

Exercise Sheet 3

Exercise 9. Traffic model I: We study the one-dimensional traffic model

$$\partial_t \rho + \partial_x (\rho V(\rho)) = 0 \text{ for } (t, x) \in [0, \infty[\times \mathbb{R}; \quad \rho(0, x) = \rho_0(x),$$

where $V(\rho) = \frac{v_{\max}}{\rho_{\max}}(\rho_{\max} - \rho)$. For a given solution $\rho : [0, T] \times \mathbb{R} \rightarrow [0, \rho_{\max}]$ we want to investigate the paths $x = X(t, \xi)$ of individual vehicles, where $t \in [0, T]$ and $\xi = X(0, \xi)$ is the starting position of the vehicle labeled by ξ . Thus, we have to solve the ODE

$$\dot{x}(t) = V(\rho(t, x(t))) \text{ and } x(0) = \xi.$$

We study two cases.

(a) *Entering of a traffic jam from behind:* For $0 < \rho_- < \rho_+ \leq \rho_{\max}$ we have the solution

$$\rho(t, x) = \begin{cases} \rho_+ & \text{for } x \geq c_* t, \\ \rho_- & \text{for } x < c_* t, \end{cases} \quad \text{with } c_* = \frac{v_{\max}}{\rho_{\max}}(\rho_{\max} - \rho_+ - \rho_-).$$

Calculate $X(t, \xi)$ explicitly.

(b) *Starting from rest after traffic light switched to green:* With $s(t) = -\ell$ for $t \in [0, \ell/v_{\max}]$ and $s(t) = v_{\max}t - 2\sqrt{\ell v_{\max}t}$ for $t \geq \ell/v_{\max}$ the traffic density is given as

$$\rho(t, x) = \begin{cases} \rho_{\max} & \text{for } t \in [0, \ell/v_{\max}] \text{ and } x \in [-\ell, -v_{\max}t], \\ \frac{\rho_{\max}}{2} \left(1 - \frac{x}{v_{\max}t}\right) & \text{for } x \in [s(t), v_{\max}t], \\ 0 & \text{otherwise.} \end{cases}$$

Calculate all $X(t, \xi)$ and discuss the relative distance of the first and the last vehicle in the queue.

Exercise 10. Traffic model II: For the PDE $\partial_t \rho + \partial_x (\rho - \rho^2) = 0$ we consider the initial condition $\rho(0, x) = f(x)$ with

$$f(x) = \begin{cases} 1/2 & \text{for } x < -\ell, \\ 1 & \text{for } x \in [-\ell, 0], \\ 0 & \text{for } x > 0. \end{cases}$$

- (a) Construct the entropy solution for $t \in [0, 2\ell]$ by piecing together known solutions.
- (b) For $t > 2\ell$ derive an ODE for the curve $s(t)$ such that $\rho(t, x) = 1/2$ for $x < s(t)$. Use the Rankine-Hugoniot relation and check the entropy condition.
- (c) Give the full entropy solution and determine $\max \rho(t, \cdot)$ explicitly.

please turn

Exercise 11 [in written form] Burgers equation (named after Johannes Martinus Burgers, 1895-1981):

$$\partial_t u + u \partial_x u = 0 \quad (\text{inviscid case}).$$

We consider the piecewise constant functions

$$\tilde{u}(t, x) = \begin{cases} u_+ & \text{for } x > ct, \\ u_- & \text{for } x < ct, \end{cases} \quad \hat{u}(t, x) = \begin{cases} u_1 & \text{for } x < c_1 t, \\ u_2 & \text{for } c_1 t < x < c_2 t, \\ u_3 & \text{for } x > c_2 t \end{cases}$$

- (a) Show that \tilde{u} is a weak solution if and only if $2c = u_+ + u_-$. Under what conditions does the shock satisfy the entropy condition?
- (b) Given u_1 and u_3 (and hence $\hat{u}(0, x)$) find all u_2 , c_1 , and c_2 such that $c_1 < c_2$ and that \hat{u} is a weak solution for $t > 0$. When do we have shocks satisfying the entropy condition?
- (c) Construct a piecewise constant solution $u^{(1)}$ satisfying $u^{(1)}(0, x) = f_1(x)$ with $f_1(x) = 6$ for $x < -1$, $f_1(x) = 4$ for $|x| < 1$, and $f_1(x) = 2$ for $x > 1$.
- (d) Give a second initial condition f_2 such that the solution $u^{(2)}$ satisfies $u^{(2)}(t, x) = u^{(1)}(t, x)$ for $t \geq 4$ and $x \in \mathbb{R}$.

Exercise 12. General jump conditions: Let $A : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^d$ and $B : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ define the quasilinear PDE in **conservation form**

$$\begin{aligned} (\operatorname{div} A(\cdot, u(\cdot)))(x) &= B(x, u(x)) \text{ for } x \in \Omega \subset \mathbb{R}^d, \\ \text{with } (\operatorname{div} A(\cdot, u(\cdot)))(x) &:= \sum_{j=1}^d \partial_{x_j} \alpha_j(x) \text{ where } \alpha_j(x) = A_j(x, u(x)) \end{aligned}$$

- (a) Write the conservation-form equation in the quasilinear form $a(x, u) \cdot \nabla u = b(x, u)$ by assuming that everything is sufficiently smooth.
- (b) A function $u \in L^\infty(\Omega)$ is called weak solution, if

$$\int_{\Omega} A(x, u(x)) \cdot \nabla \phi(x) + B(x, u(x)) \phi(x) \, dx = 0 \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

Show that classical solutions are weak solutions.

- (c) Let \mathcal{C} be a smooth hypersurface separating Ω into the two pieces Ω_+ and Ω_- . Let $u : \Omega \rightarrow \mathbb{R}$ be such that the restrictions $u_\pm = u|_{\Omega_\pm}$ lie in $C^1(\overline{\Omega_\pm})$. Derive the *jump relations*

$$\left(A(y, u_+(y)) - A(y, u_-(y)) \right) \cdot \nu(y) = 0 \quad \text{for all } y \in \mathcal{C},$$

where ν is the normal vector to \mathcal{C} . (Hint: First show that in Ω_+ and Ω_- we have classical solutions. Then do integration by parts in Ω_+ and Ω_- separately and compare the boundary terms.)

- (c) Discuss the jump relations in the special two-dimensional case

$$\partial_t(\alpha(x, u)) + \partial_x(\beta(x, u)) = B(t, x, u),$$

if \mathcal{C} is given in the form $x = s(t)$.