Exercise Sheet 2

Exercise 7 (Hölder spaces) - oral

Let $G \subseteq \mathbb{R}^d$ be a bounded domain. For $\alpha \in (0,1]$, we define the Hölder spaces

$$C^\alpha(G) := \{f \in C^0(G) \mid f \text{ bounded}, \ h_\alpha(f) < \infty\},$$
$$h_\alpha(f) := \sup \left\{ \frac{|f(s) - f(t)|}{|s-t|^\alpha} \ \middle| \ s, t \in G, \ s \neq t \right\}$$

and the Hölder norms $\|f\|_\alpha := \|f\|_\infty + h_\alpha(f)$.

(a) Show that $(C^\alpha(G), \|\cdot\|_\alpha)$ is a real normed vector space.

(b) Now let $G := [0,1] \subseteq \mathbb{R}$. For $\alpha \in (0,1]$, construct a function $g_\alpha \in C^\alpha(G)$ such that $g_\alpha \notin C^\beta(G)$ for every $\beta > \alpha$. Moreover, construct a function $g_* \in C^0(G)$ such that $g_* \notin C^\alpha(G)$ for every $\alpha > 0$.

(c) Show that, for $0 < \alpha < \beta \leq 1$, the norm $\|\cdot\|_\beta$ is stronger than the norm $\|\cdot\|_\alpha$.

(d) Show that $(C^\alpha(G), \|\cdot\|_\alpha)$ is a Banach space.

Note: Often, the Hölder space as defined above is denoted by $C^{0,\alpha}$ as to be able to distinguish the Hölder space for $\alpha = 1$ from the space $C^1(G)$ of continuously differentiable functions.

Solution

(a): We show that $C^\alpha(G)$ is a linear subspace of the space of bounded $C^0(G)$-functions. Let $\lambda \in \mathbb{R}$ and let $f, g \in C^\alpha(G)$. Then for all $s, t \in G$ with $s \neq t$ it holds

$$\frac{|(f+g)(s) - (f+g)(t)|}{|s-t|^\alpha} \leq \frac{|f(s) - f(t)|}{|s-t|^\alpha} + \frac{|g(s) - g(t)|}{|s-t|^\alpha} \leq h_\alpha(f) + h_\alpha(g) < \infty,$$
$$\frac{|(\lambda f)(s) - (\lambda f)(t)|}{|s-t|^\alpha} = |\lambda| \cdot \frac{|f(s) - f(t)|}{|s-t|^\alpha}$$

we obtain $f + g \in C^\alpha(G)$ with $h_\alpha(f+g) \leq h_\alpha(f) + h_\alpha(g)$ and by taking the supremum of both sides in the second line it follows $h_\alpha(\lambda f) = |\lambda| \cdot h_\alpha(f) < \infty$, hence $\lambda f \in C^\alpha(G)$. As $0 \in C^\alpha(G)$, the Hölder space is indeed a linear subspace of $C^0(G)$ and h_α is a semi-norm on it by the previous considerations. Hence, $\|\cdot\|_\alpha$ is a norm on $C^\alpha(G)$ (as it is the sum of the norm $\|\cdot\|_\infty$ and the seminorm h_α).

(b): We claim that $g_\alpha(s) := s^\alpha$ has the desired property. As for this, we first prove that $|a+b|^\alpha \leq |a|^\alpha + |b|^\alpha$ for all $a, b \in \mathbb{C}$. Let $c := |a|$ and $d := |b|$. Because of $|a+b| \leq |a| + |b| = c + d$ and thus $|a+b|^\alpha \leq (c+d)^\alpha$ it suffices to prove $(c+d)^\alpha \leq c^\alpha + d^\alpha$ for $c, d \geq 0$. If $c = d = 0$, this is trivial. If $(c,d) \neq (0,0)$, note that $c \leq (c^\alpha + d^\alpha)^{\frac{1}{\alpha}}$ implies $c^{1-\alpha} \leq (c^\alpha + d^\alpha)^{\frac{1-\alpha}{\alpha}}$, as $1 - \alpha \geq 0$. Analogously this holds for d. Hence

$$c \cdot (c^\alpha + d^\alpha) \leq c^\alpha \cdot (c^\alpha + d^\alpha)^{\frac{1}{\alpha}},$$
$$d \cdot (c^\alpha + d^\alpha) \leq d^\alpha \cdot (c^\alpha + d^\alpha)^{\frac{1}{\alpha}}$$

and adding these two inequalities yields

$$(c+d)\cdot(c^\alpha+d^\alpha) \le (c^\alpha+d^\alpha)\cdot(c^\alpha+d^\alpha)^{\frac{1}{\alpha}},$$

hence $c+d \le (c^\alpha+d^\alpha)^{\frac{1}{\alpha}}$ after division by $c^\alpha+d^\alpha>0$, which proves the assertion.

Now, for all $s,t\in[0,1]$ we obtain $s^\alpha = |(s-t)+t|^\alpha \le |s-t|^\alpha + t^\alpha$ and analogously for s and t interchanged, which implies $|g_\alpha(s)-g_\alpha(t)| = |s^\alpha - t^\alpha| \le |s-t|^\alpha$. Hence $h_\alpha(g_\alpha)\le 1$ (in fact, it equals 1 because of $\frac{|g_\alpha(1)-g_\alpha(0)|}{|1-0|^\alpha}=1$). But, for $1\ge\beta>\alpha$, we have

$$\frac{|g_\alpha(s)-g_\alpha(0)|}{|s-0|^\beta} = s^{\alpha-\beta} \xrightarrow{s\searrow 0} \infty,$$

i.e. $g_\alpha \notin C^\beta([0,1])$.

As for the second counterexample, consider

$$g_*(s) := \begin{cases} -\frac{1}{\ln s} & \text{for } s\in(0,1] \\ 0 & \text{for } s=0 \end{cases}.$$

Then g_* is obviously continuous, but $\frac{|g_*(s)-g_*(0)|}{|s-0|^\alpha} = -\frac{1}{s^\alpha \ln s} \xrightarrow{s\searrow 0} \infty$ for every $\alpha>0$.

(c): Let $1\ge\beta>\alpha$. We have to show $C^\beta(G)\subseteq C^\alpha(G)$ and that $\|\cdot\|_\beta$ is stronger than the restriction $\|\cdot\|_\alpha|_{C^\beta(G)}$ of the norm $\|\cdot\|_\alpha$ to the Hölder space $C^\beta(G)$. As G is bounded, $\operatorname{diam}(G) := \sup\{|s-t| \mid s,t\in G\}$ is a (finite) nonnegative real number. Let $K:=\max\{1,\operatorname{diam}(G)^{\beta-\alpha}\}$, let $f\in C^\beta(G)$ and let $s,t\in G$. If $|s-t|\le 1$, then it holds

$$|f(s)-f(t)| \le h_\beta\cdot|s-t|^\beta \le h_\beta(f)\cdot|s-t|^\alpha \le h_\beta(f)K\cdot|s-t|^\alpha.$$

If, on the other hand $|s-t|>1$, we obtain

$$\begin{aligned}|f(s)-f(t)| &\le h_\beta\cdot|s-t|^\beta = h_\beta(f)\cdot|s-t|^{\beta-\alpha}\cdot|s-t|^\alpha \\ &\le h_\beta(f)\cdot\operatorname{diam}(G)^{\beta-\alpha}\cdot|s-t|^\alpha \le h_\beta(f)K\cdot|s-t|^\alpha.\end{aligned}$$

In conclusion, $f\in C^\alpha(G)$ with $h_\alpha(f)\le K\cdot h_\beta(f)$ and thus

$$\|f\|_\alpha = 1\cdot\|f\|_\infty + h_\alpha(f) \le K\cdot\|f\|_\infty + K\cdot h_\beta(f) = K\cdot\|f\|_\beta,$$

which shows that $\|\cdot\|_\beta$ is stronger than $\|\cdot\|_\alpha|_{C^\beta(G)}$.

(d): Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $C^\alpha(G)$. Then this is in particular a Cauchy sequence in the space of bounded continuous functions on G with respect to the norm $\|\cdot\|_\infty$. As this is a Banach space, this sequence converges uniformly to some bounded $f\in C^0(G)$. Let now $\varepsilon>0$. Then there exists some $n_0(\varepsilon)\in\mathbb{N}$ such that for all $m,n\ge n_0(\varepsilon)$ we have $\|f_m-f_n\|_\alpha<\varepsilon$, hence in particular $h_\alpha(f_m-f_n)<\varepsilon$. For all $s,t\in G$ we obtain

$$|(f_m(s)-f_n(s))-(f_m(t)-f_n(t))| \le \varepsilon\cdot|s-t|^\alpha.$$

As $f_m\to f$ uniformly, this convergence is also pointwise, hence

$$|(f(s)-f_n(s))-(f(t)-f_n(t))| \le \varepsilon\cdot|s-t|^\alpha$$

for all $s, t \in G$ and all $n \geq n_0(\varepsilon)$. This shows $h_\alpha(f - f_n) \leq \varepsilon$ for all $n \geq n_0(\varepsilon)$. In particular, we get $f - f_{n_0(1)} \in C^\alpha(G)$ and hence $f = (f - f_{n_0(1)}) + f_{n_0(1)} \in C^\alpha(G)$. In addition, we have proved that for every $\varepsilon > 0$ there is an index $n_0(\varepsilon) \in \mathbb{N}$ such that $h_\alpha(f - f_n) \leq \varepsilon$ for all $n \geq n_0(\varepsilon)$, i.e. $h_\alpha(f - f_n) \to 0$ for $n \to \infty$. As $f_n \to f$ uniformly, this implies

$$\|f_n - f\|_\alpha = \|f_n - f\|_\infty + h_\alpha(f_n - f) \xrightarrow{n\to\infty} 0,$$

which shows that $(f_n)_{n\in\mathbb{N}}$ converges to f in the normed space $(C^\alpha(G), \|\cdot\|_\alpha)$.

Exercise 8 (Banach's fixed point theorem) - oral

Let $(X, \|\cdot\|)$ be a Banach space. Prove the following:

(a) Let $M \subseteq X$ be closed and $\Phi : M \to M$ be a contraction with contraction constant $\kappa < 1$. Then Φ has a unique fixed point x^*, and for every $x_0 \in M$ we have for the iterations $x_{n+1} = \Phi(x_n)$ the a priori estimate

$$\|x_n - x^*\| \leq \frac{\kappa^n}{1-\kappa} \cdot \|x_0 - \Phi(x_0)\|.$$

(b) Let $\Psi : X \to X$ be a map such that there exist $R > 0, \kappa \in (0, 1)$ and $y \in X$ such that for all $x_1, x_2 \in \overline{B_R(y)} = \{x \in X \mid \|x - y\| \leq R\}$

$$\|\Psi_(x_1) - \Psi(x_2)\| \leq \kappa\|x_1 - x_2\| \qquad \text{and} \quad \|y - \Psi(y)\| \leq (1-\kappa)R.$$

Then Ψ has a fixed point in $\overline{B_R(y)}$.

(c) Choose a norm in $\mathbb{R}^2$ of the form $\|(x_1, x_2)\| = |x_1| + a|x_2|$ such that for the map

$$\Psi(x) := \left(\frac{1}{2}x_1 - \frac{1}{16} + 4x_2^2, \frac{1}{16}\sin x_1 + \frac{1}{2}x_2\right)$$

a fixed point near 0 can be found according to (b).

Solution

(a): If $\Phi(x_1) = x_1$ and $\Phi(x_2) = x_2$ for $x_1, x_2 \in M$, then

$$\|x_1 - x_2\| = \|\Phi(x_1) - \Phi(x_2)\| \leq \kappa \cdot \|x_1 - x_2\|,$$

i.e. $(1-\kappa) \cdot \|x_1 - x_2\| \leq 0$, hence $\|x_1 - x_2\| \leq 0$ as $1 - \kappa > 0$. This shows $x_1 = x_2$.

Now let $x_0 \in M$ be arbitrary. We claim that the iterating sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. First of all, for all $k \in \mathbb{N}_0$ it holds $\|x_{k+1} - x_k\| \leq \kappa^k \cdot \|x_1 - x_0\|$. For all $m < n$ we then obtain

$$\begin{aligned}\|x_n - x_m\| &\leq \sum_{k=m}^{n-1} \|x_{k+1} - x_k\| \leq \|x_1 - x_0\| \cdot \sum_{k=m}^{n-1} \kappa^k \leq \|x_1 - x_0\| \cdot \frac{(1-\kappa^n) - (1-\kappa^m)}{1-\kappa} \\ &= \|x_1 - x_0\| \cdot \frac{\kappa^m - \kappa^n}{1-\kappa}.\end{aligned}$$

As $\left(\frac{\kappa^n}{1-\kappa}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence, this shows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence as well. Hence this sequence has a limit x^* in the Banach space X. As M is closed and $x_n \in M$ for all $n \in \mathbb{N}$, it follows $x^* \in M$. Moreover,

$$\Phi(x^*) = \Phi\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} \Phi(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$$

due to the continuity of Φ. If we let $n \to \infty$ in the estimate above, we obtain $\|x^* - x_m\| \leq \frac{\kappa^m}{1-\kappa} \cdot \|x_1 - x_0\|$, as desired.

(b): By (a) it suffices to show $\Psi\left(\overline{B_R(y)}\right) \subseteq \overline{B_R(y)}$. So let $x \in \overline{B_R(y)}$, i.e. $\|x-y\| \leq R$. Then $\|\Phi(x) - \Phi(y)\| \leq \kappa \cdot \|x-y\| \leq \kappa \cdot R$, hence

$$\|\Phi(x) - y\| \leq \|\Phi(x) - \Phi(y)\| + \|\Phi(y) - y\| \leq \kappa \cdot R + (1-\kappa)\cdot R = R,$$

i.e. $\Phi(x) \in \overline{B_R(y)}$.

(c): As to see which constant $a > 0$ we can choose to get a contraction constant κ as in (b), we just estimate $\|\Phi(x) - \Phi(y)\|$ as good and as far as possible for all $x, y \in \overline{B_R(0)}$ (i.e. $|x_1| + a|x_2| \leq R$ and analogously for y) and try to relate it to $\|x-y\|$:

$$\begin{aligned}
\|\Phi(x) - \Phi(y)\| &= \left|\frac{1}{2}(x_1 - y_1) + 4\left(x_2^4 - y_2^4\right)\right| + a \cdot \left|\frac{1}{16}(\sin x_1 - \sin y_1) + \frac{1}{2}(x_2 - y_2)\right| \\
&\leq \frac{1}{2}|x_1 - y_1| + 4\left|x_2^4 - y_2^4\right| + \frac{a}{16}|\sin x_1 - \sin y_1| + \frac{a}{2}|x_2 - y_2| \\
&\leq \left(\frac{1}{2} + \frac{a}{16}\right)\cdot|x_1 - y_1| + \left(4|x_2 + y_2|\cdot\left|x_2^2 + y_2^2\right| + \frac{a}{2}\right)\cdot|x_2 - y_2| \\
&\leq \left(\frac{1}{2} + \frac{a}{16}\right)\cdot|x_1 - y_1| + \left(4(|x_2| + |y_2|)\cdot\left(x_2^2 + y_2^2\right) + \frac{a}{2}\right)\cdot|x_2 - y_2| \\
&\leq \left(\frac{1}{2} + \frac{a}{16}\right)\cdot|x_1 - y_1| + \left(4\cdot\frac{2R}{a}\cdot\frac{2R^2}{a^2} + \frac{a}{2}\right)\cdot|x_2 - y_2| \\
&\leq \left(\frac{1}{2} + \frac{a}{16}\right)\cdot|x_1 - y_1| + \left(\frac{16R^3}{a^4} + \frac{1}{2}\right)\cdot a|x_2 - y_2| \\
&\leq \kappa\cdot|x_1 - y_1| + \kappa\cdot a|x_2 - y_2| = \kappa\cdot\|x-y\|
\end{aligned}$$

for $\kappa := \max\left\{\frac{1}{2} + \frac{a}{16}, \frac{16R^3}{a^4} + \frac{1}{2}\right\}$. We have to make sure that $\kappa < 1$ and $\|\Phi(0)\| = \frac{1}{16} \leq (1-\kappa)\cdot R$. Now $\kappa < 1$ is equivalent to $a < 8$ and $32R^3 < a^4$, whereas the second condition means

$$1 \leq 16\cdot(1-\kappa)\cdot R = R\cdot\min\left\{8 - a, 8 - \frac{16^2R^3}{a^4}\right\},$$

i.e. $1 \leq R(8-a)$ and $1 \leq 8R - \frac{16^2R^4}{a^4}$. Note that these two inequalities imply the first two as

$$8 - a \geq \frac{1}{R} > 0 \qquad \text{and} \qquad 1 - \frac{32R^3}{a^4} \geq \frac{1}{8R} > 0.$$

Actually, the second two inequalities are equivalent to

$$\frac{4R}{\sqrt[4]{8R-1}} \leq a \leq 8 - \frac{1}{R} = \frac{8R-1}{R}.$$

If we put $R = \frac{1}{4}$, this means $1 \leq a \leq 4$, i.e. we can for example choose $R = \frac{1}{4}$ and $a \in [1,4]$ arbitrarily (or any other pair (R,a) of positive real numbers satisfying these inequalities).