Exercise Sheet 6

For one possible solution of exercise 24 we will need some facts about distances of subsets in metric spaces, which we are going to recall briefly: Let (X, d) be a metric space. For $x \in X$ and a subset $T \subseteq X$ we define the distance from x to T with respect to the metric d by

$$\operatorname{dist}(x, T) := \inf\{d(x, y) \mid y \in T\} \in [0, \infty].$$

For $T = \varnothing$ this should mean $\operatorname{dist}(x, \varnothing) = \infty$. More generally, if S and T are two subsets of X we put

$$\operatorname{dist}(S, T) := \inf\{d(x, y) \mid x \in S, y \in T\} \in [0, \infty].$$

Again, if one of the sets is empty, this expression shall be ∞. The most important property for our purposes is the following:

Lemma: Let (X, d) be a metric space. If S is a compact subset of X and T is a closed subset of X with $S \cap T = \varnothing$, then $\gamma := \operatorname{dist}(S, T) > 0$.

Proof: This is proved by showing the contraposition, i.e.: If S is compact, T is closed and $\gamma = \operatorname{dist}(S, T) = 0$, then $S \cap T \neq \varnothing$. First note that S and T are both nonempty (otherwise their distance could not be 0). By the definition of the distance γ as an infimum over some subset of $\mathbb{R}$, we can find sequences $(x_n)_{n\in\mathbb{N}}$ of elements of S and $(y_n)_{n\in\mathbb{N}}$ of elements of T such that the sequence $(d(x_n, y_n))_{n\in\mathbb{N}}$ of real numbers converges to $\gamma = 0$. As S is compact, there exists a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ with limit $x \in S$. It then holds

$$d(y_{n_k}, x) \leq d(y_{n_k}, x_{n_k}) + d(x_{n_k}, x)$$

for all $k \in \mathbb{N}$ and both summands on the right hand side converge to 0 as $k \to \infty$. Hence, $(y_{n_k})_{k\in\mathbb{N}}$ is a sequence of elements of T which also converges to x. This implies $x \in T$, as T is closed. Thus, $x \in S \cap T$ and therefore the intersection is nonempty. ∎

Exercise 24 (Local Lebesgue spaces) - oral

For a domain $\Omega \subseteq \mathbb{R}^d$ and $p \in [1, \infty]$ we define

$$L^p_{\text{loc}}(\Omega) := \{f : \Omega \to \mathbb{R} \mid f \text{ measurable and } \forall\, \omega \Subset \Omega \ :\ f|_\omega \in L^p(\omega)\},$$

where „$\Subset$“ means compactly contained.

(a) Show that within the system

$$SN_\Subset = \{p_\omega : L^p_{\text{loc}}(\Omega) \to \mathbb{R} \mid \omega \Subset \Omega\} \quad \text{with} \quad p_\omega(f) := \|f|_\omega\|_{L^p(\omega)}$$

of semi-norms, there is a countable family $SN_\mathbb{N} = \{\widetilde{p}_k \mid k \in \mathbb{N}\} \subseteq SN_\Subset$ such that for every $\omega \Subset \Omega$ there exist $k \in \mathbb{N}$ and $C_k > 0$ with $p_\omega(f) \leq C_k \widetilde{p}_k(f)$ for all f.

(b) Construct a metric on $(L^p_{\text{loc}}, \mathfrak{T}_{SN_\Subset})$ which induces the topology $\mathfrak{T}_{SN_\Subset}$.

(c) Let $\Omega = (0, 1)$. Construct three examples for a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \to f$ in L^2_{loc} such that, respectively,

1. $f \notin L^2(\Omega)$.
2. $f \in L^2(\Omega)$ but $\|f_n\|_{L^2(\Omega)} \to \infty$.
3. $f \in L^2(\Omega)$, $\|f_n\|_{L^2(\Omega)}$ bounded and $\|f_n - f\|_{L^2(\Omega)} \geq \frac{1}{4}$.

Solution

(a): We will use the above considerations for the special metric space $(\mathbb{R}^d, d_2)$, where d_2 is the Euclidean metric induced by the Euclidean norm $\|\cdot\|_2$. The idea to solve this part of the exercise is to exhaust Ω by a countable family $(K_n)_{n\in\mathbb{N}}$ of compact subsets of Ω (exhaust means $\Omega = \bigcup_{n\in\mathbb{N}} K_n$) such that for all $\omega \Subset \Omega$ there exists an index $n \in \mathbb{N}$ with $\omega \subseteq K_n$. This is realized as follows:

For every $n \in \mathbb{N}$ (convention here: $\mathbb{N} = \{1, 2, 3, \ldots\}$, i.e. $0 \notin \mathbb{N}$) define the set

$$A_n := \left\{ x \in \Omega \;\middle|\; \operatorname{dist}(x, \partial\Omega) \geq \frac{1}{n} \right\},$$

where

$$\partial\Omega = \overline{\Omega} \backslash \mathring{\Omega} = \overline{\Omega} \backslash \Omega = \left\{ y \in \mathbb{R}^d \;\middle|\; \forall\, \varepsilon > 0 \;:\; B_\varepsilon(y) \cap \Omega \neq \varnothing \quad \text{and} \quad B_\varepsilon(y) \cap \left(\mathbb{R}^d \setminus \Omega\right) \neq \varnothing \right\}$$

is the boundary of Ω (recall that Ω is open, hence $\mathring{\Omega} = \Omega$). We claim that A_n is a closed subset of $\mathbb{R}^d$. In fact, if $(x_m)_{m\in\mathbb{N}}$ is a sequence of elements of A_n which converges in $\mathbb{R}^d$ to some x, then for all $y \in \partial\Omega$ and for all $m \in \mathbb{N}$ it holds

$$\frac{1}{n} \leq \operatorname{dist}(x_m, \partial\Omega) \leq d_2(x_m, y) \leq d_2(x_m, x) + d_2(x, y).$$

If we let m tend to ∞ in the gained estimate

$$\frac{1}{n} \leq d_2(x_m, x) + d_2(x, y)$$

we obtain $\frac{1}{n} \leq d_2(x, y)$. Since this is true for all $y \in \partial\Omega$, we get $\frac{1}{n} \leq \operatorname{dist}(x, \partial\Omega)$. To conclude $x \in A_n$ it therefore remains to show $x \in \Omega$. As x is the limit of a sequence of elements of Ω, x is at least an element of the closure $\overline{\Omega}$. But, as we have seen before, x has positive distance to the boundary $\partial\Omega$, hence $x \in \overline{\Omega} \setminus \partial\Omega = \mathring{\Omega} = \Omega$.

In conclusion, we have shown that A_n is closed. Hence,

$$K_n := A_n \cap \overline{B_n(0)}$$

is closed and bounded and thus compact (by Heine-Borel, see basic analysis lectures). Note that these compact sets K_n form a chain: $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots$.

Now, if $\omega \Subset \Omega$ is compactly contained in our open set Ω, the closure $\overline{\omega}$ is a compact subset of Ω (this is just the definition of „compactly contained“). In particular, $\overline{\omega}$ is bounded, so that there exists an index $n_1 \in \mathbb{N}$ with $\overline{\omega} \subseteq \overline{B_{n_1}(0)}$. Moreover, $\partial\Omega = \overline{\Omega} \cap (\mathbb{R}^d \setminus \Omega)$ is the intersection of two closed subsets of $\mathbb{R}^d$ and therefore itself a closed set. By the lemma above, $\operatorname{dist}(\overline{\omega}, \partial\Omega)$ is then > 0, hence even $\geq \frac{1}{n_2}$ for some $n_2 \in \mathbb{N}$. If we put $n := \max\{n_1, n_2\}$, we obtain

$$\operatorname{dist}(x, \partial\Omega) \geq \operatorname{dist}(\overline{\omega}, \partial\Omega) \geq \frac{1}{n_2} \geq \frac{1}{n}$$

for all $x \in \overline{\omega}$, hence $\overline{\omega} \subseteq A_n$. In conclusion, it holds $\omega \subseteq \overline{\omega} \subseteq A_n \cap \overline{B_n(0)} = K_n$. In particular, for all $f \in L^p_{\text{loc}}(\Omega)$, we have

$$p_\omega(f) = \|f|_\omega\|_{L^p(\omega)} \leq \|f|_{K_n}\|_{L^p(K_n)} = p_{K_n}(f).$$

Therefore we can put $\widetilde{p}_k := p_{K_k}$ and $C_k := 1$ for all $k \in \mathbb{N}$ to get the desired countable set of semi-norms.

Note that we did not need the fact that the chain $K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots$ is exhaustive for Ω, i.e. $\Omega = \bigcup_{n\in\mathbb{N}} K_n$. But actually, we will use this in (b). However, the proof is immediate from our previous observations: If $x \in \Omega$ is arbitrary, then $\omega := \{x\}$ is compact and therefore compactly contained in Ω, hence already contained in some K_n.

(b): Firstly, the countable system $SN_{\mathbb{N}}$ is Hausdorff, because if $f \in L^p_{\text{loc}}(\Omega)$ is a function with $p_{K_n}(f) = 0$ for all $n \in \mathbb{N}$, then $f|_{K_n}$ is zero almost everywhere on K_n, i.e. there is a zero set $N_n \subseteq K_n$ with $f|_{K_n \setminus N_n} = 0$. If we put $N := \bigcup_{n\in\mathbb{N}} N_n$, then f is also equal to the zero function on the sets

$$\bigcup_{n\in\mathbb{N}} (K_n \setminus N_n) \supseteq \bigcup_{n\in\mathbb{N}} (K_n \setminus N) = \left(\bigcup_{n\in\mathbb{N}} K_n \right) \setminus N = \Omega \setminus N.$$

Being a countable union of zero sets, N is also a zero set, hence $f = 0$ almost everywhere on Ω.

As $SN_{\mathbb{N}}$ is a countable Hausdorff system of semi-norms, there is a metric d' on L^p_{loc} with $\mathfrak{T}_{d'} = \mathfrak{T}_{SN_{\mathbb{N}}}$ (see the lectures). It therefore suffices to show $\mathfrak{T}_{SN_{\mathbb{N}}} = \mathfrak{T}_{SN_{\Subset}}$. Of course, $SN_{\mathbb{N}} \subseteq SN_{\Subset}$ implies $\mathfrak{T}_{SN_{\mathbb{N}}} \subseteq \mathfrak{T}_{SN_{\Subset}}$. Conversely, let $U \in \mathfrak{T}_{SN_{\Subset}}$ be open with respect to the topology induced by $SN_{\Subset}$ and let $f \in U$ be arbitrary. Then there exist $\varepsilon > 0$ and $\omega_1, \dots, \omega_m \Subset \Omega$ such that

$$\mathcal{O}_{\omega_1,\dots,\omega_m,\varepsilon} = \{g \in L^p_{\text{loc}} \mid p_{\omega_i}(g - f) < \varepsilon \qquad \forall\, 1 \leq i \leq m\} \subseteq U.$$

By (a) we then find indices $k_1, \dots, k_m \in \mathbb{N}$ with $\omega_i \subseteq K_{k_i}$, hence $p_{\omega_i} \leq p_{K_{k_i}}$ for all i. For $g \in L^p_{\text{loc}}(\Omega)$, the relation $p_{K_{k_i}}(g - f) < \varepsilon$ for $1 \leq i \leq m$ then implies $p_{\omega_i}(g - f) < \varepsilon$ for $1 \leq i \leq m$, hence

$$\mathcal{O}_{K_{k_1},\dots,K_{k_m},\varepsilon} \subseteq \mathcal{O}_{\omega_1,\dots,\omega_m,\varepsilon} \subseteq U.$$

Because of $p_{K_{k_i}} \in SN_{\mathbb{N}}$ and since $f \in U$ was arbitrary, this shows $U \in \mathfrak{T}_{SN_{\mathbb{N}}}$.

(c): We will construct sequences using the intervals $\left(0, \frac{1}{n}\right)$. Notice that for all $\omega \Subset \Omega = (0, 1)$ the closure $\overline{\omega}$ has a minimum $a \in (0, 1)$ and thus $\omega \subseteq [a, 1)$, so that for every such ω there exists $n_0 \in \mathbb{N}$ with $\left(0, \frac{1}{n}\right) \cap \omega = \varnothing$ for all $n \geq n_0$.

1. Let $f(x) := f_n(x) := \frac{1}{\sqrt{x}}$ for all $n \in \mathbb{N}$. Then clearly $f_n \to f$ in $L^2_{\text{loc}}((0, 1))$ and f does not lie in $L^2((0, 1))$.

2. Let $f_n(x) := n \cdot \chi_{\left(0, \frac{1}{n}\right)}$. For an arbitrary $\omega \Subset \Omega$ let n_0 be as constructed above. Then $f_n|_\omega = 0$ and thus $p_\omega(f_n) = 0$ for all $n \geq n_0$. Hence $f_n \to f := 0$ in $L^2_{\text{loc}}((0,1))$. Moreover, $\|f_n\|_{L^2((0,1))} = \sqrt{n}$.

3. Let $f_n(x) := \frac{\sqrt{n}}{4} \cdot \chi_{\left(0, \frac{1}{n}\right)}$. By the same argument as before, we have $f_n \to f := 0$. But now $\|f_n - f\|_{L^2((0,1))} = \|f_n\|_{L^2((0,1))} = \frac{1}{4}$ is bounded and always $\geq \frac{1}{4}$.