R}^d)) \cap \mathrm{C}^1(\mathbb{R}, \mathrm{H}^{2k-2}(\mathbb{R}^d)) \cap \cdots \cap \mathrm{C}^{k-1}(\mathbb{R}, \mathrm{H}^2(\mathbb{R}^d)) \cap \mathrm{C}^k(\mathbb{R}, \mathrm{L}^2(\mathbb{R}^d)).$$

**Exercise 48 (in written form). An explicit solution for the wave equation.** On $\Omega = \mathbb{R}^3$ consider the wave equation $u_{tt} = \Delta u$ with $u(0,x) = u^0(x)$ and $u_t(0,x) = u^1(x)$.

(a) Calculate the explicit solution for $u^0 \equiv 0$ and $u^1(x) = 1$ for $|x| \leq R$ and $u^1(x) = 0$ for $|x| > R$. What is the support of $u(t,\cdot)$? Where is $u$ continuous? (Hint: The initial condition is radially symmetric.)

(b) Establish the decay $\|u(t)\|_{\mathrm{L}^\infty} = O(t^{-\alpha})$ for $t \to \infty$ for a suitable $\alpha > 0$.

(c) Show that the difference between the potential energy $E_{\mathrm{pot}}(t)$ and the kinetic energy $E_{\mathrm{kin}}(t)$ decays to 0 like $O(t^{-\beta})$ for a suitable $\beta > 0$, where $E_{\mathrm{pot}}(t) = \int_{\mathbb{R}^3} \frac{1}{2}\big|\nabla u(t,x)\big|^2 \,\mathrm{d}x$ and $E_{\mathrm{kin}}(t) = \int_{\mathbb{R}^3} \frac{1}{2}\big|u_t(t,x)\big|^2 \,\mathrm{d}x$.

**Exercise 49. Finite speed of propagation in general wave equations.** On $\Omega = \mathbb{R}^d$ consider sufficiently smooth solutions of the general homogeneous wave equation

$$\rho(x)u_{tt} = \operatorname{div}\big(A(x)\nabla u(t,x)\big) - c(x)u(t,x),$$

where $\rho, c \in \mathrm{BC}(\mathbb{R}^d)$, $A \in \mathrm{BC}^1(\mathbb{R}^d, \mathbb{R}^{d\times d})$ such that $\rho(x) \geq \rho_{\min} > 0$, $c(x) \geq 0$, and $A(x)\xi\cdot\xi \geq A_{\min}|\xi|^2$ with $A_{\min} > 0$.

(a) Define the regions $R(t) = \{\, x \in \mathbb{R} \mid |x - x_*| < r_* - vt \,\}$ for $x_* \in \mathbb{R}^d$, $r_*, v > 0$, and $t \in ]0, r_*/v[$. Show

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R(t)} \frac{\rho}{2}u_t^2 + \frac{1}{2}A\nabla u\cdot\nabla u + \frac{c}{2}u^2 \,\mathrm{d}x = \int_{\partial R(t)} A\nabla u\cdot\nu\, u_t - v\big(\frac{\rho}{2}u_t^2 + \frac{1}{2}A\nabla u\cdot\nabla u + \frac{c}{2}u^2\big)\,\mathrm{d}a.$$

(b) Construct the smallest $v$, such that $u(t,x) = 0$ for all $x \in R(t)$ whenever $u(0,x) = u_t(0,x)$ for $x \in R(0)$. Conclude further that $\operatorname{sppt}((u(0,\cdot), u_t(0,\cdot)) \subset B_R(x_*)$ implies $\operatorname{sppt}((u(t,\cdot), u_t(t,\cdot)) \subset B_{R+v|t|}(x_*)$ for all $t \in \mathbb{R}$.

**Please hand in the solutions of the written exercise by Tuesday, $12^{\text{th}}$ of July 2011, 12:00 h.**