

### Exercise Sheet 6

**Exercise 21. Poisson's formula for a disc.** (Siméon D. Poisson, 1781 – 1840, Paris).

Let  $\Omega = B_R(0) \subset \mathbb{R}^2$ ,  $g \in C^0(\partial\Omega)$ , and

$$u(x) = \int_{|y|=R} P(x, y) g(y) da \quad \text{with } P(x, y) = \frac{R^2 - |x|^2}{2\pi R |x - y|^2}. \quad (\text{PI})$$

- (a) Show that (PI) defines a function  $u \in C^2(\Omega)$  satisfying  $\Delta u = 0$  in  $\Omega$ .  
 (b) Establish  $u \in C(\overline{\Omega})$  and  $u(y) = g(y)$  for  $y \in \partial\Omega$ .  
 (Hint: Show  $P(x, y) \geq 0$ ,  $\int_{\partial\Omega} P(x, y) da = 1$  for all  $x \in \Omega$ , and  $P(x, y) \rightarrow 0$  for  $x \rightarrow y_* \in \partial\Omega \setminus \{y\}$ . Polar coordinates  $x = r(\cos \phi, \sin \phi)$  and  $y = R(\cos \psi, \sin \psi)$  may come in handy.)

**Exercise 22. Hölder spaces** (Otto L. Hölder, 1859 – 1937, Leipzig, studied in Berlin)

For a smooth bounded domain  $\Omega \subset \mathbb{R}^d$  the Hölder spaces with  $\alpha \in ]0, 1[$  are given via

$$C^\alpha(\overline{\Omega}) = \{ u \in C^0(\overline{\Omega}) \mid \|u\|_{C^\alpha} := \|u\|_{L^\infty} + [u]_\alpha < \infty \} \quad \text{with } [u]_\alpha := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

It is well-known that  $(C^\alpha(\overline{\Omega}), \|\cdot\|_{C^\alpha})$  is a Banach space.

- (a) Show that the space  $h^\alpha(\overline{\Omega})$ , which is defined as the closure of  $C^2(\overline{\Omega})$  in  $C^\alpha(\overline{\Omega})$ , is strictly smaller than  $C^\alpha(\overline{\Omega})$ . Hint: Consider  $u(x) = x^\alpha$  in  $C^\alpha([0, 1])$  first.  
 (b) Establish for all  $\beta \in ]\alpha, 1[$  the continuous embeddings  $C^\beta(\overline{\Omega}) \subset h^\alpha(\overline{\Omega}) \subset C^\beta(\overline{\Omega})$ , and show that the first embedding is strict and dense.

**Exercise 23. Maximum principle.** Assume that  $\Omega \subset \mathbb{R}^d$  is a bounded domain (i.e. open and connected). Moreover, let  $u \in C^0(\overline{\Omega})$  be subharmonic, i.e.  $\forall x \in \Omega \forall r > 0$  mit  $B_r(x) \subset \Omega$  :  $u(x) \leq \frac{1}{\omega_d r^{d-1}} \int_{\partial B_r(x)} u(y) da(y)$ . Show the following statements:

- (a) *Weak form of the maximum principle:*  $u$  attains its maximum on  $\partial\Omega$ .  
 (b) *Strong form of the maximum principle:* If  $u$  attains the maximum (also) in the interior of  $\Omega$ , then  $u$  is constant.

**Exercise 24. General properties of Green's functions.** Assume that  $\Omega \subset \mathbb{R}^d$  is a bounded domain with  $C^2$  boundary. Denote by  $G : \Omega \times \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$  the Green's function for the Laplace operator on  $\Omega$  for pure Dirichlet boundary conditions. Derive the following properties:

- (i) For all  $x, y \in \overline{\Omega}$  we have  $G(x, y) = G(y, x)$  and further  $\Delta_x G(x, y) = \Delta_y G(x, y) = 0$  for  $x \neq y$ . (Hint:  $\int_\Omega u_1 \Delta u_2 dx = \int_\Omega u_2 \Delta u_1 dx$  and choose suitable  $f_j = \Delta u_j$ .)  
 (ii)  $\int_{\partial\Omega} \nabla_y G(x, y) \cdot \nu(y) da(y) = 1$  for all  $x \in \Omega$ .  
 (iii)  $G(x, y) < 0$  for  $x, y \in \Omega$ . (Hint: Use a maximum principle for  $G(x, \cdot) = 0$ .)  
 (iv) For  $y \in \partial\Omega$  we have  $\nabla_y G(x, y) = \alpha(x, y)\nu(y)$  for some  $\alpha(x, y) \geq 0$ , i.e.  $\nabla_y G(x, y) \cdot \nu(y) \geq 0$ .

**Prize Task (prize = 20 Euro book coupon)**

(Not solved in tutorial! Prize is for the best solution turned in A. Mielke by July 7, 2011.)

For some  $d \geq 2$  find a function  $f \in C_c^0(\mathbb{R}^d)$ , such that the solution  $u$  of the Poisson problem  $\Delta u = f$  given via the convolution  $u = K_d * f$  does not lie in  $C^2(\mathbb{R}^d)$ .