Höhere Analysis II: Partial Differential Equations
Summer Term 2011
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Exercise Sheet 9

Exercise 34 (in written form). Pioncaré–Friedrichs' inequality. A domain $\Omega \subset \mathbb{R}^d$ is said to satisfy a PIONCARÉ–FRIEDRICHS' INEQUALITY, if there exists a constant $C > 0$ such that

$$(*) \qquad C \int_\Omega u(x)^2 \,\mathrm{d}x \leq \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x \text{ for all } u \in \mathrm{C}_\mathrm{c}^\infty(\Omega).$$

The largest such C is called the Friedrichs constant $C_{\text{Fried}}(\Omega)$ of the domain.

(a) Show that (*) holds if and only if the same inequality holds for all $u \in \mathrm{H}_0^1(\Omega)$.

(b) Show that every bounded domain satisfies a Friedrichs inequality. For unbounded domains show that both cases can occur.

(c) Using Fourier series find the Friedrichs' constant for the domain $\Omega =]0, \ell_1[\times \cdots \times]0, \ell_d[$.

(d) For $1 \leq k < d$ consider a bounded domain $\Sigma \subset \mathbb{R}^k$ as cross-section for the domain $\Omega = \Sigma \times \mathbb{R}^{d-k} \subset \mathbb{R}^d$. Express $C_{\text{Fried}}(\Omega)$ in terms of $C_{\text{Fried}}(\Sigma)$.

Exercise 35. A bilinear form with Robin boundary conditions. Consider a bounded LIpschitz domain $\Omega \subset \mathbb{R}^d$ with nontrivial Dirichlet boundary Γ_D such that a Poincaré inequality holds. The symmetric bilinear form

$$B(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) + \alpha u(x) v(x) \,\mathrm{d}x + \int_{\partial\Omega \setminus \Gamma_\mathrm{D}} \beta u(x) v(x) \,\mathrm{d}a \quad \text{with } \alpha, \rho \in \mathbb{R}$$

is defined on the Hilbert space $\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega) = \{\, u \in \mathrm{H}^1(\Omega) \mid u|_{\Gamma_\mathrm{D}} = 0 \text{ in the trace sense} \,\}$.

(a) Show that for α, $\beta > 0$ the bilinear form is symmetric, bounded, and coercive. For the weak problem $B(u,v) = \int_\Omega f v \,\mathrm{d}x$ write the associated linear problem $L_{\alpha,\beta} u = f$ in strong form (in particular, the boundary conditions are needed).

(b) Let $\Omega =]-1, 1[\subset \mathbb{R}^1$. For $\alpha = 0$ and $\beta \in \mathbb{R}$ calculate all eigenpairs (λ_j, ϕ_j) for $L_{0,\beta}$ (i.e. $L_{0,\beta}\phi_j = \lambda_j \phi_j$),

(c) Consider the two-dimensional domain $\Omega =]0, \pi[^2$ with the Dirichlet boundary $\Gamma_\mathrm{D} = \{\, (x_1, x_2) \mid x_1 \in \{0, \pi\},\ x_2 \in]0, \pi[\,\}$. Construct all eigenfunctions having the form $u(x) = a(x_1) b(x_2)$. Is the set of these eigenfunctions a complete ONS?

(please turn)

Exercise 36. Parabolic problem on a half-line

(a) Use the reflection principle to construct a solution formula in the form $u(t,x) = \int_0^\infty K_{\mathrm{D}}(t,x,y)u_0(y)\,\mathrm{d}y$ for the heat equation in the half-line $\Omega =]0,\infty[$:

$$\begin{gathered}\partial_t u = \partial_x^2 u \text{ for } (t,x) \in]0,\infty[\times \Omega, \\ \text{(IC) } u(0,x) = u_0(x) \text{ for } x \in \Omega, \quad \text{(BC) } u(t,0) = 0 \text{ for } t > 0.\end{gathered} \tag{Dir}$$

Find similarly the kernel K_{N} if the Dirichlet boundary conditions (BC) are replaced by the Neumann boundary conditions $\partial_x u(t,0) = 0$.

(b) Assume $u_0 \in \mathrm{C}_{\mathrm{c}}^0(\Omega)$. In which case do the solutions satisfy energy conservation $\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty u(t,x)\,\mathrm{d}x = 0$ and in which case entropy production $\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \eta(u(t,x))\,\mathrm{d}x \geq 0$? Here $\eta : \mathbb{R} \to \mathbb{R}$ is a concave entropy density ($\eta'' \leq 0$) with $\eta(0) = 0$.

Exercise 37. Temperature problems when taking a shower.

How much colder is the water after shampooing your hair?

We model the water pipe by the one-dimensional interval $\Omega =]0,l[$, where the water reservoir with fixed temperature 60°C is at $x = l$, whereas $x = 0$ is the outlet. The water is flowing with velocity $v = 0$ during shampooing or $v > 0$ during running water. The equations are:

$$\begin{aligned}&cu_t = ku_{xx} + cvu_x - \alpha(u-20) \quad \text{for } t > 0,\ x \in \Omega, \\ &u_x(t,0) = 0 \quad u(t,l) = 60 \quad \text{for } t > 0.\end{aligned}$$

Here $\alpha > 0$ models the heat losses through the boundary of the pipe into the wall or the air, which are at room temperature 20°C.

(a) Let $c = k = l = 1$ and $\alpha = 4$ and calculate the steady states (time-independent solutions) for the two different flow velocities $v = 0$ and $v = 3$. Draw a picture!

(b) Show that the solution for $v = 0$ is always colder than the solution for $v > 0$.
Hint: Use maximum principles and so on.

Please turn in solution of written exercises by Tuesday, 23th of June 2011, 12:00 h.