Exercise Sheet 5

Exercise 20 (Hermite Polynomials) - oral

The *Hermite polynomials* are defined by $H_n(x) := (-1)^n \cdot \mathrm{e}^{x^2} \cdot \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{-x^2}\right)$. They satisfy the differential equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 \tag{1}$$

and are orthogonal with respect to the weightet scalar product

$$\langle f, g\rangle_w = \int_{\mathbb{R}} f(x)g(x)\mathrm{e}^{-x^2}\,\mathrm{d}x,$$

i.e. they satisfy

$$\langle H_m, H_n\rangle_w = C_n \cdot \delta_{mn} \tag{2}$$

with a constant C_n depending on n.

(a) Show (1). Moreover, explicitly calculate H_n and C_n for $n = 0, 1, 2, 3$.

(b) Prove (2) and calculate C_n for general $n \in \mathbb{N}$. Conclude that the rescaled polynomials $\frac{1}{\sqrt{C_n}} \cdot H_n$ are (weighted) orthonormal.

(c) Use the Hermite polynomials to construct functions ψ_n which are orthonormal with respect to the usual L^2-scalar product. Establish a differential equation for the ψ_n.

Remark: The ψ_n are called *Hermite functions*. They play an important role as solutions of the Schrödinger equation of the quantised harmonic oscillator.

Note: Of course, the rescaled polynomials were ment to be $\frac{1}{\sqrt{C_n}} \cdot H_n$ instead of $C_n \cdot H_n$. Moreover, note that the scalar product $\langle\cdot,\cdot\rangle_w$ is defined on the space

$$\left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; f \text{ measurable and } \int_{\mathbb{R}} |f(x)|^2 \cdot \mathrm{e}^{-x^2}\,\mathrm{d}x < \infty \right\},$$

factorized by the usual relation of „equality almost everywhere“.

Solution

(a): For all $x \in \mathbb{R}$ it holds

$$H_n'(x) = (-1)^n \cdot 2x\mathrm{e}^{x^2} \cdot \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{-x^2}\right) + (-1)^n \cdot \mathrm{e}^{x^2} \cdot \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left(\mathrm{e}^{-x^2}\right) = 2xH_n(x) - H_{n+1}(x).$$

This gives the inductive formula

$$H_{n+1}(x) = 2xH_n(x) - H_n'(x) \quad \forall\, x \in \mathbb{R}, \quad \text{i.e.} \quad H_{n+1} = 2\,\mathrm{id}_{\mathbb{R}} \cdot H_n - H_n'. \tag{3}$$

Using this (or just calculating the derivatives explicitly), we get

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2, \qquad H_3(x) = 8x^3 - 12x.$$

Obviously, H_0 satisfies the differential equation (1). We now use induction to show this for all $n \in \mathbb{N}$. In fact, by inserting the inductive formula, we get for all $n \in \mathbb{N}$

$$\begin{aligned}
&H''_{n+1}(x) - 2xH'_{n+1}(x) + 2(n+1)H_{n+1}(x) \\
&= \big(2H'_n(x) + 2H'_n(x) + 2xH''_n(x) - H'''_n(x)\big) - 2x \cdot \big(2H_n(x) + 2xH'_n(x) - H''_n(x)\big) \\
&\quad + \big(2(n+1)2xH_n(x) - 2(n+1)H'_n(x)\big) \\
&= -\big(H'''_n(x) - 2H'_n(x) - 2xH'''_n(x) + 2nH'_n(x)\big) \\
&\quad + 2x \cdot \big(H''_n(x) - 2H_n(x) - 2xH'_n(x) + 2(n+1)H_n(x)\big) \\
&= -\big(H''_n(x) - 2xH'_n(x) + 2nH_n(x)\big)' + 2x\big(H''_n(x) - 2xH'_n(x) + 2nH_n(x)\big) \\
&= -0 + 2x \cdot 0 = 0.
\end{aligned}$$

Moreover, $C_0 = \int_{\mathbb{R}} \mathrm{e}^{-x^2}\, \mathrm{d}x = \sqrt{\pi}$ (see basic analysis courses) and this implies (by using integration by parts several times)

$$C_1 = 2\sqrt{\pi}, \qquad C_2 = 8\sqrt{\pi}, \qquad C_3 = 48\sqrt{\pi}.$$

(b): Let us first show the auxiliary result

$$H'_{n+1} = 2(n+1) \cdot H_n \tag{4}$$

for all $n \in \mathbb{N}$. If $n = 0$, this is obviously true. Using induction, we get for $n \geq 1$

$$\begin{aligned}
H'_{n+1}(x) &\overset{(3)}{=} (2xH_n(x) - H'_n(x))' = 2H_n(x) + 2xH'_n(x) - H''_n(x) \\
&= 2H_n(x) + 4xnH_{n-1}(x) - 2nH'_{n-1}(x) = 2H_n(x) + 2n \cdot (2xH_{n-1} - H'_{n-1}(x)) \\
&\overset{(3)}{=} 2H_n(x) + 2nH_n(x) = 2(n+1)H_n(x).
\end{aligned}$$

If we put $H_{-1} := 0$, then (4) obviously also holds for $n = -1$ and (3) then implies

$$H_{n+1} = 2\,\mathrm{id}_{\mathbb{R}} \cdot H_n - H'_n = 2\,\mathrm{id}_{\mathbb{R}} \cdot H_n - 2n \cdot H_{n-1} \qquad \forall\, n \geq 0. \tag{5}$$

Now, we want to show that for all $m \neq n$ in $\mathbb{N}$ we have $\langle H_m, H_n \rangle_w = 0$. To see this, we use induction over $k := m+n$, i.e. we show that for all $k \in \mathbb{N}$ the following condition is fulfilled: For all $(m,n) \in (\mathbb{N} \cup \{-1\})^2$ with $m < n$ (w.l.o.g.!) and $m+n = k$ we have $\langle H_m, H_n \rangle_w = 0$. If $k = 0$, then we necessarily have $m = -1$ and $n = 0$, hence the scalar product is 0 as $H_{-1} = 0$. If $k = 1$, then $m = -1, n = 2$ or $m = 0, n = 1$. In the first case, the scalar product is again 0 and in the second case we have $\langle H_0, H_1 \rangle_w = \int_{\mathbb{R}} 2x\mathrm{e}^{-x^2}\, \mathrm{d}x = 0$ (the integrand is an odd function). As for the induction step, let now $k \geq 2$ and let $-1 \leq m < n$ satisfy $m + n = k$. W.l.o.g. we can again assume $m \geq 0$. Moreover, note that neccessarily $n \geq 2$ (as $k \geq 2$). Then we get

$$\langle H_m, H_n \rangle_w \overset{(5)}{=} \langle H_m, 2\,\mathrm{id}_{\mathbb{R}} \cdot H_{n-1} \rangle_w - 2(n-1) \cdot \langle H_m, H_{n-2} \rangle_w.$$

Now, integration by parts yields

$$
\begin{aligned}
\langle H_m, 2\,\mathrm{id}_{\mathbb{R}} \cdot H_{n-1}\rangle_w &= \int_{\mathbb{R}} H_m(x)H_{n-1}(x)\cdot 2x\mathrm{e}^{-x^2}\,\mathrm{d}x \\
&= \lim_{a\to\infty} \Big[-H_m(x)H_{n-1}(x)\mathrm{e}^{-x^2}\Big]_{-a}^{a} + \int_{\mathbb{R}} (H_m\cdot H_{n-1})'(x)\cdot \mathrm{e}^{-x^2}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} (H_m\cdot H_{n-1})'(x)\cdot \mathrm{e}^{-x^2}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} H_m'(x)\cdot H_{n-1}(x)\cdot \mathrm{e}^{-x^2}\,\mathrm{d}x + \int_{\mathbb{R}} H_m(x)\cdot H_{n-1}'(x)\cdot \mathrm{e}^{-x^2}\,\mathrm{d}x \\
&= \langle H_m', H_{n-1}\rangle_w + \langle H_m, H_{n-1}'\rangle_w \\
&\overset{(4)}{=} 2m\cdot\langle H_{m-1}, H_{n-1}\rangle + 2(n-1)\cdot\langle H_m, H_{n-2}\rangle_w \\
&= 2m\cdot\langle H_{m-1}, H_{n-1}\rangle + 2(n-1)\cdot\langle H_m, H_{n-2}\rangle_w. \qquad (6)
\end{aligned}
$$

By inserting this in the previous equation for the scalar product of H_m and H_n, we obtain

$$\langle H_m, H_n\rangle_w = 2m\cdot\langle H_{m-1}, H_{n-1}\rangle = 0$$

by the induction hypothesis.

Using this, we can calculate C_n inductively (for $0 \le n \le 3$ see (a)). Let $n \ge 2$. Then

$$
\begin{aligned}
C_n &= \langle H_n, H_n\rangle_w = \langle 2\,\mathrm{id}_{\mathbb{R}}\cdot H_{n-1} - 2(n-1)H_{n-2}, 2\,\mathrm{id}_{\mathbb{R}}\cdot H_{n-1} - 2(n-1)H_{n-2}\rangle_w \\
&= \langle 2\,\mathrm{id}_{\mathbb{R}}\cdot H_{n-1}, 2\,\mathrm{id}_{\mathbb{R}}\cdot H_{n-1}\rangle_w - 2\cdot 2(n-1)\cdot\langle H_{n-2}, 2\,\mathrm{id}_{\mathbb{R}}\cdot H_{n-1}\rangle_w \\
&\quad + 4(n-1)^2\cdot\langle H_{n-2}, H_{n-2}\rangle_w.
\end{aligned}
$$

By our calculations above, the second term is given by

$$
\begin{aligned}
\langle H_{n-2}, 2\,\mathrm{id}_{\mathbb{R}}\cdot H_{n-1}\rangle_w &\overset{(6)}{=} 2(n-2)\cdot\langle H_{n-3}, H_{n-1}\rangle + 2(n-1)\cdot\langle H_{n-2}, H_{n-2}\rangle_w \\
&= 0 + 2(n-1)C_{n-2} = 2(n-1)C_{n-2}
\end{aligned}
$$

and by using integration by parts for the first term we obtain

$$
\begin{aligned}
&\langle 2\,\mathrm{id}_{\mathbb{R}}\cdot H_{n-1}, 2\,\mathrm{id}_{\mathbb{R}}\cdot H_{n-1}\rangle_w = \int_{\mathbb{R}} (H_{n-1}(x))^2 2x\cdot 2x\mathrm{e}^{-x^2}\,\mathrm{d}x = \int_{\mathbb{R}} \left(2x(H_{n-1}(x))^2\right)'\mathrm{e}^{-x^2}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} 2(H_{n-1}(x))^2\mathrm{e}^{-x^2}\,\mathrm{d}x + \int_{\mathbb{R}} 2H_{n-1}(x)H_{n-1}'(x)\cdot 2x\mathrm{e}^{-x^2}\,\mathrm{d}x \\
&= 2\cdot\langle H_{n-1}, H_{n-1}\rangle_w + \int_{\mathbb{R}} (2H_{n-1}H_{n-1}')'(x)\cdot\mathrm{e}^{-x^2}\,\mathrm{d}x \\
&= 2\cdot C_{n-1} + \int_{\mathbb{R}} (2H_{n-1}H_{n-1}')'(x)\cdot\mathrm{e}^{-x^2}\,\mathrm{d}x \\
&= 2\cdot C_{n-1} + \int_{\mathbb{R}} 2(H_{n-1}'(x))^2\cdot\mathrm{e}^{-x^2}\,\mathrm{d}x + \int_{\mathbb{R}} 2H_{n-1}(x)H_{n-1}''(x)\cdot\mathrm{e}^{-x^2}\,\mathrm{d}x \\
&= 2\cdot C_{n-1} + 2\cdot\langle H_{n-1}', H_{n-1}'\rangle_w + 2\langle H_{n-1}, H_{n-1}''\rangle_w \\
&\overset{(4)}{=} 2\cdot C_{n-1} + 2\cdot(2(n-1))^2\cdot\langle H_{n-2}, H_{n-2}\rangle_w + 2\cdot 2(n-1)\cdot 2(n-2)\langle H_{n-1}, H_{n-3}\rangle_w \\
&= 2\cdot C_{n-1} + 2\cdot(2(n-1))^2\cdot C_{n-2} + 0.
\end{aligned}
$$

In conclusion, we get

$$\begin{aligned} C_n &= 2 \cdot C_{n-1} + 2 \cdot (2(n-1))^2 \cdot C_{n-2} - 2 \cdot 2(n-1) \cdot 2(n-1) C_{n-2} + 4(n-1)^2 \cdot C_{n-2} \\ &= 2 \cdot C_{n-1} + 4(n-1)^2 \cdot C_{n-2}. \end{aligned}$$

We now claim that $C_n = 2^n \cdot n! \cdot \sqrt{\pi}$ (of course, this is not trivial to guess, so one could look up the result or use a computer). This is easily proved by induction over n, because for $n = 0, 1$ this follows from (a) and for $n \geq 2$ the induction hypothesis implies

$$\begin{aligned} C_n &= 2 \cdot C_{n-1} + 4(n-1)^2 \cdot C_{n-2} = \sqrt{\pi} \cdot \left(2 \cdot 2^{n-1} \cdot (n-1)! + 4(n-1)^2 \cdot 2^{n-2} \cdot (n-2)!\right) \\ &= \sqrt{\pi} \cdot 2^n \cdot (n-1)! \cdot (1 + n - 1) = \sqrt{\pi} \cdot 2^n \cdot n!. \end{aligned}$$

In conclusion, $\left\{ \frac{1}{\sqrt{2^n \cdot n! \cdot \sqrt{\pi}}} \cdot H_n \;\middle|\; n \in \mathbb{N} \right\}$ is an orthonormal system with respect to the weighted scalar product $\langle \cdot, \cdot \rangle$.

(c): By definition of the weighted scalar product, the functions ψ_n defined by

$$\psi_n(x) := \frac{1}{\sqrt{2^n \cdot n! \cdot \sqrt{\pi}}} \cdot H_n(x) \cdot \mathrm{e}^{\frac{-x^2}{2}}$$

form an orthonormal system with respect to the standard scalar product on $L^2(\mathbb{R})$. Moreover, we have

$$\begin{aligned} \psi_n'(x) &= \frac{1}{\sqrt{2^n \cdot n! \cdot \sqrt{\pi}}} \cdot (H_n'(x) - xH_n(x)) \cdot \mathrm{e}^{\frac{-x^2}{2}} \\ \psi_n''(x) &= \frac{1}{\sqrt{2^n \cdot n! \cdot \sqrt{\pi}}} \cdot ((H_n''(x) - xH_n'(x) - H_n(x)) - x(H_n'(x) - xH_n(x))) \cdot \mathrm{e}^{\frac{-x^2}{2}} \\ &= \frac{1}{\sqrt{2^n \cdot n! \cdot \sqrt{\pi}}} \cdot \left(H_n''(x) - 2xH_n'(x) + \left(x^2 - 1\right) H_n(x)\right) \cdot \mathrm{e}^{\frac{-x^2}{2}}. \end{aligned}$$

As H_n satisfies the differential equation $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$, we obtain

$$\psi_n''(x) = \frac{1}{\sqrt{2^n \cdot n! \cdot \sqrt{\pi}}} \cdot \left(x^2 - 1 - 2n\right) \cdot H_n(x) \cdot \mathrm{e}^{\frac{-x^2}{2}} = \left(x^2 - 1 - 2n\right) \cdot \psi_n(x).$$

Hence, ψ_n satisfies the differential equation

$$\psi_n''(x) + \left(2n + 1 - x^2\right) \psi_n(x) = 0.$$