

Exercise Sheet 8

Exercise 30. A corner singularity. In \mathbb{R}^2 use the polar coordinates $x = r(\cos \phi, \sin \phi)$ and define the three-quarter disc $\Omega = \{r(\cos \phi, \sin \phi) \in \mathbb{R}^2 \mid r \in]0, 1[, \phi \in]0, 3\pi/2[\}$.

- (a) Check that $w(x) = r^{2/3} \sin(2\phi/3)$ is harmonic in Ω and lies in $H^1(\Omega) \setminus H^2(\Omega)$.
 (b) Find a function $f \in C^0(\overline{\Omega}) \subset L^2(\Omega)$ such that the unique solution u of the DIRICHLET problem $\Delta u = f$ in Ω and $u|_{\partial\Omega} = 0$ lies in $H^1(\Omega)$ but not in $H^2(\Omega)$. (Hint: Determine the values of w on the boundary $\partial\Omega$ and write $u = w - v$ for a smooth function v .)

Exercise 31. Lax-Milgram lemma in unbounded domains. Let $\Omega \subset \mathbb{R}^d$ be any domain (i.e., open and connected). On $H = H_0^1(\Omega)$ define the bilinear form $a : H \times H \rightarrow \mathbb{R}$ via $B(u, v) = \int_{\Omega} \nabla u(x) \cdot A(x) \nabla v(x) + c(x)u(x)v(x) dx$, with $A \in L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{d \times d})$ and $c \in L^\infty(\Omega)$. Further there is an $\alpha > 0$ with $\xi \cdot A(x)\xi \geq \alpha|\xi|^2$ for all $x \in \Omega$ and all $\xi \in \mathbb{R}^d$.

- (a) Show that B is a symmetric and continuous bilinear form. Give sufficient conditions on c that guarantee that B is also coercive.
 (b) Consider the Schrödinger operator $A_\lambda u = -\Delta u + Vu + \lambda u$ in $H_0^1(\Omega)$ with $\Omega = \mathbb{R} \times]0, \pi[$, where $V(x) = 1/(1+x_1^2)$. Show that the bilinear form B_λ associated with A_λ is coercive for $\lambda > -1$ but not for $\lambda \leq -1$.

Exercise 32. The Rellich-Kondrachov Theorem of compact embedding: For a bounded domain $\Omega \subset \mathbb{R}^d$ each bounded sequence $(u_n)_{n \in \mathbb{N}}$ in $H_0^1(\Omega)$ has a subsequence $(u_{n_l})_{l \in \mathbb{N}}$ and a limit $u \in H_0^1(\Omega)$ such that $u_{n_l} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u_{n_l} \rightarrow u$ in $L^2(\Omega)$.

- (a) Prove this result for the case $\Omega =]0, \pi[$ using a suitable orthonormal Fourier basis $\{\phi_k \mid k \in \mathbb{N}^d\}$ and by looking at convergence of the coefficients.
 (b) Generalize the result to all bounded domains. (Hint: Embed $H_0^1(\Omega)$ into $H_0^1(]-R, R[^d)$.)

Exercise 33 (in written form). Céa's Lemma (JEAN CÉA 1962) Consider a symmetric, bounded and coercive bilinear form $B : H \times H \rightarrow \mathbb{R}$ on a real Hilbert space H :

$$B(u, v) = B(v, u), \quad |B(u, v)| \leq C_1 \|u\| \|v\|, \quad B(u, u) \geq c_0 \|u\|^2.$$

- (a) Let the approximation space H_N be a closed subspace of H . For $F \in H^* = \text{Lin}(H, \mathbb{R})$ consider the solution $u \in H$ and the approximation $u_N \in H_N$ defined via

$$B(u, v) = F(v) \text{ for all } v \in H \quad \text{and} \quad B(u_N, v_N) = F(v_N) \text{ for all } v_N \in H_N,$$

respectively. Show that u_N is uniquely defined and satisfies the orthogonality relation

$$B(u - u_N, u - u_N) + B(w_N - u_N, w_N - u_N) = B(u - w_N, u - w_N) \quad \text{for all } w_N \in H_N.$$

- (b) Derive Céa's estimate, which says that u_N is, up to a constant, as good as the H_N -best approximant:

$$\|u - u_N\| \leq \sqrt{C_1/c_0} \inf \{ \|u - w_N\| \mid w_N \in H_N \}. \quad (\text{Céa})$$

- (c) Let $H = H_0^1(]0, \pi[)$ and $H_N = \text{span}\{x \mapsto \sin(kx) \mid k = 1, \dots, N\}$. For $a \in C^1(]0, \pi[)$ with $a(x) \in [1, 2]$ and $f \in L^2(]0, \pi[)$ set $B(u, v) = \int_0^\pi a(x)u'(x)v'(x) dx$ and $F(v) = \int_0^\pi f(x)v(x) dx$. For u and u_N as in (a) show

$$\|u_N - u\|_{H^1} \leq \frac{1}{N} C_2 \|u\|_{H^2} \quad \text{and} \quad \|u\|_{H^2} \leq C_a \|f\|_{L^2}.$$

Hint: Combine Exercise 28 and (Céa).

Please turn in solution of written exercises by Tuesday, 16th of June 2011, 12:00 h.