

## *Exercise Sheet 10*

**Exercise 38. Resolvents and eigenvalues of elliptic operators.** Consider a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^d$ . Further,  $A \in L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{d \times d})$ ,  $a \in L^\infty(\Omega; \mathbb{R}^d)$  and  $\alpha, \rho \in L^\infty(\Omega)$  are given such that for  $\delta > 0$  we have  $A(x)\xi \cdot \xi \geq \delta|\xi|^2$  and  $\rho(x) \geq \delta$  for all  $x \in \Omega$  and  $\xi \in \mathbb{R}^d$ . On the complex Hilbert space  $H_0^1(\Omega) \oplus iH_0^1(\Omega)$  we define the sesquilinear form

$$B(u, v) = \int_{\Omega} A(x) \nabla u(x) \cdot \overline{\nabla v(x)} + a(x) \cdot \nabla u(x) \overline{v(x)} + \alpha(x) u(x) \overline{v(x)} dx$$

and call  $\lambda \in \mathbb{C}$  an eigenvalue of  $(B, \rho)$ , if there exists  $u \in H_0^1(\Omega) \setminus \{0\}$  such that

$$B_\lambda(u, v) := B(u, v) - \lambda \int_{\Omega} \rho u \overline{v} dx = 0 \quad \text{for all } v \in H_0^1(\Omega).$$

If for some  $\lambda \in \mathbb{C}$  there exists  $c_0 > 0$  such that  $|B_\lambda(u, u)| \geq c_0 \|u\|_{H^1}^2$ , then  $\lambda$  lies in the resolvent set of  $B$ . Our aim is to show that the resolvent set is large.

(a) Find  $C_1 > 0$ ,  $c_0 > 0$ , and  $\lambda_* \in \mathbb{R}$  such that  $|B(u, v)| \leq C_1 \|u\|_{H^1} \|v\|_{H^1}$  and  $B_\lambda(u, u) \geq c_0 \|u\|_{H^1}^2$  holds for all  $u, v \in H_0^1(\Omega)$  and  $\text{Re } \lambda \leq \lambda_*$ .

(b) Construct positive numbers  $\beta, \gamma$  such that the set  $\{\lambda \in \mathbb{C} \mid |\text{Im } \lambda| \geq \gamma(\text{Re } \lambda + \beta)\}$  is contained in the resolvent set. For this consider  $B_{\theta, \lambda}(u, v) = e^{i\theta} B_\lambda(u, v)$  and bound  $|B_{\theta, \lambda}(u, u)|$  from below via  $\text{Re}(B_{\theta, \lambda}(u, u))$ .

### **Exercise 39. Gronwall's lemma [1919]**

(after THOMAS HAKON GRÖNWALL (1877-1932)).

(a) Differential form: Let  $y \in W^{1,1}([0, T])$  with  $y(t) \geq 0$  and  $\alpha \in L^1([0, T])$  be such that  $\dot{y}(t) \leq \alpha(t)y(t)$ . Show the Gronwall estimate  $y(t) \leq y(0)e^{\int_0^t \alpha(s) ds}$ .

(b) Generalize (a) to the case  $\dot{y}(t) \leq \alpha(t)y(t) + \beta(t)$  for  $\beta \in L^1([0, T])$  with  $\beta \geq 0$ .

(c) Integrated form:

Assume that  $z \in C^0([0, T])$  with  $z(t) \geq 0$  satisfies  $z(t) \leq z_0 + \int_0^t \alpha_0 z(s) + \beta(s) ds$  with  $\alpha_0 \geq 0$  and  $\beta$  as in (b). Conclude  $z(t) \leq z_0 e^{\alpha_0 t} + \int_0^t e^{\alpha_0(t-s)} \beta(s) ds$ .

(d) Assume that  $y \in W^{1,1}([0, \infty[)$  and  $\phi \in L^1([0, \infty[)$  satisfy  $\phi(t) \geq 0$  and  $\frac{d}{dt} y^2 \leq 2\phi y$  for all  $t \geq 0$ . Show that  $y$  remains bounded by  $|y(0)| + \int_0^\infty \phi(t) dt$ .

(please turn)

**Exercise 40 (in written form). A priori estimates).**

For a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^d$  we consider the general parabolic equation

$$\begin{aligned} c(x)u_t &= \operatorname{div}(A(x)\nabla u(x)) + a(x) \cdot \nabla u(x) + \alpha(x)u(x) + f(t, x), & (t, x) \in ]0, \infty[ \times \Omega, \\ u(0, x) &= u_0(x) \text{ in } \Omega, & (A(x)\nabla u(t, x)) \cdot \nu(x) + \beta(x)u(x) = 0 \text{ for } t > 0, x \in \partial\Omega, \end{aligned}$$

where the divergence is understood in the weak sense. For the coefficients assume  $A \in L^\infty(\Omega, \mathbb{R}_{\text{sym}}^{d \times d})$ ,  $a \in L^\infty(\Omega, \mathbb{R}^d)$ ,  $c, \alpha \in L^\infty(\Omega)$ ,  $\beta \in L^\infty(\partial\Omega)$  with  $\beta \geq 0$ . Moreover,  $A(x)\xi \cdot \xi \geq A_{\min}|\xi|^2$  and  $c(x) \geq c_{\min}$  with  $A_{\min}, c_{\min} > 0$  and  $f \in \text{BC}([0, \infty[, L^2(\Omega))$ .

(a) Let  $E_0(t) = \int_\Omega c(x)u(t, x)^2 dx$  and assume that  $u$  is a sufficiently smooth solution. Derive the estimate

$$\dot{E}_0(t) + c_1 \|\nabla u\|_{L^2(\Omega)}^2 \leq C_2 E_0(t) + C_3 \|f(t)\|_{L^2(\Omega)}^2,$$

where the constants  $c_1, C_2, C_3$  may only depend on the coefficients.

(b) Use the Gronwall lemma to show the a priori estimate

$$\|u(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla u(s)\|_{L^2(\Omega)}^2 ds \leq C_4 (\|u(0)\|_{L^2(\Omega)}^2 + F^2) e^{C_5 t},$$

where  $F = \sup\{\|f(t)\|_{L^2(\Omega)} \mid t \geq 0\}$ .

**Exercise 41. Quadratic membrane.** On the square  $\Omega = ]0, \pi[ \times ]0, \pi[$  we consider the wave equation  $u_{tt} = \Delta u = u_{x_1 x_1} + u_{x_2 x_2}$  with DIRICHLET conditions  $u(t, \cdot)|_{\partial\Omega} = 0$ .

(a) Determine a complete orthonormal system  $\{\Phi_j \mid j \in \mathbb{N}\}$  in  $L^2(\Omega)$ , such that  $u(t, x) = \cos(\omega_j t)\Phi_j(x)$  provides a solution to the wave equation for suitable  $\omega_j$ .

(b) Show that for  $\omega^2 = 10$  there exist two different eigenfunctions  $\Phi_j$ , such that  $\Phi_i(x_1, x_2) = \Phi_j(x_2, x_1) \neq \Phi_j(x_1, x_2)$  holds for  $(x_1, x_2) \in \Omega$ . Discuss the nodal lines (= zero set) of the solution  $u(t, \cdot) = \cos(\sqrt{10}t)(\alpha\Phi_i + \beta\Phi_j)$ . In particular study the cases  $\alpha/\beta \in \{\infty, 1, -1, 0\}$ .

**Please turn in solution of written exercises by  
Tuesday, 30<sup>th</sup> of June 2011, 12:00 h.**

**Prize Task (prize = 20 Euro book coupon for the best solution turned in to  
A. Mielke by July 7, 2011.)**

For some  $d \geq 2$  find a function  $f \in C_c^0(\mathbb{R}^d)$ , such that the solution  $u$  of the Poisson problem  $\Delta u = f$  given via the convolution  $u = K_d * f$  does not lie in  $C^2(\mathbb{R}^d)$ .