Exercise Sheet 5

Exercise 17. Theorem of Cauchy (1789-1857) & Kovalevskaya (1850-1891) For $\sigma \in \{-1, 1\}$ construct the solution of the following Cauchy problems via power-series expansion:

$$u_{xx} + \sigma u_{yy} = 0, \quad u(x,0) = u_0(x) = \tfrac{x^2-1}{(x^2+1)^2}, \; u_y(x,0) = u_1(x) = \tfrac{2-6x^2}{(x^2+1)^3}.$$

(a) Check that the Theorem of Cauchy–Kovalevskaya is applicable in both cases.

(b) Find the power-series expansion explicitly. (Hint: Write u_0 as real part and u_1 as imaginary part of simple complex-valued functions.) Discuss the radius of convergence and the singularities of the explicit solutions.

Exercise 18. On the analyticity of solutions for the heat equation. Consider the following Cauchy problem for the one-dimensional heat equation in a neighborhood $\mathcal{U}$ of $(t_0, x_0) = 0$:

$$u_t = u_{xx}, \qquad u(0,x) = u_0(x).$$

(a) Use one of the previous exercises to contruct a solution in $\mathcal{U} \cap ([0, \infty[\, \times \mathbb{R})$.

(b) For $r > 0$ show that the function $u_0(x) = \sum_{k=1}^{\infty} r^k \cos(kx)$ is analytic on $\mathbb{R}$.

(c) Calculate the formal expansion $u(t,x) = \sum_{n=0}^{\infty} g_n(x) t^k$ for u_0 in (b) and show that there is no analytical solution for the corresponding Cauchy problem.

Exercise 19. The Korteweg–de Vries equation describes the evolution of the water surface in cases where the surface remains sufficiently smooth and surface tension can be neglected:

$$u_t + 6uu_x + u_{xxx} = 0 \quad \text{for } t, x \in \mathbb{R}. \tag{1}$$

(a) Construct all nontrivial solution of the form $u(t,x) = U(x-ct)$ satisfying $U(\xi) \to 0$ for $|\xi| \to \infty$. (These solitary waves are used to model tsunamis.)

(b) Transform the equation into a first order system $A_0 \mathbf{w}_t + A_1 \mathbf{w}_x = \mathbf{g}(\mathbf{w})$ and find all characteristic curves. Can the initial value problem with $u(0,x) = u_0(x)$ be solved with the theorem of Cauchy-Kovalevskaya?

Exercise 20 [written] Helmholtz equation (Hermann von Helmholtz 1821-1894, Berlin).

The Helmholtz equation for a frequency $\omega \geq 0$ reads

$$\Delta u + \omega^2 u = 0 \quad \text{on } \Omega. \tag{2}$$

(a) Show that the solutions u of (2) give rise to oscillating solutions $v(t,x)$ of the wave equation $v_{tt} = \Delta v$.

(b) Derive an ODE for the function $g : [0, \infty[\, \to \mathbb{R}$ such that $u(x) = g(|x|)$ satisfies (2).

(c) Let $d = 3$ and $U_1(x) = \sin(\omega|x|)/|x|$ and $U_2(x) = \cos(\omega|x|)/|x|$. In what sense are U_1 and U_2 solutions of (2).

Please turn in solution of "written exercise" by Tuesday, 17. of May 2011, 12:00 h.