Exercise Sheet 4

Lemma: Let $b > 0$ and for $j \in \mathbb{N}$ let $g_j(t) = \sqrt{\frac{2}{b}} \cdot \sin\left(\frac{\pi}{b} jt\right)$ for all $t \in (0,b)$. Then $\{g_j \mid j \in \mathbb{N}\}$ is a complete ONS in $L^2((0,b))$.

Proof: The idea of the proof is quite easy: Use the cONS of $L^2((0,2\pi))$ from exercise 13 and transfer it to $L^2((-b,b))$. To get rid of all the cosinus functions, use the following trick: Extend functions u defined on $(0,b)$ to functions $\tilde{u}$ on the interval $(-b,b)$ such that they are odd (i.e. $\tilde{u}(-t) = -\tilde{u}(t)$) and expand them in their Fourier series. Because $\tilde{u}$ is odd, all the Fourier coefficients corresponding to cosinus-terms will be zero. In the following, we make this idea more explicit and more precise.

Consider the (clearly well-defined) operator $\Phi : L^2((0,2\pi)) \to L^2((-b,b))$ which assigns to a function $f \in L^2((0,2\pi))$ a new function $\Phi(f) \in L^2((-b,b))$ defined by

$$\Phi(f)(t) := \sqrt{\frac{\pi}{b}} \cdot f\left(\frac{\pi}{b} \cdot t + \pi\right).$$

This map Φ is an isometry of Hilbert spaces, because for $f, g \in L^2((0,2\pi))$ integration by substitution (with $x = \frac{\pi}{b} \cdot t + \pi$) yields

$$\begin{aligned}\langle \Phi(f), \Phi(g) \rangle_{L^2((-b,b))} &= \int_{-b}^{b} \sqrt{\frac{\pi}{b}} \cdot f\left(\frac{\pi}{b} \cdot t + \pi\right) \cdot \sqrt{\frac{\pi}{b}} \cdot g\left(\frac{\pi}{b} \cdot t + \pi\right) \, \mathrm{d}t \\ &= \int_0^{2\pi} f(x) g(x) \, \mathrm{d}x = \langle f, g \rangle_{L^2((0,2\pi))}.\end{aligned}$$

Consequently, Φ carries the complete orthonormal system $\{e_j \mid j \in \mathbb{Z}\}$ of $L^2((0,2\pi))$ considered in exercise 13 to a complete orthonormal system $\{f_j \mid j \in \mathbb{Z}\}$ in $L^2((-b,b))$ given as follows (we have changed some signs to get a similar structure of the cONS with no negative signs):

$$\begin{aligned}
f_0(t) &:= \Phi(e_0)(t) = \sqrt{\frac{1}{2b}}, \\
f_j(t) &:= \Phi(e_j)(t) = \sqrt{\frac{1}{b}} \cdot \cos\left(\frac{j\pi}{b} \cdot t + j\pi\right) = \sqrt{\frac{1}{b}} \cdot \cos\left(\frac{j\pi}{b} \cdot t\right) \qquad \text{for} \quad j \in 2\mathbb{N}, \\
f_j(t) &:= -\Phi(e_j)(t) = -\sqrt{\frac{1}{b}} \cdot \cos\left(\frac{j\pi}{b} \cdot t + j\pi\right) = -\sqrt{\frac{1}{b}} \cdot \cos\left(\frac{j\pi}{b} \cdot t + \pi\right) \\
&= \sqrt{\frac{1}{b}} \cdot \cos\left(\frac{j\pi}{b} \cdot t\right) \qquad \text{for} \quad j \in 2\mathbb{N}_0 + 1, \\
f_{-j}(t) &:= \Phi(e_{-j})(t) = \sqrt{\frac{1}{b}} \cdot \sin\left(\frac{j\pi}{b} \cdot t + j\pi\right) = \sqrt{\frac{1}{b}} \cdot \sin\left(\frac{j\pi}{b} \cdot t\right) \qquad \text{for} \quad j \in 2\mathbb{N}, \\
f_{-j}(t) &:= -\Phi(e_{-j})(t) = -\sqrt{\frac{1}{b}} \cdot \sin\left(\frac{j\pi}{b} \cdot t + j\pi\right) = -\sqrt{\frac{1}{b}} \cdot \sin\left(\frac{j\pi}{b} \cdot t + \pi\right) \\
&= \sqrt{\frac{1}{b}} \cdot \sin\left(\frac{j\pi}{b} \cdot t\right) \qquad \text{for} \quad j \in 2\mathbb{N}_0 + 1.
\end{aligned}$$

We therefore have the complete orthonormal system of $L^2((-b,b))$ given by

$$f_0(t) = \sqrt{\frac{1}{2b}}, \qquad f_j(t) = \sqrt{\frac{1}{b}} \cdot \cos\left(\frac{j\pi}{b} \cdot t\right), \qquad f_{-j}(t) = \sqrt{\frac{1}{b}} \cdot \sin\left(\frac{j\pi}{b} \cdot t\right)$$

for $j \in \mathbb{N}$. Now take any function $u \in L^2((0,b))$ and define $\tilde{u} \in L^2((-b,b))$ by

$$\tilde{u}(t) := \begin{cases} -u(-t) & \text{for } t \in (-b,0) \\ 0 & \text{for } t = 0 \\ u(t) & \text{for } t \in (0,b). \end{cases}$$

Then the sequence of finite Fourier sums

$$\tilde{u}_n := \sum_{j=-n}^{n} \langle \tilde{u}, f_j \rangle_{L^2((-b,b))} \cdot f_j$$

converges to $\tilde{u}$ in $L^2((-b,b))$. Since $\tilde{u}$ is odd and f_j is even for $j \geq 0$, the product $\tilde{u} \cdot f_j$ is still an odd function and so its integral over $(-b,b)$ is zero, i.e. $\langle \tilde{u}, f_j \rangle_{L^2((-b,b))} = 0$ for all $j \geq 0$. As for the coefficient $\langle \tilde{u}, f_{-j} \rangle_{L^2((-b,b))}$ for $j > 0$, recall that $\tilde{u}$ and f_{-j} are both odd, so their product is even, hence

$$\begin{aligned} \langle \tilde{u}, f_{-j} \rangle_{L^2((-b,b))} &= \int_{-b}^{b} \tilde{u}(t) f_{-j}(t) \, dt = 2 \cdot \int_0^b \tilde{u}(t) \cdot f_{-j}(t) \, dt \\ &= 2 \cdot \int_0^b u(t) \cdot f_{-j}(t) \, dt = 2 \cdot \langle u, f_{-j}|_{(0,b)} \rangle_{L^2((0,b))}. \end{aligned}$$

Thus we get

$$\tilde{u}_n = \sum_{j=1}^{n} \langle \tilde{u}, f_{-j} \rangle_{L^2((-b,b))} \cdot f_{-j} = \sum_{j=1}^{n} 2 \cdot \langle u, f_{-j}|_{(0,b)} \rangle_{L^2((0,b))} \cdot f_{-j}$$

and this converges to $\tilde{u}$ in $L^2((-b,b))$. But then $\tilde{u}_n|_{(0,b)}$ must converge to $\tilde{u}|_{(0,b)} = u$ in $L^2((0,b))$, too, as is provided by

$$\begin{aligned} \|\tilde{u}_n|_{(0,b)} - \tilde{u}|_{(0,b)}\|^2_{L^2((0,b))} &= \int_0^b |\tilde{u}_n(t) - \tilde{u}(t)|^2 \, dt \\ &\leq \int_{-b}^0 |\tilde{u}_n(t) - \tilde{u}(t)|^2 \, dt + \int_0^b |\tilde{u}_n(t) - \tilde{u}(t)|^2 \, dt \\ &= \int_{-b}^b |\tilde{u}_n(t) - \tilde{u}(t)|^2 \, dt = \|\tilde{u}_n - \tilde{u}\|^2_{L^2((-b,b))} \xrightarrow{n \to \infty} 0. \end{aligned}$$

Because of $g_j = \sqrt{2} \cdot f_{-j}|_{(0,b)}$, we get

$$\begin{aligned} \tilde{u}_n|_{(0,b)} &= \sum_{j=1}^{n} \langle \tilde{u}, f_{-j} \rangle_{L^2((-b,b))} \cdot f_{-j}|_{(0,b)} = \sum_{j=1}^{n} 2 \cdot \langle u, f_{-j}|_{(0,b)} \rangle_{L^2((0,b))} \cdot f_{-j}|_{(0,b)} \\ &= \sum_{j=1}^{n} \left\langle u, \sqrt{2} f_{-j}|_{(0,b)} \right\rangle_{L^2((0,b))} \cdot \sqrt{2} f_{-j}|_{(0,b)} = \sum_{j=1}^{n} \langle u, g_j \rangle_{L^2((0,b))} \cdot g_j. \end{aligned}$$

In summary, we have shown that for every $u \in L^2((0,b))$ the sequence of the functions $\sum_{j=1}^{n} \langle u, g_j \rangle_{L^2((0,b))} \cdot g_j \in L^2((0,b))$ converges to u in the normed space $\big(L^2((0,b)), \|\cdot\|_{L^2((0,b))}\big)$.
By exercise 9, it thus remains to prove that the g_j form an ONS to conclude that they in fact form a complete ONS. But this is clear from

$$\begin{aligned}\delta_{jk} &= \langle f_{-j}, f_{-k} \rangle_{L^2((-b,b))} = \int_{-b}^{b} f_{-j}(t) f_{-k}(t)\ \mathrm{d}t = 2 \cdot \int_{0}^{b} f_{-j}(t) f_{-k}(t)\ \mathrm{d}t \\ &= \left\langle \sqrt{2} f_{-j}|_{(0,b)}, \sqrt{2} f_{-k}|_{(0,b)} \right\rangle_{L^2((0,b))} = \langle g_j, g_k \rangle_{L^2((0,b))}.\end{aligned}$$

for all $j, k > 0$ (f_{-j} and f_{-k} are odd and so their product is again even). ∎

Exercise 15 (A Differential Operator) - oral

On the interval $\Omega = (0,1)$, we define the differential operator

$$L : Y \to X, \qquad u \mapsto u'' \qquad\qquad \text{with} \qquad Y \subseteq X = C^0\left(\overline{\Omega}\right)$$

where $Y := \left\{ u \in C^2\left(\overline{\Omega}\right) \;\middle|\; u(0) = 0, \quad u'(1) = 0 \right\}$.

(a) Determine all eigen-pairs $(\lambda_k, u_k) \in \mathbb{R} \times Y$ (i.e. pairs such that $Lu_k = \lambda_k u_k$).

(b) Show that for all $u, v \in Y$, $\langle Lu, v\rangle = \langle u, Lv\rangle$ holds, where $\langle\cdot,\cdot\rangle$ denotes the standard L^2-scalar product.

(c) Conclude that there are no eigen-pairs (λ, u) with $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and that eigenfunctions with respect to different eigenvalues are orthogonal on each other.

(d) Show that the normed eigenfunctions form a complete orthonormal system.

Solution for (d): It is not mentioned for which space the normed eigenfunctions should form a complete ONS. However, we will see that they do so for all of the three spaces $L^2((0,1)), X$ and Y.

The normed eigenfunctions u_k (to the eigenvalues $-\frac{(2k+1)^2\pi^2}{4}$) are given by

$$u_k(t) = \sqrt{2}\cdot\sin\left(\frac{(2k+1)\pi}{2}\cdot t\right)$$

for $k \geq 0$. These are pairwise orthogonal by (c) (or also by the following considerations). By the previous Lemma (for $b = 2$), the functions $g_j \in L^2((0,2))$ for $j \in \mathbb{N}$ defined by $g_j(t) = \sin\left(\frac{\pi}{2} jt\right)$ form a complete ONS of $L^2((0,2))$. Now we use just the same method as before to get rid of those g_j with j even. For an arbitrary $h \in L^2((0,1))$ we define $\tilde{h} \in L^2((0,2))$ by

$$\tilde{h}(t) := \begin{cases} h(t) & \text{for } t \in (0,1) \\ 0 & \text{for } t = 1 \\ h(2-t) & \text{for } t \in (1,2). \end{cases}$$

The key property of $\tilde{h}$ is the fact that it is axially symmetric with respect to the axis $\{(1,y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$, i.e. „$t = 1$“ (in other words: the function $t \mapsto \tilde{h}(t+1)$ from $(-1,1)$ to $\mathbb{R}$ is even). Once again, the sequence $\left(\tilde{h}_n\right)_{n\in\mathbb{N}}$ of the functions

$$\tilde{h}_n = \sum_{j=1}^{n} \left\langle \tilde{h}, g_j \right\rangle_{L^2((0,2))} \cdot g_j$$

converges to $\tilde{h}$ in $L^2((0,2))$. If $j = 2k$ is even, then $g_j = g_{2k}$ is given by $g_{2k}(t) = \sin(k\pi t)$ and this function is then point symmetric to the point $(1,0) \in \mathbb{R}^2$, because of

$$g_{2k}(2-t) = \sin(2k\pi - k\pi t) = \sin(-k\pi t) = -\sin(k\pi t) = -g_{2k}(t)$$

for all $t \in (0,2)$ (in other words: the function $t \mapsto g_{2k}(t+1)$ from $(-1,1)$ to $\mathbb{R}$ is odd). Hence,

$$\left\langle \tilde{h}, g_{2k} \right\rangle_{L^2((0,2))} = \int_0^2 \tilde{h}(t) g_{2k}(t)\ \mathrm{d}t = 0.$$

On the other hand, if $j = 2k + 1$ is odd, then for all $t \in (0, 2)$ we have

$$\begin{aligned} g_{2k+1}(2-t) &= \sin\left(\frac{\pi}{2}(2k+1)(2-t)\right) = \sin\left((2k+1)\pi - \frac{\pi}{2}(2k+1)t\right) \\ &= \sin\left(\pi - \frac{\pi}{2}(2k+1)t\right) = \sin\left(\frac{\pi}{2}(2k+1)t\right) = g_{2k+1}(t), \end{aligned}$$

i.e. g_{2k+1} is then axially symmetric with respect to the line „$t = 2$“ as well. If now a function $v \in L^2((0,2))$ also has this property (e.g. $v = \tilde{h}$), so does the product vg_{2k+1}, hence

$$\begin{aligned} \langle v, g_{2k+1}\rangle_{L^2((0,2))} &= \int_0^2 v(t)g_{2k+1}(t)\ \mathrm{d}t = 2 \cdot \int_0^1 v(t)g_{2k+1}(t)\ \mathrm{d}t \\ &= 2 \cdot \langle v|_{(0,1)}, g_{2k+1}|_{(0,1)}\rangle_{L^2((0,1))}. \end{aligned}$$

In particular, it holds

$$\begin{aligned} \delta_{jk} &= \langle g_{2l+1}, g_{2k+1}\rangle_{L^2((0,2))} = 2 \cdot \langle g_{2l+1}|_{(0,1)}, g_{2k+1}|_{(0,1)}\rangle_{L^2((0,1))} \\ &= \left\langle \sqrt{2}g_{2l+1}|_{(0,1)}, \sqrt{2}g_{2k+1}|_{(0,1)}\right\rangle_{L^2((0,1))} = \langle u_l, u_k\rangle_{L^2((0,1))}. \end{aligned}$$

for all $j, k \geq 0$. Thus, the u_k form in fact an ONS. Moreover, $\tilde{h}_n \to \tilde{h}$ in $L^2((0,2))$ particularly implies $\tilde{h}_{2n+1}\Big|_{(0,1)} \to \tilde{h}\Big|_{(0,1)} = h$ in $L^2((0,1))$ for $n \to \infty$, where

$$\begin{aligned} \tilde{h}_{2n+1}\Big|_{(0,1)} &= \sum_{j=1}^{2n+1} \left\langle \tilde{h}, g_j\right\rangle_{L^2((0,2))} \cdot g_j|_{(0,1)} = \sum_{k=0}^{n} 2 \cdot \left\langle \tilde{h}\Big|_{(0,1)}, g_{2k+1}|_{(0,1)}\right\rangle_{L^2((0,2))} \cdot g_{2k+1}|_{(0,1)} \\ &= \sum_{k=0}^{n} \left\langle h, \sqrt{2}g_{2k+1}|_{(0,1)}\right\rangle_{L^2((0,2))} \cdot \sqrt{2}g_{2k+1}|_{(0,1)} = \sum_{k=0}^{n} \langle h, u_k\rangle_{L^2((0,2))} \cdot u_k. \end{aligned}$$

In conclusion, $\{u_k \mid k \in \mathbb{N}_0\}$ is in fact a complete ONS in $L^2((0,1))$ (and therefore also in the pre-Hilbert spaces X and Y endowed with the L^2-scalar product).

For part (c) of exercise 15, we will use the following Proposition (see basic analysis lectures or books):

Proposition

Let $a < b$ be real numbers, let $(f_n)_{n\in\mathbb{N}}$ be a series of functions $f_n : [a,b] \to \mathbb{R}$ (e.g. $f_n = \sum_{j=1}^{n} u_j$) and let the following conditions be fulfilled:

(a) Every f_n is differentiable on $[a,b]$ and the sequence $(f_n')_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$ (to some function g on $[a,b]$).

(b) There is a $x_0 \in [a,b]$ such that $(f_n(x_0))_{n\in\mathbb{N}}$ is a convergent sequence of real numbers (i.e. the sequence $(f_n)_{n\in\mathbb{N}}$ converges „pointwise in at least one point").

Then the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly converging to a function $f : [a,b] \to \mathbb{R}$, the limit function f is differentiable and we have $f = g$, i.e. $(f_n')_{n\in\mathbb{N}}$ converges (uniformly) to f'.

Exercise 16 (Heat Conduction) - oral

Let $\kappa > 0$, $\Omega := (0,b)$ und $u_0 : \Omega \to \mathbb{R}$ be given. We are looking for a function $u : [0,\infty) \times [0,b] \to \mathbb{R}$ such that

$$\text{(HE)} \quad \frac{\partial u}{\partial t} = \kappa \cdot \frac{\partial^2}{\partial x^2} \quad \text{in } (0,\infty) \times \Omega,$$
$$\text{(IC)} \quad u(0,x) = u_0(x) \quad \text{for } x \in \Omega, \qquad \text{(BC)} \quad u(t,0) = u(t,b) = 0 \quad \text{for } t > 0.$$

(a) Show that there are solutions of (HE) and (BC) in the form $u(t,x) = a(t)\sin(\mu x)$ with a and μ yet to be determined.

(b) Choose a suitable orthonormal system $\{e_j \mid j \in \mathbb{N}\}$ in $L^2((0,b))$ such that there are solutions of (HE) and (BC) in the form $u(t,x) = \sum_{j=1}^{n} a_j(t)e_j(x)$.

(c) How can we find, for arbitrary $u_0 \in L^2((0,b))$, a solution of (HE) which also satisfies (BC) and (IC)? (Additional question: Why is u thus constructed for $(t,x) \in (0,\infty) \times (0,b)$ suitably often differentiable?)

Solution

(a) $u \equiv 0$ is the trivial solution. One gets nontrivial solutions of the form $u(t,x) = a_0 \cdot \mathrm{e}^{-\kappa \frac{j^2\pi^2}{b^2} t} \cdot \sin\left(\frac{\pi}{b} jx\right)$ with $a_0 \in \mathbb{R}$ and $j \in \mathbb{N}$ arbitrary (we skip the calculation).

(b) Just choose $e_j(x) := \sqrt{\frac{2}{b}} \cdot \sin\left(\frac{\pi}{b} jx\right)$ for $j \in \mathbb{N}$. This is a complete (!) ONS by the Lemma above. We could then, for example, take $a_j(t) := \mathrm{e}^{-\kappa \frac{j^2\pi^2}{b^2} t}$ to obtain solutions $u_j(t,x) := a_j(t)e_j(x)$ like in (a). Since all the conditions ((HE) and (IC)) are linear in u, finite sums of such solutions will again yield solutions of (HE) and (IC).

(c) Aside from (HE), (IC) and (BC) there are no other conditions posed to the function u (except for the fact that from (HE) we just see that u should apparently be two times (partially) differentiable on $(0,\infty)\times\Omega$). So, if $\tilde{u}$ is a solution of (HE) and (IC) (e.g. one of the functions in (b)), then u defined by

$$u(t,x) := \begin{cases} u_0(x) & \text{for } (t,x)\in\{0\}\times(0,b) \\ \tilde{u}(t,x) & \text{otherwise} \end{cases},$$

clearly satisfies (HE), (IC) and (BC) (and is a C^∞-function on $(0,\infty)\times(0,b)$ if $\tilde{u}$ is just a function as in (b)). Although this is a correct solution of the exercise (because, for example, there was not given any specification of a function space u should belong to), this was of course not the intention of the exercise. So, let's take it a bit more serious:

Since the functions e_j defined in (b) form a complete orthonormal system, we have $u_0 = \sum\limits_{j=1}^{\infty} \langle u_0, e_j\rangle e_j$ (this equation means that the series on the right hand side converges to u_0 in $L^2((0,b))$). We would now like to define

$$u(t,x) := \sum_{j=1}^{\infty} \langle u_0, e_j\rangle \cdot a_j(t) e_j(x) = \sum_{j=1}^{\infty} \langle u_0, e_j\rangle \cdot u_j(t,x)$$

with $a_j(t) = \mathrm{e}^{-\kappa\frac{j^2\pi^2}{b^2}t}$ as in (b). So we have to show the convergence of this series for all $(t,x)\in[0,\infty)\times[0,b]$. For $t=0$ this is not always possible, because the Fourier series of u_0 need not converge pointwise, but at least it converges pointwise almost everywhere on $(0,b)$ to u_0. Of course, (IC) is then just satisfied almost everywhere on Ω, but that's already the best possible result for this part. (So we should actually change the condition (IC) to „for almost all $x\in\Omega$“).

Now we concentrate on $t>0$. To show the pointwise convergence of the series on $(0,\infty)\times[0,b]$, it suffices to show that this series is uniformly converging on $[a,\infty)\times[0,b]$ for all $a>0$. To cover the additional question, too, we consider a more general case: Let $p:\mathbb{R}\to\mathbb{R}$ be a polynomial. We want to show that the series $\sum\limits_{j=1}^{\infty} \langle u_0, e_j\rangle\cdot p(j)\cdot u_j$ (i.e. the sequence $(S_n)_{n\in\mathbb{N}}$ defined by $S_n := \sum\limits_{j=1}^{n} \langle u_0, e_j\rangle\cdot p(j)\cdot u_j$) converges uniformly on $[a,\infty)\times[0,b]$. Firstly, it holds

$$|u_j(t,x)| = \mathrm{e}^{-\kappa\frac{j^2\pi^2}{b^2}t}\cdot\left|\sqrt{\frac{2}{b}}\cdot\sin\left(\frac{\pi}{b}jx\right)\right| \leq \sqrt{\frac{2}{b}}\cdot\mathrm{e}^{-\kappa\frac{j^2\pi^2}{b^2}a}$$

for all $(t,x)\in[a,\infty)\times[0,b]$. For simplification we put $c := \kappa\frac{\pi^2}{b^2}a > 0$, so that we have shown $\left\|u_j|_{[a,\infty)\times[0,b]}\right\|_\infty \leq \sqrt{\frac{2}{b}}\cdot\mathrm{e}^{-cj^2}$. We thus get

$$\left\|\langle u_0, e_j\rangle\cdot p(j)\cdot u_j|_{[a,\infty)\times[0,b]}\right\|_\infty \leq |\langle u_0, e_j\rangle|\cdot|p(j)|\cdot\sqrt{\frac{2}{b}}\cdot\mathrm{e}^{-cj^2}$$

for all $j\in\mathbb{N}$. By Weierstraß' M-test (in german sometimes called „Weierstraßsches Majorantenkriterium“) it therefore suffices to show that the series of real numbers

$$\sqrt{\frac{2}{b}}\cdot\sum_{j=1}^{\infty} |\langle u_0, e_j\rangle|\cdot|p(j)|\cdot\mathrm{e}^{-cj^2}$$

converges. This is provided by the root test, because for $j \in \mathbb{N}$ we have

$$\sqrt[j]{|\langle u_0, e_j\rangle| \cdot |p(j)| \cdot \mathrm{e}^{-cj^2}} = \sqrt[j]{|\langle u_0, e_j\rangle|} \cdot \sqrt[j]{|p(j)|} \cdot \mathrm{e}^{-cj}.$$

By Bessel's estimate the series $\sum_{j=1}^{\infty} |\langle u_0, e_j\rangle|^2$ converges, so that $|\langle u_0, e_j\rangle|$ has limit 0 and therefore $\sqrt[j]{|\langle u_0, e_j\rangle|}$ is at least bounded (e.g. by 1 for almost all j). The second term $\sqrt[j]{|p(j)|}$ converges to 1 as $j \to \infty$ as is known from basic analysis. As the third factor e^{-cj} converges to 0, so does the whole product. Therefore the series in question is in fact convergent (by the root test).

We have thus shown that $\sum_{j=1}^{\infty} \langle u_0, e_j\rangle \cdot u_j$ converges uniformly on $[a, \infty) \times [0, b]$ for all $a > 0$ and therefore pointwise on $(0, \infty) \times [0, b]$. In conclusion, u exists and is even continuous on every interval $[a, \infty) \times [0, b]$ (u is the uniform limit of the sequence of continuous functions S_n) and hence on all of $(0, \infty) \times [0, b]$. The general considerations above now yield even more:

Firstly, we can show by induction over $k \in \mathbb{N}_0$ that all k-th partial derivatives of u do exist on $(0, \infty) \times [0, b]$ and are continuous and that for $m = (m_1, m_2) \in \mathbb{N}_0^2$ with $|m| := m_1 + m_2 = k$ it holds (with $\partial^m = \frac{\partial^{m_1}}{\partial t^{m_1}} \frac{\partial^{m_2}}{\partial x^{m_2}}$ as usual in analysis)

$$\partial^m u = \lim_{n\to\infty} \partial^m s_n = \sum_{j=1}^{\infty} \langle u_0, e_j\rangle \cdot \partial^m u_j = \sum_{j=1}^{\infty} \langle u_0, e_j\rangle \cdot \frac{\partial^{m_1} a_j}{\partial t^{m_1}} \frac{\partial^{m_2} e_j}{\partial x^{m_2}}$$

pointwise on $(0, \infty) \times [0, b]$ (and even uniformly on $[a, \infty) \times [0, b]$), where we put $s_n := \sum_{j=1}^{\infty} \langle u_0, e_j\rangle \cdot u_j$ here. The base case $k = 0$ is already established. We now assume $k \geq 0$ and want to show that all $(k+1)$-st partial derivatives of u exist. Let $m = (m_1, m_2) \in \mathbb{N}_0^2$ be arbitrary with $|m| = m_1 + m_2 = k$. By induction hypothesis, the partial derivative $\partial^m u$ exists, is continuous and is given by the (pointwise converging) series

$$\partial^m u = \lim_{n\to\infty} \partial^m s_n = \sum_{j=1}^{\infty} \langle u_0, e_j\rangle \cdot \partial^m u_j = \sum_{j=1}^{\infty} \langle u_0, e_j\rangle \cdot \frac{\partial^{m_1} a_j}{\partial t^{m_1}} \frac{\partial^{m_2} e_j}{\partial x^{m_2}}.$$

One easily sees that $\frac{\partial^{m_1} a_j}{\partial t^{m_1}} \frac{\partial^{m_2} e_j}{\partial x^{m_2}}$ is of the form $p_m(j) u_j = p_m(j) \cdot a_j e_j$ or of the form $p_m(j) \hat{u}_j = p_m(j) \cdot a_j \hat{e}_j$ (where $\hat{e}_j(x) := \cos\left(\frac{\pi}{b} jx\right)$) with a suitable polynomial p_m. Let $t \in [a, \infty)$ be fixed. The partial sums

$$\partial^m s_n = \sum_{j=1}^{n} \langle u_0, e_j\rangle \cdot \partial^m u_j = \sum_{j=1}^{n} \langle u_0, e_j\rangle \cdot p_m(j) \cdot u_j$$

(or with $\hat{u}_j$ instead of u_j) are obviously differentiable with respect to x on $[0, b]$ (t is still fixed) and moreover, the series $\sum_{j=1}^{\infty} \langle u_0, e_j\rangle \cdot \partial_x \partial^m u_j = \sum_{j=1}^{\infty} \langle u_0, e_j\rangle \cdot p_{(m_1, m_2+1)}(j) \cdot u_j$ (or with $\hat{u}_j$ instead of u_j), i.e. the sequence $(\partial_x \partial^m s_n)_{n\in\mathbb{N}}$, which we obtain by derivating partially with respect to x term by term, is also uniformly convergent on $[a, \infty) \times [0, b]$ (see above) and thus in particular on $\{t\} \times [0,b]$. Hence, by the Proposition $\partial^m u$ is partially differentiable with respect to x on $\{t\} \times [0,b]$ and the derivation is given by

$$\partial_x \partial^m u = \lim_{n\to\infty} \partial_x s_n = \sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot \partial_x \partial^m u_j$$

on $\{t\} \times [0,b]$. Since t was arbitrary, this holds on all of $(0,\infty) \times [0,b]$ (and even uniformly on $[a,\infty) \times [0,b]$). For the partial derivative of $\partial^m u$ in a point (t,x) with respect to t, one argues analogously (let x be fixed, choose a and d with $0 < a < t < d$ and then use the Proposition on the set $[a,d] \times \{x\}$).

Now we have to check that the constructed function $u : [0,\infty) \times [0,b] \to \mathbb{R}$ (which is in fact just defined on $((0,\infty) \times [0,b]) \cup (\{0\} \times A)$ where $A \subseteq [0,b]$ is such that $(0,b) \setminus A$ is a zero set with respect to the Lebesgue measure) which we have just shown to be a C^∞-function on $(0,\infty) \times (0,b)$ actually satisfies the required conditions, i.e. (HE), (IC) and (BC). We have already discussed (IC) above, because

$$u(0,x) = \sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot a_j(0) e_j(x) = \sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot e_j(x)$$

is just the Fourier series of u with respect to the complete ONS $\{e_j \mid j \in \mathbb{N}\}$. As $e_j(0) = e_j(b) = 0$ for all $j \in \mathbb{N}$, we obviously have $u(t,0) = u(t,b) = 0$ for all $t > 0$, which is (BC). Moreover, every $s_n = \sum_{j=1}^{n} \langle u_0, e_j \rangle \cdot u_j$ satisfies condition (HE) by (b). As we have just discussed how to derivate u partially (namely term by term), we get

$$\frac{\partial u}{\partial t} = \partial_t u = \lim_{n\to\infty} \partial_t s_n = \lim_{n\to\infty} \kappa \cdot \partial_x^2 s_n = \kappa \cdot \partial_x^2 u = \kappa \cdot \frac{\partial^2 u}{\partial x^2}$$

on $(0,\infty) \times (0,b)$. In conclusion, u has all the desired properties.