Exercise Sheet 4

Exercise 13 [in written form]. Eikonal equation:

The eikonal equation in a medium with varying refraction index n reads

$$|\nabla u|^2 - n(x) = 0.$$

The associated equation for the characteristic strips reads

$$|p|^2 - n(x) = 0, \quad x' = 2p, \quad z' = 2n(x), \quad p' = -\nabla_x n(x),$$

which we want to solve with the initial data $u(x_1, 0) = 0$ and $\partial_{x_2} u(x_1, 0) > 0$.

We consider a two-dimensional problem with $\Omega = \mathbb{R}^2$, where the refraction index only depends on x_1, i.e. $n(x) = \mu(x_1)$. This corresponds to a layered material.

(a) Show that along characteristic strips the equation equation for $x_1(s)$ takes the form $x_1'' + 2\mu'(x_1) = 0$. Find a function of (x_1, x_1') that is constant along characteristic strips.

(b) *Focusing case:* We consider the case $\mu(x_1) = 2 - \frac{1}{1+x_1^2}$. Show that all characteristics starting at $x(0, \xi) = (\xi, 0)$ oscillate periodically around the x_2 axis and thus form a caustic. Try to estimate the smallest value $x_2 > 0$, for which two different characteristics meat in $(0, x_2^*)$. (Because of the reduced refractive index for $x_1 \approx 0$, the material acts as a wave guide, i.e. light is guided along the x_2-axis.)

(c) [Voluntary] Write a computer program for solving the characteristic strips numerically. Plot a few characteristics $s \mapsto x(s; \xi)$ for the the case in (b) as well as for the defocusing case $\mu(x_1) = 1 + \frac{1}{1+x_1^2}$.

Exercise 14. Explicit solutions for the wave equation: We consider the Cauchy problem

$$u_{tt} = u_{xx} + bu \text{ for } (t, x) \in \mathbb{R}^2, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),$$

where b is a real parameter.

(a) Use a power-series expansion to find the solution of the CAUCHY problem for $u_0(x) = e^{\alpha x}$ and $u_1 \equiv 0$, where $\alpha \in \mathbb{C}$.

(b) Use the linearity to find the solution for $u_0(x) = \frac{1}{h}\big(e^{(\alpha+h)x} - e^{\alpha x}\big)$ and $u_1 \equiv 0$. Consider and justify the limit $h \to 0$ to find the solution v for the CAUCHY data $v_0(x) = x \sin(kx)$ and $v_1 \equiv 0$ mit $k \in \mathbb{R}$.

(c) Return to the solution u from Part (a). Show that $w = u_t$ is again a solution. What are the CAUCHY data for w at $t = 0$? Give the solution for the CAUCHY data $u_0 \equiv 0$ and $u_1(x) = e^{\alpha x}$.

please turn

Exercise 15. Viscous conservation laws: For $\varepsilon > 0$ the semilinear parabolic equation

$$u_t + f'(u)u_x = \varepsilon u_{xx} \quad \text{or} \quad u_t + \big(f(u) - \varepsilon u_x\big)_x = 0$$

is called a viscous conservation law, as it is in conservation form and also contains the viscous term εu_{xx}. The positive sign of the viscosity ε is remanent to introducing a positive direction of time.

(a) We are looking for traveling waves $u(t, x) = U\big((x-ct)/\varepsilon\big)$. Derive an ODE of the form $U' = g(U)$, which has to be satisfies by such waves. Give its explicit form in the case of $f(u) = \frac{1}{2}u^2$.

(b) A wave U from (a) is called *front* if it satisfies $U(\xi) \to u_\pm$ for $\xi \to \pm\infty$. Discuss the existence of such fronts and relate it to the entropy condition and the Rankine-Hugoniot condition.

Exercise 16. Weak solutions for the wave equation:

We consider the linear wave equation $u_{tt} = u_{xx}$, which has the classical solutions $u(t, x) = g(x+t) + h(x-t)$ for $g, f \in \mathrm{C}^2(\mathbb{R})$. We define weak solutions $u \in \mathrm{C}^{\mathrm{Lip}}(\mathbb{R}^2)$ via the condition

$$\forall\, \phi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^2) : \quad \int_{\mathbb{R}^2} \big(u_t\phi_t - u_x\phi_x\big)\,\mathrm{d}t\,\mathrm{d}x = 0,$$

and distributional solutions $u \in \mathrm{L}^\infty(\mathbb{R}^2)$ via the condition

$$\forall\, \phi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^2) : \quad \int_{\mathbb{R}^2} \big(u\phi_{tt} - u\phi_{xx}\big)\,\mathrm{d}t\,\mathrm{d}x = 0.$$

We use here that for every function $u \in \mathrm{C}^{\mathrm{Lip}}(\mathbb{R}^2)$ the gradient ∇u is a well-defined function in $\mathrm{L}^\infty(\mathbb{R}^2)$ (Rademacher's theorem).

(a) Show that weak and distributional solutions satisfying additionally $u \in \mathrm{C}^2(\mathbb{R}^2)$ are classical solutions.

(b) For which g and h is $u(t, x) = g(x+t) + h(x-t)$ a weak or distributional solution? [Hint: Use linearity to discuss g and h independently.]

(c) Find the solution with the initial data $u(0, x) = f_0(x)$ and $u_t(0, x) \equiv 0$, where f_0 is the piecewise affine and periodic function with $f_0(x) = 1 - |x-1|$ for $x \in [-1, 3]$ and $f_0(x+4k) = f_0(x)$ for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$.

(d) Show that the solution u in (c) satisfies $u(t, 0) = 0 = u(t, 2)$ for $t \in \mathbb{R}$ and $u(t+p, x) = u(t, x)$ for a suitable period p. Hence it describes the vibration of a guitar string, which is clamped at the points $x_0 = 0$ and $x_1 = 2$ and released from an initial triangle position. Draw a picture or plot u for $(t, x) \in [0, 10] \times [0, 2]$ and discuss the relation to experiments.