Höhere Analysis II: Partial Differential Equations
Summer Term 2011
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Exercise Sheet 11

Exercise 42. Non-uniqueness for the heat equation We want to show that the equation $u_t = u_{xx}$ has a solution $u \in \mathrm{C}^\infty(\mathbb{R}\times\mathbb{R})$ such that $u(t,x) = 0$ for $t \leq 0$ and $x \in \mathbb{R}$, while $u(t,0) \neq 0$ for $t > 0$. We construct u in the form

$$u(t,x) = \sum_{k=1}^{\infty} g_k(t)x^k, \text{ where } \ g_1 \equiv 0, \quad g_0 = g \in \mathrm{C}^\infty(\mathbb{R}) \text{ with } g(t) = 0 \text{ for } t \leq 0.$$

(a) Show by formal calculations that u solves the heat equation if $g_k' = (k+2)(k+1)g_{k+2}$ and that a solution should have the form

$$u(t,x) = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k)!}\, x^{2k}.$$

(b) To show that the formula in (a) is convergent for $t > 0$ we choose the special function g with $g(t) = \mathrm{e}^{-1/t^2}$. Show the estimate

$$|g^{(k)}(t)| \leq \frac{k!}{(Ct)^k}\, \mathrm{e}^{-1/(2t^2)}, \quad k \in \mathbb{N}, \quad t > 0.$$

Use the holomorphic extension of g to $\{\, t \in \mathbb{C} \mid \mathrm{Re}\, t > 0 \,\}$ and $g^{(k)}(t) = \frac{1}{2\pi}\oint_\Gamma \frac{g(\tau)}{(\tau-t)^{k+1}}\,\mathrm{d}\tau$.

(c) Prove that u defined in (a) is a C^∞-solution of the heat equation.

Exercise 43. Smoothing properties of the heat equation.

Consider the heat equation $u_t = u_{xx}$ on the interval $\Omega = \,]0,2\pi[$ with periodic boundary conditions $u(t,2\pi) = u(t,0)$ and $u_x(t,2\pi) = u_x(t,0)$. The initial-value problem has the solution

$$u(t,\cdot) = \mathrm{e}^{tA}u_0 \quad \text{ with } \ \mathrm{e}^{tA}v := \sum\nolimits_{n\in\mathbb{Z}} \mathrm{e}^{-n^2 t}\langle v, E_n\rangle_{\mathrm{L}^2} E_n,$$

where $E_n(x) = (2\pi)^{-1/2}\mathrm{e}^{\mathrm{i}nx}$ forms a complete ONS in $\mathrm{L}^2(\Omega)$.

(a) Show that the operator e^{tA} is a bounded linear operator from $\mathrm{L}^2(\Omega)$ to $\mathrm{L}^2(\Omega)$ which has operator norm 1.

(b) For $k \in \mathbb{N}$ construct C_k and α_k such that $\|\mathrm{e}^{tA}\|_{\mathrm{L}^2(\Omega)\to\mathrm{H}^k_{\mathrm{per}}(\Omega)} \leq C_k(1{+}t^{-\alpha_k})$ holds for all $t > 0$.

(please turn)

Exercise 44. The Cole-Hopf transformation. We consider the linear heat equation

$$u_t = k\Delta u \tag{1}$$

and the nonlinear transformation $u = \phi(v)$ or $v = \psi(u)$.

(a) Find a function g such that v solves

$$v_t = k\Delta v + g(v)|\nabla v|^2 \tag{2}$$

if and only if u solves (1). Which ϕ give $g \equiv$ const.?

(b) Use (a) to show that any solution of the initial value problem $v_t = \Delta v + b|\nabla v|^2$, $v(0,x) = v_0(x)$ with $v_- \le v_0(x) \le v_+$ satisfies $v_- \le v(t,x) \le v_+$ and $|\nabla v(t,x)| \le C/t^{1/2}$ for all $t > 0$ and $x \in \mathbb{R}^d$, where C depends only on v_- and v_+.

Lemma of Lagrange 1736–1813: Consider $a \in \mathbb{R}$ and a function $m \in \mathrm{C}(\mathbb{R})$, for which there exists $C > 0$ such that $m(y) \le C - |y|/C$ for $y \in \mathbb{R}$. Then

$$I(\varepsilon) = 2\varepsilon \log\Big(a \textstyle\int_{\mathbb{R}} \exp\big(m(y)/(2\varepsilon)\big)\,\mathrm{d}y\Big) \;\longrightarrow\; m_* := \max\{\, m(y) \mid y \in \mathbb{R}\,\} \text{ for } \varepsilon \to 0^+.$$

Exercise 45 (in written form). Lax-Oleinik formula for entropy solutions.
(Peter Lax (*1926, Abel Prize 2005) and Olga Oleinik (1925-2001))
The viscous Burgers equation is given via

$$u_t + uu_x = \varepsilon u_{xx}, \quad u(0,x) = f(x). \tag{3}$$

(a) Show that $(t,x) \mapsto v(t,x) = \int_{-\infty}^{x} u(t,\xi)\,\mathrm{d}\xi$ solves $v_t + \frac{1}{2}v_x^2 = \varepsilon v_{xx}$, use the Cole-Hopf transform, and the explicit solution of the heat equation $w_t = \varepsilon w_{xx}$ to establish the following exact solution formula for the viscous Burgers equation (3):

$$\begin{aligned} u(t,x) &= \tfrac{\partial}{\partial x} V_\varepsilon(t,x) \;\text{ with } F(x) = \textstyle\int_{-\infty}^{x} f(\xi)\,\mathrm{d}\xi \text{ and} \\ V_\varepsilon(t,x) &= -2\varepsilon \log\Big((2\varepsilon\pi t)^{-1/2} \textstyle\int_{\mathbb{R}} \exp\big(-(x-y)^2/(4\varepsilon t) - F(y)/(2\varepsilon)\big)\,\mathrm{d}y\Big), \end{aligned}$$

(b) Use the Lemma of Lagrange for passing to the limit of vanishing viscosity, i.e. $\varepsilon \to 0$, and derive the Lax-Oleinik formula

$$u(t,x) = \partial_x V(t,x) \;\text{ with } V(t,x) = \min\{\, \tfrac{(x-y)^2}{2t} + F(y) \mid y \in \mathbb{R}\,\}, \tag{4}$$

which is the unique entropy solution for the initial value problem for the Burgers equation (3) with $\varepsilon = 0$.

Conclusion (for proofs see the book of Evans): *For the Burgers equation the entropy solutions and the solutions obtained via the vanishing-viscosity limit are identical.*

Please hand in the solutions of the written exercise by
Tuesday, 5th of July 2011, 12:00 h.