Exercise Sheet 3

Exercise 9 (Complete Orthonormal Systems) - written

Let $\mathcal{O} = \{e_j \mid j \in \mathbb{N}\}$ be an orthonormal system in $(X, \langle\cdot,\cdot\rangle)$. Show that the following are equivalent.

(i) $\mathcal{O}$ is complete, i.e. $\operatorname{span}\mathcal{O}$ is dense in X.

(ii) $\langle x, e_j\rangle = 0$ for all $j \in \mathbb{N}$ implies $x = 0$.

(iii) For all $x \in X$, $\sum_{j=1}^{\infty} |\langle x, e_j\rangle|^2 = \|x\|^2$.

(iv) For all $x \in X$, $\sum_{j=1}^{n} \langle x, e_j\rangle e_j \xrightarrow{n\to\infty} x$.

Remark: Although it is not stated here explicitly, $(X, \langle\cdot,\cdot\rangle)$ has to be a Hilbert space to ensure that the four conditions above are equivalent (see the lectures, where X in fact was required to be complete).

Proof: In the Theorem concerning the existence of $Q_{\mathcal{O}}$ (in the lectures, the ONS was called S instead of $\mathcal{O}$), it has already been shown that (iii) and (iv) are consequences of (i). By going through step 3 of the proof of this Theorem, one actually finds the proofs for the implications (i)⇒(ii)⇒(iv)⇒(iii), or rather we can at least extract the ideas from what was done there:

(i)⇒(ii): Let $y \in X$ satisfy $\langle y, e_j\rangle = 0$ for all $j \in \mathbb{N}$. Since $\operatorname{span}\mathcal{O}$ is dense in X, there is a sequence $(s_m)_{m\in\mathbb{N}}$ consisting of elements $s_m \in \operatorname{span}\mathcal{O}$, which converges to y. As y is orthogonal to every e_j, the same holds for all finite linear combinations of the e_j, i.e. y is actually orthogonal to every element of $\operatorname{span}\mathcal{O}$. In particular, y and s_m are always orthogonal, hence

$$\|y\|^2 = \langle y, y\rangle = \left\langle y, \lim_{m\to\infty} s_m \right\rangle = \lim_{m\to\infty} \langle y, s_m\rangle = \lim_{m\to\infty} 0 = 0.$$

This shows $y = 0$.

(ii)⇒(iv): Let $x \in X$ be arbitrary. Since X is complete (this is the only point, where the completeness of X is needed!), the series $\sum\limits_{j\in\mathbb{N}} \langle x, e_j\rangle e_j$ is converging to some $z = Q_{\mathcal{O}}(x) \in X$ (see the lectures, or rather more precisely, the first step in the proof mentioned above). Now, $y := x - z = x - Q_{\mathcal{O}}(x)$ is orthogonal to each e_k (for $k \in \mathbb{N}$):

$$\langle y, e_k\rangle = \left\langle x - \sum_{j\in\mathbb{N}} \langle x, e_j\rangle e_j, e_k \right\rangle = \langle x, e_k\rangle - \sum_{j\in\mathbb{N}} \langle x, e_j\rangle \cdot \langle e_j, e_k\rangle = \langle x, e_k\rangle - \langle x, e_k\rangle = 0.$$

By condition (ii), this implies $y = 0$, hence $x = z = Q_{\mathcal{O}}(x)$.

(iv)⇔(iii): Let $x \in X$ be fixed. If we put $s_n := \sum\limits_{j=1}^{n} \langle x, e_j\rangle e_j$ for every $n \in \mathbb{N}$, we obtain

$$\|x - s_n\|^2 = \left\| x - \sum_{j=1}^{n} \langle x, e_j\rangle e_j \right\|^2 = \|x\|^2 - \sum_{j=1}^{n} |\langle x, e_j\rangle|^2$$

(for, $x - s_n$ is orthogonal to s_n, hence $\|x\|^2 = \|x - s_n\|^2 + \|s_n\|^2$). (iv) is then equivalent to the fact that the left hand side converges to zero. This happens if and only if the right hand side converges to zero, which is obviously equivalent to Parseval's identity $\|x\|^2 = \sum_{j=1}^{\infty} |\langle x, e_j\rangle|^2$, which is (iii).

To finish the proof, it now remains to show (iv)$\Rightarrow$(i).

<u>(iv)$\Rightarrow$(i):</u> As every $x \in X$ is the limit of the sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n = \sum_{j=1}^{n} \langle x, e_j\rangle e_j \in \operatorname{span}\mathcal{O}$, this is immediate. ∎