Exercise Sheet 7

Exercise 25 (in written form). Weak derivatives. For $\alpha \in \mathbb{N}_0^d$ a function $g_\alpha \in \mathrm{L}^1(\Omega)$ is called the weak α-derivative of $u \in \mathrm{L}^p(\Omega)$ if

$$\forall\, \phi \in \mathrm{C}_\mathrm{c}^\infty(\Omega): \quad \int_\Omega u(x)\, \mathrm{D}^\alpha \phi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g_\alpha(x)\, \phi(x)\,\mathrm{d}x$$

We then write $\mathrm{D}^\alpha u := g_\alpha$ (in the sense of distributions).

(a) Show that the weak derivative is unique and that for $u \in \mathrm{C}^n(\overline{\Omega})$ all weak derivatives up to order n exist and coincide with the classical derivatives $\mathrm{D}^\alpha_{\mathrm{cl}} u$.

(b) Let $\Omega = B_1(0) \subset \mathbb{R}^d$ with $d \geq 2$ and $u(x) = 1/|x|$. For which p do we have $u \in \mathrm{L}^p(\Omega)$ and for which q does the classical gradient $\nabla_{\mathrm{cl}} u$ lie in $\mathrm{L}^q(\Omega)$? Show that for these values the weak gradient ∇u coincides with $\nabla_{\mathrm{cl}} u$.

Exercise 26. Sobolev spaces. Let $p \in [1, \infty[$. For general domains $\Omega \subset \mathbb{R}^d$ one defines the Sobolev spaces $W^{n,p}(\Omega)$ using the weak derivatives as follows:

$$W^{n,p}(\Omega) = \{\, u \in \mathrm{L}^p(\Omega) \mid \forall\, \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq n\ \exists\, g_\alpha \in \mathrm{L}^p(\Omega):\ \mathrm{D}^\alpha u = g_\alpha \,\},$$

$$\|u\|_{n,p} = \Big(\sum_{|\alpha| \leq n} \|\mathrm{D}^\alpha u\|^p_{\mathrm{L}^p} \Big)^{1/p}.$$

Moreover, $W^{n,p}_0(\Omega)$ is defined as the completion of $\mathrm{C}_\mathrm{c}^\infty(\overline{\Omega})$ with respect to the norm $\|\cdot\|_{n,p}$.

For bounded domains $\Omega \subset \mathbb{R}^d$ we define $\widetilde{W}^{n,p}(\Omega)$ as the completion of $\mathrm{C}^\infty(\overline{\Omega})$ with respect to the norm $\|\cdot\|_{n,p}$.

(a) Show the inclusion $\widetilde{W}^{n,p}(\Omega) \subset W^{n,p}(\Omega)$. (Hint: For Cauchy sequences $(u_k)_{k\in\mathbb{N}}$ in $W^{n,p}(\Omega)$ consider the limits of $\mathrm{D}^\alpha u_k$.)

(b) Show that $W^{n,p}(\Omega)$ is a Banach space.

(Remark: For bounded domains Ω with Lipschitz boundary, one has $\widetilde{W}^{n,p}(\Omega) = W^{n,p}(\Omega)$.)

Exercise 27. Dirichlet **problem on the unit disc.** Let $\Omega = \{\, x \in \mathbb{R}^2 \,|\, |x| < 1 \,\}$ and consider the Dirichlet problem

$$\Delta u(x) = 0 \text{ for } x \in \Omega, \quad u(y) = g(y) \text{ for } y = (\cos\phi, \sin\phi) \in \partial\Omega =: \mathbb{S}. \tag{DP}$$

(a) Construct for $g_N : \phi \mapsto \mathrm{Re}\big(\sum_{n=0}^N c_n \mathrm{e}^{\mathrm{i}n\phi}\big)$ the associated solution u_N of (DP) and calcualte the norm of u_N in $\mathrm{L}^2(\Omega)$, $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^2(\Omega)$ explicitly. (Hint: $x \mapsto \mathrm{Re}\big(c_n(x_1 + \mathrm{i}x_2)^n\big)$ solves (DP) for suitable boundary data and satisfies some orthogonality condition.)

(b) Investigate via the limit $N \to \infty$ for which functions $g : \partial\Omega \to \mathbb{R}$ the solution of (DP) exists and lies in $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^2(\Omega)$, respectively. For $s \in [0, \infty[$ use the function spaces

$$\mathrm{H}^s_{\mathrm{per}}(\mathbb{S}) = \{\, f \in \mathrm{L}^2(\mathbb{S}) \mid \exists\, (c_k)_{k\in\mathbb{Z}}\colon \sum_{k\in\mathbb{Z}} (1+|k|^2)^s |c_k|^2 < \infty \text{ and } f(\phi) = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}k\phi} \,\}.$$

(please turn)

Exercise 28. Fourier series and Hilbert spaces.
Let $\Omega =]0, 2\pi[$ and $k \in \mathbb{N}_0$ and consider the Hilbert spaces

$$\mathrm{H}^k(\Omega) = \{ f \in \mathrm{L}^2(\Omega) \mid f, f', ..., f^{(k)} \in \mathrm{L}^2(\Omega) \},$$
$$\mathrm{H}^k_{\mathrm{per}}(\Omega) = \{ f \in \mathrm{H}^k(\Omega) \mid f^{(j)}(0) = f^{(j)}(2\pi) \text{ for } j = 0, ..., k{-}1 \},$$
$$\mathrm{H}^k_0(\Omega) = \text{closure of } \mathrm{C}^\infty_\mathrm{c}(\Omega) \text{ in } \mathrm{H}^k(\Omega).$$

Further let $S_n(t) = s_n \sin(nt)$ and $C_n(t) = c_n \cos(nt)$. Then, $\mathrm{L}^2(\Omega)$ has the following three complete orthonormal systems (cONS)

$$O_1 = \{ C_n, S_m \mid m \in \mathbb{N},\ n \in \mathbb{N}_0 \}, \quad O_2 = \{ C_{m/2} \mid m \in \mathbb{N}_0 \}, \quad O_3 = \{ S_{m/2} \mid m \in \mathbb{N} \}.$$

(a) Show that O_3 lies in $\mathrm{H}^1_0(\Omega)$ and that it is a cONS in $\mathrm{L}^2(\Omega)$. *(Results from classical Fourier theorie may be used.)*

(b) For O_1 show that $f = \sum_1^\infty a_m S_m + \sum_0^\infty b_n C_n \in \mathrm{L}^2(\Omega)$ lies in $\mathrm{H}^1_{\mathrm{per}}(\Omega)$ if and only if $\sum_1^\infty l^2(|a_l|^2 + |b_l|^2)$ is finite and that in this case we may differentiate the series representation of f term by term.

(c) For O_2 show that $f = \sum_0^\infty b_m C_{m/2}$ lies in $\mathrm{H}^1(\Omega)$ if and only if $\sum_0^\infty m^2 b_m^2 < \infty$. For O_3 show that $f = \sum_1^\infty a_m S_{m/2}$ lies in $\mathrm{H}^1_0(\Omega)$ if and only if $\sum_1^\infty m^2 a_m^2 < \infty$.

(General hints: Use, without proof, that $\mathrm{H}^{k+1}(]0, 2\pi[) \subset \mathrm{C}^k([0, 2\pi])$ for $k = 0, 1, \ldots$. Compare the series differentiated term by terms with a suitable new expansion of the derivative. Take care of boundary terms in integrations by parts.)

Exercise 29 (in written form). General elliptic equation via Fourier transform in $\mathbb{R}^d$.
Let $A \in \mathbb{R}^{d \times d}$ be symmetric and positive definite, $b \in \mathbb{R}^d$, $c \in]0, \infty[$, and $f \in \mathrm{L}^2(\mathbb{R}^d)$.

(a) Show that the elliptic equation

$$-\sum_{i,j=1}^{d} A_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u + \sum_{k=1}^{d} b_k \frac{\partial}{\partial x_k} u + c\, u = f \text{ auf } \mathbb{R}^d$$

has a unique solution $u \in \mathrm{H}^2(\mathbb{R}^d)$.

(b) For $\lambda \in \mathbb{C}$ (complex plane) consider the so-called *resolvent equation*

$$-\sum_{i,j=1}^{d} A_{ij} \frac{\partial^2}{\partial x_i \partial x_j} U + \sum_{k=1}^{d} b_k \frac{\partial}{\partial x_k} U + c\, U - \lambda U = f \text{ auf } \mathbb{R}^d.$$

Multiply the equation by U, integrate by parts and show that for $\lambda \in]-\infty, 0]$ the unique solution (since (a) is applicable) satisfies $\|U\|_{\mathrm{L}^2} \leq \frac{1}{c-\lambda} \|f\|_{\mathrm{L}^2}$.

(c) Using Fourier transform, try to characterize the set of $\lambda \in \mathbb{C}$ for which the resolvent equation has a unique solution. Give estimates for $\|U\|_{\mathrm{L}^2}$ in terms of λ and $\|f\|_{\mathrm{L}^2}$.

There is no tutorial on 1st of June, 2011.
Please turn in solution of "written exercises" by Tuesday, 7th of June 2011, 12:00 h.

Prize Task (prize = 20 Euro book coupon)
(Not solved in tutorial! Prize is for the best solution turned in to A. Mielke by July 7, 2011.)
For some $d \geq 2$ find a function $f \in \mathrm{C}^0_c(\mathbb{R}^d)$, such that the solution u of the Poisson problem $\Delta u = f$ given via the convolution $u = K_d * f$ does not lie in $\mathrm{C}^2(\mathbb{R}^d)$.