Höhere Analysis II: Partial Differential Equations
Summer Term 2011
Alexander Mielke, Donat Wegner
14. April 2011

Exercise Sheet 1

Exercise 1 (Navier–Stokes equation) We consider the Navier–Stokes equation

$$\partial_t u + (u\cdot\nabla)u + \nabla p = \Delta u, \qquad \operatorname{div} u = 0 \qquad \text{for } (t,x) \in (0,T) \times \Omega$$

in the two-dimensional strip $\Omega = \mathbb{R} \times (0,h)$. Here $u(t,x) \in \mathbb{R}^2$ describes the velocity of the fluid particles, and $p(t,x) \in \mathbb{R}$ is the pressure field. By Δu we denote the componentwise Laplace-Operator, and $(u\cdot\nabla)v = \sum_1^d u_j \partial_j v$. Moreover, we impose the boundary conditions $u = 0$ on the boundary $\partial\Omega$ of the flow domain Ω.

(a) Show that for all $k \in \mathbb{N}$ there are solutions of the form $u(t,x) = \mathrm{e}^{\alpha t} \sin(k\pi x_2/h)\Phi$ for suitable $\alpha \in \mathbb{R}$ and $\Phi \in \mathbb{R}^2$.

(b) Determine all solutions (u,p), for which u does not depend on (t, x_1).

(c) For general solutions show that the mass flux in x_1 direction is independent of x_1, i.e., $\int_0^h u_1(t,x_1,\eta)\,\mathrm{d}\eta = c(t)$. (Hint: Apply Gauß's theorem to a suitable subdomain of Ω.)

(d) Find a simple partial differential equation characterizing all solutions (u,p), for which u is independent of $x_1 \in \mathbb{R}$.

Exercise 2 (Schrödinger equation) A free quantummechanical particle is governed by the complex-valued Schrödinger equation without potential. The wave function $\Psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ has to satisfy

$$\mathrm{i}\frac{\partial}{\partial t}\Psi = \Delta\Psi. \tag{SE}$$

(a) We look for solutions Ψ of (SE) in the form

$$\Psi(t,x) = \gamma \mathrm{e}^{-a(t)|x|^2 + b(t)\cdot x + c(t)} \qquad \text{for } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^d.$$

Derive ordinary differential equations

$$\dot a = f(a,b,c), \qquad \dot b = g(a,b,c), \qquad \text{and } \dot c = h(a,b,c),$$

for $a : \mathbb{R} \to \mathbb{C}$, $b : \mathbb{R} \to \mathbb{C}^d$, and $c : \mathbb{R} \to \mathbb{C}$ such that this Ψ solves (SE).

(b) Write $a(t) = \alpha(t) + \mathrm{i}\beta(t)$ and show that $\dot a = f(a,b,c)$ gives the ODE

$$\dot\alpha = r(\alpha,\beta) \quad \text{and} \quad \dot\beta = s(\alpha,\beta).$$

Conclude that $\alpha(0) > 0$ implies that $\alpha(t) > 0$ for all t. Show that all solutions can be given explicity. (Hint: For a one obtains a complex Bernoulli equation.)

(c) Provide the general solution for the ODE system in (a).

(d) The function $\rho(t,\cdot) = |\Psi(t,\cdot)|^2 \geq 0$ has the physical interpretation of a density distribution of the particle's position in $\mathbb{R}^d$ at time $t \in \mathbb{R}$. Discuss the motion of the average position $X(t) \in \mathbb{R}^d$ and the variance $\sigma(t) > 0$ of the distribution.

please turn

Exercise 3 (Separation of variables) Consider the three PDEs

$$\text{(i) } u_{tt} = u_{xx}, \qquad \text{(ii) } u_{tt} + u_{xx} = 0, \qquad \text{(iii) } u_t = u_{xx}.$$

(a) For all three PDEs construct all possible solutions with *separated variables*, i.e. they have the form $u(t,x) = V(t)W(x)$ for some real-valued, twice differentiable functions V and W. (Hint: After inserting the ansatz into the PDE, separate the variables to the different sides of "=". Argue then that both sides must be constant.)

(b) For the heat equation (iii) find all solutions of (a) satisfying additionally $W(0) = 0 = W(1)$. From this, construct a solution of (iii) that additionally satisfies the boundary conditions $u(t,0) = u(t,1) = 0$ and the initial condition $u(0,x) = \frac{\pi^2}{17}\sin(\pi x) - 1.2\sin(4\pi x)$.

(c) For the wave equation (i) find all solution of (a) satisfying additionally $W(0) = 0 = W(\ell)$, where $\ell > 0$ is given. Show that all these solutions satisfy $u(t+2n\ell, x) = u(t,x)$ for $n \in \mathbb{Z}$.

Exercise 4 [written] (Heat equation) We want to investigate the heat equation on $\Omega = (t,x) \in (0,\infty) \times \mathbb{R}^d$:

$$u_t = \Delta u = \sum_{j=1}^{d} \frac{\partial^2}{\partial {x_j}^2} u. \tag{HE}$$

(a) The function $H(t,x) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-|x|^2/(4t)\right)$ is called *heat kernel*. Show that $u(t,x) = H(t,x)$ is a solution of (HE).

(b) For $f \in \mathrm{C}^0_\mathrm{c}(\mathbb{R}^d) = \{\, f \in \mathrm{C}^0(\mathbb{R}^d) \mid \operatorname{sppt}(f) \text{ compact in } \mathbb{R}^d \,\}$ check that

$$u(t,x) = \int_{y \in \mathbb{R}^d} H(t, x-y) f(y) \,\mathrm{d}y \tag{S}$$

is a solution of (HE), too. (Please, justify the interchange of limit processes $\int$ and ∂!)

(c) For the solutions u in (S) and $t > 0$ derive the estimates

$$\inf\{\, f(x) \mid x \in \mathbb{R}^d \,\} \le \inf\{\, u(t,x) \mid x \in \mathbb{R}^d \,\} \le \sup\{\, u(t,x) \mid x \in \mathbb{R}^d \,\} \le \sup\{\, f(x) \mid x \in \mathbb{R}^d \,\}.$$

Moreover, establish the conservation of the total heat energy, i.e. $\int_{\mathbb{R}^d} u(t,x)\,\mathrm{d}x$ is independent of the time t.

(d) Establish for u in (S) and all $x \in \mathbb{R}^d$ the limit $\lim_{t \nearrow 0} u(t,x) = f(x)$. (Hint: Substitute $y = x + t^\alpha z$ for a suitable α.)

Turn in solutions for written exercises by Tuesday 19. April 2011, 12:00 h.