Exercise Sheet 2

Exercise 5. Fundamental lemma of the calculus of variations:

Consider an open domain $\Omega \subset \mathbb{R}^d$, a scalar function $b \in \mathrm{C}(\Omega)$ and a vector field $v \in \mathrm{C}^1(\Omega; \mathbb{R}^d)$.

(a) Assume that for all closed balls $B_r(x) \subset \Omega$ (as test volumes) we have $\int_{B_r(x)} b(y)\,\mathrm{d}y = 0$. Conclude that $b \equiv 0$.

(b) Assume that for all closed balls $B_r(x) \subset \Omega$ we have the relations $\int_{B_r(x)} b(y)\,\mathrm{d}y = \int_{\partial B_r(x)} v(\eta) \cdot \nu(\eta)\,\mathrm{d}a(\eta)$. Conclude $b = \operatorname{div} v$ in Ω.

(c) Assume that $\int_\Omega b(y)\psi(y)\,\mathrm{d}y = 0$ for all $\psi \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$. Show that $b \equiv 0$. (Here $\mathrm{C}_\mathrm{c}^\infty(\Omega)$ denotes the space of all infinitely often differentiable functions $\psi : \Omega \to \mathbb{R}$ such that the support $\mathrm{sppt}(\psi) = \mathrm{closure}\big(\{\, x \in \Omega \mid \psi(x) \neq 0 \,\}\big)$ is compact and contained in Ω.)

Exercise 6. Linear transport problem:

For constants $a, b \in \mathbb{R}$ we consider the scalar, linear equation

$$u_t + a u_x = bu \quad \text{for } t > 0 \text{ and } x \in \Omega = \,]0, \infty[$$

together with the Cauchy data $u(0, x) = f(x)$.

(a) Show that for $a \leq 0$ there is a unique solution for this Cauchy problem (recall $t > 0$). Give the solution explicitly.

(b) Provide all solutions of the above Cauchy problem for the case $a > 0$. Confirm *local uniqueness* near $(t, x) = (0, x_*)$ and show *global nonuniqueness*.

(c) Under what conditions does the above Cauchy problem (with $t > 0$) has a solution that satisfies the additional *boundary condition* $u(t, 0) = h(t)$?

Exercise 7. Scalar equation of first order:

(a) Find the solution of the semilinear PDE $u_t + u_y = -u^3$, which satisfies the initial condition $u(0, y) = (1{+}y^4)^{1/2}$.

(b) For arbitrary, differentiable $f : \mathbb{R} \to R$ give the solution of the Cauchy problem

$$u_t + y u_y = -y^3, \quad u(0, y) = f(y).$$

please turn

Exercise 8 [in written form]. Transformation of quasilinear equations:

We consider the quasilinear problem

$$a(x, u(x)) \cdot \nabla u(x) = g(x, u(x)) \text{ for } x \in \Omega \subset \mathbb{R}^d, \tag{Q}$$

where the function $u : \Omega \to \mathbb{R}$ is to be determined.

(a) Consider a coordinate change $x = \Phi(y)$ with a bijective mapping $\Phi : \widetilde{\Omega} \to \Omega$, where Φ and Φ^{-1} are in C^1. We let $v(y) = u(\Phi(y)) = (u \circ \Phi)(y)$. Which equation holds for v, if u is a solution of (Q)?

(b) Moreover, consider a bijection $\psi : \mathbb{R} \to \mathbb{R}$, such that ψ and ψ^{-1} are in C^1. Which equation holds for $w : \Omega \to \mathbb{R}; x \mapsto \psi(u(x))$, if u is a solution of (Q)?

(c) For the special case $\partial_t u + \widetilde{a}(t, x, u) \cdot \nabla_x u = b(t, x, u)$ give the equations for v and w from (a) und (b) in the form $\partial_t v + \ldots$ and $\partial_t w + \ldots$, respectively. Here, the coordinate change Φ in (a) should not depend on t.

(d) Which bijections Φ and ψ transform the equation $\partial_t u + x_2 \partial_{x_1} u + \partial_{x_2} u = 1 - u$ into the equation $\partial_t w + \partial_{x_1} w = 1$? Construct the general solution w and provide a formula for the general solution u.

Please turn in solution of "written exercise" by
Tuesday, 26. of April 2011, 12:00 h.

Shift in Tutorial Class on Wednesday: **Beginn is 11:00 h.**

Official exam dates are 19.+20. of September 2011.
Other dates only after individual agreement.