Exercise Sheet 7

Exercise 28 (Schwartz Functions) - oral

Let $\mathrm{S}(\mathbb{R}^d)$ denote the set of „rapidly decreasing" or SCHWARTZ functions, defined as follows.

$$\mathrm{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d;\mathbb{R}) \mid \forall\, k,l \in \mathbb{N}_0 \;:\; p_{k,l}(f) < \infty\}$$

where $p_{k,l}(f) := \sup\{(1+|x|)^k \cdot |\mathrm{D}^\alpha f(x)| \mid x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}_0^d,\ |\alpha|_1 = l\}$.

(a) Show that $\mathrm{S}(\mathbb{R}^d)$ equipped with the system $SN = \{p_{k,l} \mid k,l \in \mathbb{N}_0\}$ of semi-norms is a complete topological vector space.

(b) Show that $C_c^\infty(\mathbb{R}^d;\mathbb{R})$ is dense in $\mathrm{S}(\mathbb{R}^d)$.

(c) The SCHRÖDINGER operator in $\mathbb{R}^d$ is defined as

$$(H_{\text{Schröd}} f)(x) = -(\Delta f)(x) + |x|^2 f(x) \qquad \text{with} \quad \Delta f = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} f.$$

Is $H_{\text{Schröd}}$, considered as a map $\mathrm{S}(\mathbb{R}^d) \to \mathrm{S}(\mathbb{R}^d)$, continuous?

Remark: We have already encountered $H_{\text{Schröd}}$ in exercise 20.

Note: If $x \in \mathbb{R}^d$, then $|x| := \|x\|_2$ stands for the usual Euclidean norm of x.

Solution

(a): We first show that $\mathrm{S}(\mathbb{R}^d)$ is a linear subspace of the real vector space $C^\infty(\mathbb{R}^d;\mathbb{R})$ and that the $p_{k,l}$ are semi-norms on this space. As for this, it obviously suffices to show that for all $(k,l) \in \mathbb{N}_0^2$, the set $U_{k,l} := \{f \in C^\infty(\mathbb{R}^d;\mathbb{R}) \mid p_{k,l}(f) < \infty\}$ is a linear subspace of $C^\infty(\mathbb{R}^d;\mathbb{R})$ and that $p_{k,l}$ is a semi-norm on $U_{k,l}$, because then $\mathrm{S}(\mathbb{R}^d) = \bigcap_{(k,l)\in\mathbb{N}_0^2} U_{k,l}$ is also a linear subspace of $C^\infty(\mathbb{R}^d;\mathbb{R})$ and every $p_{k,l}$ is then of course a semi-norm on $\mathrm{S}(\mathbb{R}^d)$.

So let $k,l \in \mathbb{N}_0$ be fixed and let $\beta \in \mathbb{R}$ and $f,g \in U_{k,l}$. Then for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha|_1 = l$ it holds

$$(1+|x|)^k \cdot |\mathrm{D}^\alpha(f+g)(x)| \le (1+|x|)^k \cdot |\mathrm{D}^\alpha f(x)| + (1+|x|)^k \cdot |\mathrm{D}^\alpha g(x)| \le p_{k,l}(f) + p_{k,l}(g)$$

for all $x \in \mathbb{R}^d$. Hence $p_{k,l}(f+g) \le p_{k,l}(f) + p_{k,l}(g) < \infty$ and therefore $f+g \in U_{k,l}$. Moreover, this shows that $p_{k,l}$ satisfies the triangle inequality on $U_{k,l}$. Now, for all $x \in \mathbb{R}^d$ we also have

$$(1+|x|)^k \cdot |\mathrm{D}^\alpha(\beta f)(x)| = |\beta| \cdot (1+|x|)^k \cdot |\mathrm{D}^\alpha f(x)|,$$

which implies that the suprema of both sides taken over all $x \in \mathbb{R}^d$ and all $\alpha \in \mathbb{N}_0^d$ with $|\alpha|_1 = l$ coincide, i.e. $p_{k,l}(\beta f) = |\beta| \cdot p_{k,l}(f) < \infty$. Hence $\beta f \in U_{k,l}$ and every $p_{k,l}$ has the scaling property. As $U_{k,l}$ clearly contains the zero function, it is in fact a linear subspace of $C^\infty(\mathbb{R}^d;\mathbb{R})$ and $p_{k,l}$ is then a semi-norm on $U_{k,l}$.

In conclusion, $\mathrm{S}(\mathbb{R}^d)$ is a linear subspace of $C^\infty(\mathbb{R}^d;\mathbb{R})$ and the system SN of semi-norms makes it into a topological vector space. It remains to show completeness. So let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{S}(\mathbb{R}^d)$, i.e. for all $k,l \in \mathbb{N}_0$ and all $\varepsilon > 0$ there exists an index $n_0 = n_0(\varepsilon,k,l) \in \mathbb{N}$ such that for all $m,n \geq n_0$ we have $p_{k,l}(f_m - f_n) < \varepsilon$. In particular, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to every semi-norm $p_{0,l}$ for $l \in \mathbb{N}_0$, where

$$p_{0,l}(h) = \max\{\|\mathrm{D}^\alpha h\|_\infty \mid \alpha \in \mathbb{N}_0^d,\ |\alpha|_1 = l\}.$$

Hence, it is also a Cauchy sequence with respect to the norm $\|\cdot\|_{V_l} := p_{0,0} + \ldots + p_{0,l}$ on the vector space

$$V_l := \{f \in C^l(\mathbb{R}^d;\mathbb{R}) \mid p_{0,r}(f) < \infty \quad \forall\, 0 \leq r \leq l\}$$

for every $l \in \mathbb{N}_0$. Now, this space is a Banach space when equipped with this norm (hopefully, you have seen this in basic analysis courses - if not, see the last two pages), so $(f_n)_{n\in\mathbb{N}}$ converges to some $g_l \in V_l$ with respect to $\|\cdot\|_{V_l}$. Obviously, if $l \leq l'$, then the convergence $f_n \xrightarrow{n\to\infty} g_{l'}$ in $V_{l'}$ implies the convergence $f_n \xrightarrow{n\to\infty} g_{l'}$ in V_l (as $\|h\|_{V_l} \leq \|h\|_{V_{l'}}$ for all $h \in V_{l'}$), hence $g_{l'} = g_l$. This shows that all limit functions $g_l \in V_l$ are equal, hence $(f_n)_{n\in\mathbb{N}}$ converges to $f := g_0$ with respect to every semi-norm $p_{0,l}$.

It now remains to show $f \in \mathrm{S}(\mathbb{R}^d)$ and $f_n \to f$ in $\mathrm{S}(\mathbb{R}^d)$. Let $k,l \in \mathbb{N}_0$ be fixed and let $\varepsilon > 0$. Then there exists $n_0(\varepsilon,k,l) \in \mathbb{N}$ such that $p_{k,l}(f_m - f_n) < \varepsilon$ for all $m,n \geq n_0(\varepsilon,k,l)$. We obtain

$$(1+|x|)^k \cdot |\mathrm{D}^\alpha f_m(x) - \mathrm{D}^\alpha f_n(x)| < \varepsilon$$

for all $x \in \mathbb{R}^d$ and all $\alpha \in \mathbb{N}_0^d$ with $|\alpha|_1 = l$. As $\mathrm{D}^\alpha f_m \xrightarrow{m\to\infty} \mathrm{D}^\alpha f$ uniformly (this is just the convergence of f_m to f with respect to $p_{0,l}$) and thus pointwise, letting m tend to ∞ yields

$$(1+|x|)^k \cdot |\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha f_n(x)| \leq \varepsilon \qquad \forall\, x \in \mathbb{R}^d \quad \forall\, \alpha \in \mathbb{N}_0^d \text{ with } |\alpha|_1 = l,$$

i.e. $p_{k,l}(f - f_n) \leq \varepsilon$ for all $n \geq n_0(\varepsilon,k,l)$. In particular for $\varepsilon = 1$, we get $f - f_{n_0(1,k,l)} \in U_{k,l}$. Because of $f_{n_0(1,k,l)} \in \mathrm{S}(\mathbb{R}^d) \subseteq U_{k,l}$ and the fact that $U_{k,l}$ is a linear subspace of $C^\infty(\mathbb{R}^d;\mathbb{R})$, we obtain

$$f = \big(f - f_{n_0(1,k,l)}\big) + f_{n_0(1,k,l)} \in U_{k,l}.$$

By letting $\varepsilon > 0$ be arbitrary once again, the estimate $p_{k,l}(f - f_n) \leq \varepsilon$ for all $n \geq n_0(\varepsilon,k,l)$ then implies $f_n \xrightarrow{n\to\infty} f$ in $U_{k,l}$ with respect to the semi-norm $p_{k,l}$. As this is true for every pair $(k,l) \in \mathbb{N}_0^2$, we obtain $f \in \mathrm{S}(\mathbb{R}^d)$ and $p_{k,l}(f - f_n) \xrightarrow{n\to\infty} 0$ for all $(k,l) \in \mathbb{N}_0^2$. In conclusion, the Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{S}(\mathbb{R}^d)$ has the limit $f \in \mathrm{S}(\mathbb{R}^d)$.

(b): Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := \mathrm{e}^{-\frac{1}{x}}$ for $x > 0$ and $g(x) := 0$ for $x \leq 0$. Then $g \in C^\infty(\mathbb{R};\mathbb{R})$ (as is well-known). Let

$$h(x) := \frac{1}{\int_0^1 g(t)g(1-t)\,\mathrm{d}t} \cdot \int_0^x g(t)g(1-t)\,\mathrm{d}t \qquad \forall\, x \in \mathbb{R}.$$

Then $h \in C^\infty(\mathbb{R};\mathbb{R})$ is increasing and $h(x) = 0$ for $x \leq 0$ and $h(x) = 1$ for $x \geq 1$. Now we put $G(x) := h(x+2)h(2-x)$, so that $G \in C_c^\infty(\mathbb{R};\mathbb{R})$ fulfills $G(x) = 0$ for $|x| \geq 2$ and $G(x) = 1$ for $|x| \leq 1$. Now let $H : \mathbb{R}^d \to \mathbb{R}$ be defined by $H(x) := G(|x|^2) = G(\|x\|_2^2)$.

Then we have $H \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$ with $H|_{\overline{B_1(0)}} \equiv 1$ and $H|_{\mathbb{R}^d \setminus B_{\sqrt{2}}(0)} \equiv 0$. (Up to here, this is just the usual construction of smooth „cut-off"-functions.)

For $m \in \mathbb{N}$, we put $H_m(x) := H\left(\frac{x}{m}\right)$ for $x \in \mathbb{R}^d$. Then $H_m \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$ with $H_m|_{\overline{B_m(0)}} \equiv 1$ and $H_m|_{\mathbb{R}^d \setminus B_{\sqrt{2}m}(0)} \equiv 0$. We now want to show that for all $f \in \mathrm{S}(\mathbb{R}^d)$ we have $H_m f \xrightarrow{m\to\infty} f$ in $\mathrm{S}(\mathbb{R}^d)$. First note that $H_m f \in C_c^\infty(\mathbb{R}^d;\mathbb{R}) \subseteq \mathrm{S}(\mathbb{R}^d)$ for all $m \in \mathbb{N}$. Now let $k, l \in \mathbb{N}_0$ be fixed and let $\varepsilon > 0$. We will now use the following steps:

(1) Every partial derivative of H has also compact support, so

$$C := \max\left\{ \left\|\mathrm{D}^\beta H\right\|_\infty \;\middle|\; \beta \in \mathbb{N}_0^d,\ |\beta|_1 \le l \right\}$$

is a finite positive constant. This implies

$$\left|\mathrm{D}^\beta H_m(x)\right| = \frac{1}{m^{|\beta|_1}} \cdot \left|\mathrm{D}^\beta H\left(\frac{x}{m}\right)\right| \le 1 \cdot C = C,$$

i.e. $\left\|\mathrm{D}^\beta H_m\right\|_\infty \le C$ for all $m \in \mathbb{N}$ and all $\beta \in \mathbb{N}_0^d$ with $|\beta|_1 \le l$.

(2) Let

$$K := \max\left\{ \sum_{\substack{(\beta,\gamma)\in(\mathbb{N}_0^d)^2\\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \;\middle|\; \alpha \in \mathbb{N}_0^d,\ |\alpha|_1 = l \right\},$$

where $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!\cdot\gamma!}$ with the usual multi-index-notation ($\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_d!$ for $\alpha = (\alpha_1, \ldots, \alpha_d)$).

(3) Let $D := \max\{p_{k+1,r}(f) \mid 0 \le r \le l\}$. For every $\gamma \in \mathbb{N}_0^d$ with $r = |\gamma|_1 \le l$, we then have

$$(1+|x|)^{k+1} \cdot |\mathrm{D}^\gamma f(x)| \le p_{k+1,r}(f) \le D, \quad \text{hence} \quad (1+|x|)^k \cdot |\mathrm{D}^\gamma f(x)| \le \frac{D}{1+|x|}.$$

for all $x \in \mathbb{R}^d$.

(4) If $u, v \in C^l(\mathbb{R}^d;\mathbb{R})$, then for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha|_1 = l$ we have

$$D^\alpha(u \cdot v)(x) = \sum_{\substack{(\beta,\gamma)\in(\mathbb{N}_0^d)^2\\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \cdot \mathrm{D}^\beta u(x) \cdot \mathrm{D}^\gamma v(x),$$

which is proved by induction over l using Leibniz' rule for differentiating products.

By the construction of H_m, we have $(H_m f)|_{B_m(0)} = f|_{B_m(0)}$, hence also $\mathrm{D}^\alpha(H_m f - f)|_{B_m(0)} = 0$. Thus we can restrict our attention to $x \in \mathbb{R}^d \setminus B_m(0)$, i.e. $|x| \ge m$ when trying to estimate terms including the difference $H_m f - f$ (as sort of factor). For all those

x and for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha|_1 = l$, we then obtain the estimate

$$\begin{aligned}
&(1+|x|)^k \cdot |\mathrm{D}^\alpha (H_m f - f)(x)| = (1+|x|)^k \cdot |\mathrm{D}^\alpha ((H_m - 1)f)(x)| \\
&\overset{(4)}{=} (1+|x|)^k \cdot \left| \sum_{\substack{(\beta,\gamma)\in(\mathbb{N}_0^d)^2 \\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \cdot \mathrm{D}^\beta (H_m - 1)(x) \cdot \mathrm{D}^\gamma f(x) \right| \\
&\leq \sum_{\substack{(\beta,\gamma)\in(\mathbb{N}_0^d)^2 \\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \cdot \left|\mathrm{D}^\beta H_m(x) - \mathrm{D}^\beta 1\right| \cdot (1+|x|)^k \cdot |\mathrm{D}^\gamma f(x)| \\
&\overset{(3)}{\leq} \sum_{\substack{(\beta,\gamma)\in(\mathbb{N}_0^d)^2 \\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \cdot \left(\left|\mathrm{D}^\beta H_m(x)\right| + \mathrm{D}^\beta 1\right) \cdot \frac{D}{1+|x|} \\
&\overset{(1)}{\leq} \sum_{\substack{(\beta,\gamma)\in(\mathbb{N}_0^d)^2 \\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \cdot (C+1) \cdot \frac{D}{1+|x|} = (C+1) \cdot \frac{D}{1+|x|} \cdot \sum_{\substack{(\beta,\gamma)\in(\mathbb{N}_0^d)^2 \\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \\
&\leq \frac{(C+1)\cdot K \cdot D}{|x|+1} \leq \frac{(C+1)\cdot K \cdot D}{m+1}.
\end{aligned}$$

Since C, K and D only depend on f, k and l, the supremum of the left hand side taken over all $x \in \mathbb{R}^d$ and all $\alpha \in \mathbb{N}_0^d$ with $|\alpha|_1 = l$ can therefore be estimated as follows:

$$p_{k,l}(H_m f - f) \leq \frac{(C+1)\cdot K \cdot D}{m+1}.$$

But this already shows $H_m f \xrightarrow{m\to\infty} f$ with respect to $p_{k,l}$. As $(k,l) \in \mathbb{N}_0^2$ was chosen arbitrarily, $H_m f$ actually converges to f with respect to all semi-norms $p_{k,l}$, i.e. $H_m f \to f$ in $\mathrm{S}(\mathbb{R}^d)$ for $m \to \infty$. This shows the assertion, because $H_m f \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$ for all $m \in \mathbb{N}$.

<u>**(c):**</u> Let $k, l \in \mathbb{N}_0$ be fixed. Then for all $\alpha \in \mathbb{N}_0^d$ and all $x \in \mathbb{R}^d$ we have

$$\begin{aligned}
(1+|x|)^k \cdot |\mathrm{D}^\alpha (H_{\text{Schröd}} f)(x)| &= (1+|x|)^k \cdot \left| |x|^2 f(x) - \sum_{j=1}^d \mathrm{D}^{\alpha+2e_j} f(x) \right| \\
&\leq (1+|x|)^k \cdot |x|^2 |f(x)| + \sum_{j=1}^d (1+|x|)^k \cdot |\mathrm{D}^{\alpha+2e_j} f(x)| \\
&\leq (1+|x|)^k \cdot (1+|x|)^2 \cdot |f(x)| + \sum_{j=1}^d p_{k,l+2}(f) \\
&\leq p_{k+2,0}(f) + d \cdot p_{k,l+2}(f) \leq d \cdot (p_{k+2,0}(f) + p_{k,l+2}(f)).
\end{aligned}$$

This shows $p_{k,l}(H_{\text{Schröd}} f) \leq d \cdot (p_{k+2,0}(f) + p_{k,l+2}(f))$, so that $H_{\text{Schröd}}$ is continuous by exercise 27 (note that $p_{0,0} = \|\cdot\|_\infty$ is a norm on $\mathrm{S}(\mathbb{R}^d)$, hence SN is a system of semi-norms which is Hausdorff).

In the following, we show that V_l is a Banach space when equipped with the norm $\|\cdot\|_{V_l}$ defined above. We start with the case $l = 1$:

Lemma: Let $d \in \mathbb{N}$ and let V_1 be the vector space of all $f \in C^1(\mathbb{R}^d;\mathbb{R})$ such that f and all its first partial derivatives are bounded functions. Then $\|f\|_{V_1} := \|f\|_\infty + \max\{\|\partial_i f\|_\infty \mid 1 \le i \le d\}$ defines a norm on V_1, which makes V_1 into a Banach space.

Proof: Of course, V_1 is a linear subspace of $C^1(\mathbb{R}^d;\mathbb{R})$ and $\|\cdot\|_{V_1}$ is surely a norm on V_1 as $\|\cdot\|_\infty$ is a norm on V_1 and $\|\cdot\|_\infty \circ \partial_i$ are semi-norms on V_1. Now let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in V_1. Then $(f_n)_{n\in\mathbb{N}}$ and all $(\partial_i f_n)_{n\in\mathbb{N}}$ are Cauchy sequences with respect to $\|\cdot\|_\infty$. As the space of all real-valued continuous bounded functions on $\mathbb{R}^d$ is a Banach space when equipped with $\|\cdot\|_\infty$, there exist continuous bounded functions $f, g_1, \ldots, g_d : \mathbb{R}^d \to \mathbb{R}$ such that $f_n \to f$ and $\partial_i f_n \to g_i$ for $n \to \infty$ uniformly (for all $1 \le i \le d$). It remains to show $\partial_i f = g_i$, i.e. $\mathrm{D}f = (g_1, \ldots, g_n)$.

To this end, let $\mathbb{R}^d$ be equipped with the $\|\cdot\|_1$-norm, so that the operator norm $\|\cdot\|_{(\mathbb{R}^d)^*}$ on the dual space is the maximum norm $\|\cdot\|_\infty$ on $\mathbb{R}^d$. Let $g := (g_1, \ldots, g_d) : \mathbb{R}^d \to (\mathbb{R}^d)^*$ and let $x \in \mathbb{R}^d$ be an arbitrary, but fixed point. For $\varepsilon > 0$ we can find some $\delta > 0$ such that for all $y \in \mathbb{R}^d$ with $\|y - x\|_1 < \delta$ we have $\|g(y) - g(x)\|_\infty < \varepsilon$ (because g is continuous). If now $y \in \mathbb{R}^d$ satisfies $\|y - x\|_1 < \delta$, the Mean Value Theorem implies that we can find some $t_n \in [0,1]$ such that the point $\xi_n = x + t_n(y - x)$ satisfies

$$|f_n(y) - f_n(x) - \mathrm{D}f_n(x)\cdot(y-x)| = |(\mathrm{D}f_n(\xi_n) - \mathrm{D}f_n(x))\cdot(y-x)|$$

$$= |(\mathrm{D}f_n(\xi_n) - g(\xi_n) + g(\xi_n) - g(x) + g(x) - \mathrm{D}f_n(x))\cdot(y-x)|$$

$$\le (\|\mathrm{D}f_n(\xi_n) - g(\xi_n)\|_\infty + \|g(\xi_n) - g(x)\|_\infty + \|g(x) - \mathrm{D}f_n(x)\|_\infty)\cdot\|y-x\|_1$$
$$\le \left(2\cdot\sup_{\xi\in\mathbb{R}^d}\|\mathrm{D}f_n(\xi) - g(\xi)\|_\infty + \|g(\xi_n) - g(x)\|_\infty\right)\cdot\|y-x\|_1.$$

(Note that the $\|\cdot\|_\infty$-norm occurring here is the maximum norm on $\mathbb{R}^d$.) This implies

$$\frac{|f_n(y) - f_n(x) - \mathrm{D}f_n(x)\cdot(y-x)|}{\|y-x\|} \le 2\cdot\max\{\|\partial_i f_n - g_i\|_\infty \mid 1 \le i \le d\} + \|g(\xi_n) - g(x)\|_\infty$$

$$\le 2\cdot\max\{\|\partial_i f_n - g_i\|_\infty \mid 1 \le i \le d\} + \max_{t\in[0,1]}\|g(x + t(y-x)) - g(x)\|_\infty.$$

If we let n tend to infinity, then the first term on the right hand side converges to 0 (as $\partial_i f_n$ converges uniformly to g_i) and we therefore get

$$\frac{|f(y) - f(x) - g(x)\cdot(y-x)|}{\|y-x\|} \le \max_{t\in[0,1]}\|g(x + t(y-x)) - g(x)\|_\infty < \varepsilon.$$

This shows that $\frac{|f(y)-f(x)-g(x)\cdot(y-x)|}{\|y-x\|}$ converges to zero for $y \to x$. Consequently, f is differentiable at x with the derivative $\mathrm{D}f(x) = g(x)$. ■

Proposition: Let $d \in \mathbb{N}$ and $l \in \mathbb{N}_0$ and let V_l be the vector space of functions $f \in C^l(\mathbb{R}^d;\mathbb{R})$ such that f and all its partial derivatives are bounded functions. Then $\|f\|_{V_l} :=$

$\sum_{j=0}^{l} \max\{\|\mathrm{D}^\alpha f\|_\infty \mid \alpha \in \mathbb{N}_0^d,\ |\alpha|_1 = j\}$ defines a norm on V_l which makes this space into a Banach space.

Proof: Again, V_l is a linear subspace of $C^l(\mathbb{R}^d;\mathbb{R})$ and $\|\cdot\|_{V_l}$ clearly defines a norm on V_l. We now use the previous lemma and induction over l for the completeness statement. For $l = 0$ this is clear and $l = 1$ is the case of the lemma. Let $l \geq 2$ and let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in V_l. Then every sequence $(\partial_i f_n)_{n\in\mathbb{N}}$ for $1 \leq i \leq d$ is a Cauchy sequence in V_{l-1} and thus converges to some $g_i \in V_{l-1}$ by the induction hypothesis. Moreover, $(f_n)_{n\in\mathbb{N}}$ is in particular a Cauchy sequence in V_1. By the lemma, it therefore converges to some $f \in V_1$. On the one hand $(\partial_i f_n)_{n\in\mathbb{N}}$ now converges uniformly to g_i and on the other hand it converges uniformly to $\partial_i f$. Hence $\partial_i f = g_i \in V_{l-1}$ for $1 \leq i \leq d$, which implies $f \in V_l$ and the estimate

$$\|f_n - f\|_{V_l} \leq \|f_n - f\|_\infty + \sum_{i=1}^{d} \|\partial_i f_n - \partial_i f\|_{V_{l-1}} = \|f_n - f\|_\infty + \sum_{i=1}^{d} \|\partial_i f_n - g_i\|_{V_{l-1}}.$$

But this converges to 0 for $n \to \infty$, thus $f_n \to f$ in V_l. ■