Höhere Analysis I: Lineare Funktionalanalysis
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Alexander Mielke, Josua Groeger, Max Bender
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Exercise Sheet 9

Exercise 34 (One-dimensional Differential Operators) - oral

Let $\Omega = (0,\ell) \subseteq \mathbb{R}$ be a bounded interval. We study weak formulations of the ordinary differential equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(a(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right) + b(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x) + c(x)u(x) = f(x) \text{ in } \Omega \text{ and } u(0) = u(\ell) = 0 \text{ (ODE).}$$

Let $a, b, c \in \mathrm{L}^\infty(\Omega)$ and $a(x) \geq \alpha$ for some $\alpha > 0$ almost everywhere in Ω.

(a) Let $\rho \in \mathrm{C}^1\left(\overline{\Omega}; (0,\infty)\right)$. Show that

$$\|f\|_\rho = \left(\int_0^\ell |f(x)|^2 \rho(x)\, \mathrm{d}x\right)^{\frac{1}{2}} \text{ and } \|u\|_{1,\rho} = \left(\int_0^\ell \left[|u(x)|^2 + |u'(x)|^2\right] \rho(x)\, \mathrm{d}x\right)^{\frac{1}{2}}$$

defines, respectively, a norm on $\mathrm{L}^2(\Omega)$ and a norm on $\mathrm{H}^1_0(\Omega)$ which, in either case, is equivalent to the respective standard norm.

(b) For $f \in \mathrm{L}^2(\Omega)$, write (ODE) in the weak form

$$\forall\, v \in \mathrm{H}^1_0(\Omega) \;:\; \mathcal{A}_\rho(u,v) := \int_0^\ell (au'v' + \beta u'v + cuv)\,\rho\, \mathrm{d}x = \langle f, v\rangle_\rho := \int_0^\ell f(x)v(x)\rho(x)\, \mathrm{d}x.$$

How must $\beta \in \mathrm{L}^\infty$ be chosen?

(c) Can ρ be chosen such that the bilinear form $\mathcal{A}_\rho$ in (b) becomes symmetric?

(d) Conclude that, under the assumption $c(x) \geq \gamma$ for some $\gamma > 0$ almost everywhere in Ω, (ODE) is always soluble. In particular, b is not required to be „small“.

Remark: As for (c), it is not enough to suppose $a, b \in \mathrm{L}^\infty(\Omega)$. In fact, we should require that $a, b \in \mathrm{C}^0\left(\overline{\Omega}\right)$.

Solution

(a): For $f, g \in \mathrm{L}^2(\Omega)$, let

$$\langle f, g\rangle_\rho := \int_0^\ell f(x)g(x)\rho(x)\, \mathrm{d}x = \langle f\sqrt{\rho}, g\sqrt{\rho}\rangle_{\mathrm{L}^2(\Omega)}.$$

This is well-defined, because $\rho \in C^1\left(\overline{\Omega}; (0,\infty)\right) \subseteq \mathrm{L}^\infty(\Omega)$ implies $f^2\rho, g^2\rho \in \mathrm{L}^1(\Omega)$, i.e. $f\sqrt{\rho}$ and $g\sqrt{\rho}$ lie in $\mathrm{L}^2(\Omega)$. As $\langle\cdot,\cdot\rangle_{\mathrm{L}^2(\Omega)}$ is a scalar product on $\mathrm{L}^2(\Omega)$, $\langle\cdot,\cdot\rangle_\rho$ is at least a symmetric and positive semidefinite bilinear form on $\mathrm{L}^2(\Omega)$. But $\langle f, f\rangle_\rho = 0$ implies $f\sqrt{\rho} = 0$ almost every and thus $f = 0$ almost everywhere because of $\rho > 0$ (everywhere). Hence $\langle\cdot,\cdot\rangle_\rho$ is a scalar product on $\mathrm{L}^2(\Omega)$ whose induced norm is just $\|\cdot\|_\rho$ because of

$$\langle f, f\rangle_\rho := \int_0^\ell |f(x)|^2 \rho(x)\, \mathrm{d}x = \|f\|_\rho$$

for all $f \in \mathrm{L}^2(\Omega)$.

Now, $(u, v) \mapsto \langle u', v' \rangle_\rho$ is obviously a symmetric, positive semidefinite bilinear form on $\mathrm{H}_0^1(\Omega)$ (notice that the differential is linear), thus $\langle u, v \rangle_{1,\rho} := \langle u, v \rangle_\rho + \langle u', v' \rangle_\rho$ is a scalar product on $\mathrm{H}_0^1(\Omega)$ (because it is a sum of a positive definite and a positive semidefinite, symmetric bilinear form). Its induced norm is given by

$$\sqrt{\langle u, u \rangle_{1,\rho}} = \sqrt{\|u\|_\rho^2 + \|u'\|_\rho^2} = \left(\int_0^\ell \ \left[|u(x)|^2 + |u'(x)|^2 \right] \rho(x) \ \mathrm{d}x \right)^{\frac{1}{2}} = \|u\|_{1,\rho}.$$

As ρ is continuous on a compact set and maps to $(0, \infty)$, it has a minimum $m > 0$ und a maximum $n > 0$. We then obtain

$$\int_0^\ell \ |f(x)|^2 m \ \mathrm{d}x \leq \int_0^\ell \ |f(x)|^2 \rho(x) \ \mathrm{d}x \leq \int_0^\ell \ |f(x)|^2 n \ \mathrm{d}x,$$

i.e. $\sqrt{m}\|f\|_{\mathrm{L}^2(\Omega)} \leq \|f\|_\rho \leq \sqrt{n}\|f\|_{\mathrm{L}^2(\Omega)}$ for all $f \in \mathrm{L}^2(\Omega)$. Hence $\| \cdot \|_\rho$ is equivalent to the usual L^2-norm. This implies

$$m\|u\|_{\mathrm{L}^2(\Omega)}^2 + m\|u'\|_{\mathrm{L}^2(\Omega)}^2 \leq \|u\|_\rho^2 + \|u'\|_\rho^2 \leq n\|u\|_{\mathrm{L}^2(\Omega)}^2 + n\|u'\|_{\mathrm{L}^2(\Omega)}^2,$$

i.e. $\sqrt{m}\|u\|_{\mathrm{H}_0^1(\Omega)} \leq \|u\|_{1,\rho} \leq \sqrt{n}\|u\|_{\mathrm{H}_0^1(\Omega)}$ for every $u \in \mathrm{H}_0^1(\Omega)$. Hence $\| \cdot \|_{1,\rho}$ is also equivalent to $\| \cdot \|_{\mathrm{H}_0^1(\Omega)}$.

(b): Let $u \in \mathrm{C}^2\left(\overline{\Omega}\right)$ (or $u \in \mathrm{H}_0^2(\Omega)$) be a solution of (ODE). For $v \in \mathrm{H}_0^1(\Omega)$, we obtain by using integration by parts and $v(0) = v(\ell) = 0$:

$$\begin{aligned}
\langle f, v \rangle_\rho &= \int_0^\ell \ f(x)v(x)\rho(x) \ \mathrm{d}x = \int_0^\ell \ (-(au')'v + bu'v + cuv)\rho \ \mathrm{d}x \\
&= -\int_0^\ell \ (au')'v\rho \ \mathrm{d}x + \int_0^\ell \ bu'v\rho \ \mathrm{d}x + \int_0^\ell \ cuv\rho \ \mathrm{d}x \\
&= [au'v\rho]_0^\ell + \int_0^\ell \ au' \cdot v'\rho \ \mathrm{d}x + \int_0^\ell \ (au')v\rho' \ \mathrm{d}x + \int_0^\ell \ bu'v\rho \ \mathrm{d}x + \int_0^\ell \ cuv\rho \ \mathrm{d}x \\
&= \int_0^\ell \ \left(au'v' + \left(a \cdot \frac{\rho'}{\rho} + b \right) u'v + cuv \right) \rho \ \mathrm{d}x.
\end{aligned}$$

If we put $\beta := a \cdot \frac{\rho'}{\rho} + b$ and

$$\mathcal{A}_\rho(u, v) = \int_0^\ell \ (au'v' + \beta u'v + cuv) \, \rho \ \mathrm{d}x$$

for $u, v \in \mathrm{H}_0^1(\Omega)$, which always equals $\int_0^\ell \ (-(au')'v + bu'v + cuv)\rho \ \mathrm{d}x$ for $u \in \mathrm{H}_0^2(\Omega)$ (by the previous calculation), then u satisfies the required integral equation.

Conversely, let $u \in \mathrm{H}_0^2(\Omega)$ satisfy this integral equation $\mathcal{A}_\rho(u, v) = \langle f, v \rangle_\rho$ for all $v \in \mathrm{H}_0^1(\Omega)$ with the bilinear form $\mathcal{A}_\rho$ just defined. Then, for $v \in \mathrm{C}_c^\infty(\Omega)$ this integral equation also holds for $\frac{v}{\rho} \in \mathrm{C}_c^1\left(\overline{\Omega}; (0, \infty)\right) \subseteq \mathrm{H}_0^1(\Omega)$, i.e.

$$\int_0^\ell \ (-(au')' + bu' + cu - f)v \ \mathrm{d}x = 0,$$

which just means $\langle -(au')' + bu' + cu - f, v\rangle_{\mathrm{L}^2(\Omega)} = 0$ for all $v \in \mathrm{C}_c^\infty(\Omega)$. Since $\mathrm{C}_c^\infty(\Omega)$ is dense in $\mathrm{L}^2(\Omega)$, this implies $-(au')' + bu' + cu - f = 0$ (almost everywhere in Ω). The boundary condition $u(0) = u(\ell) = 0$ follows from $u \in \mathrm{H}_0^2(\Omega)$.

(c): If $\beta = 0$, then $\mathcal{A}_\rho$ is clearly symmetric. $\beta = 0$ is equivalent to $a\rho' = -b\rho$. This differential equation has a solution in $\mathrm{C}^1\left(\overline{\Omega}\right)$ if $-\frac{b}{a} \in \mathrm{C}^0\left(\overline{\Omega}\right)$, but for general $a, b \in \mathrm{L}^\infty(\Omega)$ this differential equation need not have a solution. So let us suppose $a, b \in \mathrm{C}^0\left(\overline{\Omega}\right)$ and $a > 0$ everywhere (hence $a \geq \alpha$ for some $\alpha > 0$ as before). Then this differential equation has solutions of the form $\rho(x) = \rho_0 \cdot \exp\left(-\int_0^x \frac{b}{a}\,\mathrm{d}x\right)$. We are only interested in the case $\rho_0 > 0$. For such a choice of ρ, the bilinear form $\mathcal{A}_\rho$ becomes symmetric.

(d): Let ρ be chosen as in (c) so that $\mathcal{A}_\rho$ becomes symmetric. For $u, v \in \mathrm{H}_0^1(\Omega)$ we then get (by using the Cauchy-Schwarz estimate)

$$\begin{aligned}|\mathcal{A}_\rho(u,v)| &= \left|\int_0^\ell (au'v' + cuv)\rho\,\mathrm{d}x\right| = |\langle au', v'\rangle_\rho + \langle cu, v\rangle_\rho| \leq \|au'\|_\rho \|v'\|_\rho + \|cu\|_\rho \|v\|_\rho \\ &\leq \|a\|_\infty \|u'\|_\rho \|v'\|_\rho + \|c\|_\infty \|u\|_\rho \|v\|_\rho \leq (\|a\|_\infty + \|c\|_\infty)\|u\|_{1,\rho}\|v\|_{1,\rho}.\end{aligned}$$

Hence $\mathcal{A}_\rho$ is a continuous symmetric bilinear form on $\mathrm{H}_0^1(\Omega)$. Moreover, it holds

$$\begin{aligned}\mathcal{A}_\rho(u,u) &= \int_0^\ell (a(u')^2 + cu^2)\rho\,\mathrm{d}x \geq \alpha \cdot \int_0^\ell (u')^2\rho\,\mathrm{d}x + \gamma \cdot \int_0^\ell u^2\rho\,\mathrm{d}x \\ &\geq \min\{\alpha,\gamma\} \cdot \int_0^\ell ((u')^2 + u^2)\rho\,\mathrm{d}x = \min\{\alpha,\gamma\} \cdot \|u\|_{1,\rho}^2\end{aligned}$$

for all $u \in \mathrm{H}_0^1(\Omega)$. In conclusion, $\mathcal{A}_\rho$ is a scalar product on $\mathrm{H}_0^1(\Omega)$ whose induced norm - let us call it $|||\cdot|||$ - is equivalent to $\|\cdot\|_{1,\rho}$ and hence to $\|\cdot\|_{\mathrm{H}_0^1(\Omega)}$ by (a). In particular, $(\mathrm{H}_0^1(\Omega), \mathcal{A}_\rho)$ is a Hilbert space. Moreover, $F := \langle f, \cdot\rangle_\rho$ is a continuous linear functional on $\mathrm{H}_0^1(\Omega)$ with respect to $(\|\cdot\|_\rho)|_{\mathrm{H}_0^1(\Omega)}$, hence also with respect to $\|\cdot\|_{1,\rho} \geq (\|\cdot\|_\rho)|_{\mathrm{H}_0^1(\Omega)}$ and thus as well with respect to the equivalent norm $|||\cdot|||$. By the Riesz Representation Theorem we therefore find a unique $u \in \mathrm{H}_0^1(\Omega)$ with $\mathcal{A}_\rho(u,v) = F(v) = \langle f, v\rangle_\rho$ for all $v \in \mathrm{H}_0^1(\Omega)$.

It remains to be shown that $u \in \mathrm{H}_0^2(\Omega)$: By the considerations in (b), it then satisfies (ODE). $\mathcal{A}_\rho(u,v) = \langle f, v\rangle_\rho$ for all $v \in \mathrm{H}_0^1(\Omega)$ is equivalent to

$$\int_0^\ell (au'v'\rho + (cu - f)v\rho)\,\mathrm{d}x = 0.$$

Here, we assume that $a \in \mathrm{C}^1(\Omega)$. For $v \in \mathrm{C}_c^\infty(\Omega)$ this equations also holds for v replaced by $\tilde{v} := \frac{v}{a\rho} \in \mathrm{H}_0^1(\Omega)$. Therefore

$$0 = \int_0^\ell (au'\tilde{v}'\rho + (cu - f)\tilde{v}\rho)\,\mathrm{d}x = \int_0^\ell \left(u'v' - \frac{u'v(a\rho)'}{a\rho} + \frac{v}{a}(cu - f)\right)\mathrm{d}x$$

or, equivalently,

$$\int_0^\ell \left(\frac{u'(a\rho)'}{a\rho} - \frac{cu - f}{a}\right) v\,\mathrm{d}x = \int_0^\ell u'v'\,\mathrm{d}x = -\int_0^\ell uv''\,\mathrm{d}x.$$

Thus, the second (weak) derivative u'' of u exists and equals

$$u'' = -\left(\frac{u'(a\rho)'}{a\rho} - \frac{cu - f}{a}\right) \in \mathrm{L}^2(\Omega).$$

In particular, we conclude that $u \in \mathrm{H}_0^2(\Omega)$.