

### *Exercise Sheet 4*

**Lemma:** Let  $b > 0$  and for  $j \in \mathbb{N}$  let  $g_j(t) = \sqrt{\frac{2}{b}} \cdot \sin\left(\frac{\pi}{b}jt\right)$  for all  $t \in (0, b)$ . Then  $\{g_j \mid j \in \mathbb{N}\}$  is a complete ONS in  $L^2((0, b))$ .

**Proof:** The idea of the proof is quite easy: Use the cONS of  $L^2((0, 2\pi))$  from exercise 13 and transfer it to  $L^2((-b, b))$ . To get rid of all the cosine functions, use the following trick: Extend functions  $u$  defined on  $(0, b)$  to functions  $\tilde{u}$  on the interval  $(-b, b)$  such that they are odd (i.e.  $\tilde{u}(-t) = -\tilde{u}(t)$ ) and expand them in their Fourier series. Because  $\tilde{u}$  is odd, all the Fourier coefficients corresponding to cosine-terms will be zero. In the following, we make this idea more explicit and more precise.

Consider the (clearly well-defined) operator  $\Phi : L^2((0, 2\pi)) \rightarrow L^2((-b, b))$  which assigns to a function  $f \in L^2((0, 2\pi))$  a new function  $\Phi(f) \in L^2((-b, b))$  defined by

$$\Phi(f)(t) := \sqrt{\frac{\pi}{b}} \cdot f\left(\frac{\pi}{b} \cdot t + \pi\right).$$

This map  $\Phi$  is an isometry of Hilbert spaces, because for  $f, g \in L^2((0, 2\pi))$  integration by substitution (with  $x = \frac{\pi}{b} \cdot t + \pi$ ) yields

$$\begin{aligned} \langle \Phi(f), \Phi(g) \rangle_{L^2((-b, b))} &= \int_{-b}^b \sqrt{\frac{\pi}{b}} \cdot f\left(\frac{\pi}{b} \cdot t + \pi\right) \cdot \sqrt{\frac{\pi}{b}} \cdot g\left(\frac{\pi}{b} \cdot t + \pi\right) dt \\ &= \int_0^{2\pi} f(x)g(x) dx = \langle f, g \rangle_{L^2((0, 2\pi))}. \end{aligned}$$

Consequently,  $\Phi$  carries the complete orthonormal system  $\{e_j \mid j \in \mathbb{Z}\}$  of  $L^2((0, 2\pi))$  considered in exercise 13 to a complete orthonormal system  $\{f_j \mid j \in \mathbb{Z}\}$  in  $L^2((-b, b))$  given as follows (we have changed some signs to get a similar structure of the cONS with no negative signs):

$$f_0(t) := \Phi(e_0)(t) = \sqrt{\frac{1}{2b}},$$

$$f_j(t) := \Phi(e_j)(t) = \sqrt{\frac{1}{b}} \cdot \cos\left(\frac{j\pi}{b} \cdot t + j\pi\right) = \sqrt{\frac{1}{b}} \cdot \cos\left(\frac{j\pi}{b} \cdot t\right) \quad \text{for } j \in 2\mathbb{N},$$

$$\begin{aligned} f_j(t) &:= -\Phi(e_j)(t) = -\sqrt{\frac{1}{b}} \cdot \cos\left(\frac{j\pi}{b} \cdot t + j\pi\right) = -\sqrt{\frac{1}{b}} \cdot \cos\left(\frac{j\pi}{b} \cdot t + \pi\right) \\ &= \sqrt{\frac{1}{b}} \cdot \cos\left(\frac{j\pi}{b} \cdot t\right) \quad \text{for } j \in 2\mathbb{N}_0 + 1, \end{aligned}$$

$$f_{-j}(t) := \Phi(e_{-j})(t) = \sqrt{\frac{1}{b}} \cdot \sin\left(\frac{j\pi}{b} \cdot t + j\pi\right) = \sqrt{\frac{1}{b}} \cdot \sin\left(\frac{j\pi}{b} \cdot t\right) \quad \text{for } j \in 2\mathbb{N},$$

$$\begin{aligned} f_{-j}(t) &:= -\Phi(e_{-j})(t) = -\sqrt{\frac{1}{b}} \cdot \sin\left(\frac{j\pi}{b} \cdot t + j\pi\right) = -\sqrt{\frac{1}{b}} \cdot \sin\left(\frac{j\pi}{b} \cdot t + \pi\right) \\ &= \sqrt{\frac{1}{b}} \cdot \sin\left(\frac{j\pi}{b} \cdot t\right) \quad \text{for } j \in 2\mathbb{N}_0 + 1. \end{aligned}$$

We therefore have the complete orthonormal system of  $L^2((-b, b))$  given by

$$f_0(t) = \sqrt{\frac{1}{2b}}, \quad f_j(t) = \sqrt{\frac{1}{b}} \cdot \cos\left(\frac{j\pi}{b} \cdot t\right), \quad f_{-j}(t) = \sqrt{\frac{1}{b}} \cdot \sin\left(\frac{j\pi}{b} \cdot t\right)$$

for  $j \in \mathbb{N}$ . Now take any function  $u \in L^2((0, b))$  and define  $\tilde{u} \in L^2((-b, b))$  by

$$\tilde{u}(t) := \begin{cases} -u(-t) & \text{for } t \in (-b, 0) \\ 0 & \text{for } t = 0 \\ u(t) & \text{for } t \in (0, b). \end{cases}$$

Then the sequence of finite Fourier sums

$$\tilde{u}_n := \sum_{j=-n}^n \langle \tilde{u}, f_j \rangle_{L^2((-b, b))} \cdot f_j$$

converges to  $\tilde{u}$  in  $L^2((-b, b))$ . Since  $\tilde{u}$  is odd and  $f_j$  is even for  $j \geq 0$ , the product  $\tilde{u} \cdot f_j$  is still an odd function and so its integral over  $(-b, b)$  is zero, i.e.  $\langle \tilde{u}, f_j \rangle_{L^2((-b, b))} = 0$  for all  $j \geq 0$ . As for the coefficient  $\langle \tilde{u}, f_{-j} \rangle_{L^2((-b, b))}$  for  $j > 0$ , recall that  $\tilde{u}$  and  $f_{-j}$  are both odd, so their product is even, hence

$$\begin{aligned} \langle \tilde{u}, f_{-j} \rangle_{L^2((-b, b))} &= \int_{-b}^b \tilde{u}(t) f_{-j}(t) \, dt = 2 \cdot \int_0^b \tilde{u}(t) \cdot f_{-j}(t) \, dt \\ &= 2 \cdot \int_0^b u(t) \cdot f_{-j}(t) \, dt = 2 \cdot \langle u, f_{-j}|_{(0, b)} \rangle_{L^2((0, b))}. \end{aligned}$$

Thus we get

$$\tilde{u}_n = \sum_{j=1}^n \langle \tilde{u}, f_{-j} \rangle_{L^2((-b, b))} \cdot f_{-j} = \sum_{j=1}^n 2 \cdot \langle u, f_{-j}|_{(0, b)} \rangle_{L^2((0, b))} \cdot f_{-j}$$

and this converges to  $\tilde{u}$  in  $L^2((-b, b))$ . But then  $\tilde{u}_n|_{(0, b)}$  must converge to  $\tilde{u}|_{(0, b)} = u$  in  $L^2((0, b))$ , too, as is provided by

$$\begin{aligned} \|\tilde{u}_n|_{(0, b)} - \tilde{u}|_{(0, b)}\|_{L^2((0, b))}^2 &= \int_0^b |\tilde{u}_n(t) - \tilde{u}(t)|^2 \, dt \\ &\leq \int_{-b}^0 |\tilde{u}_n(t) - \tilde{u}(t)|^2 \, dt + \int_0^b |\tilde{u}_n(t) - \tilde{u}(t)|^2 \, dt \\ &= \int_{-b}^b |\tilde{u}_n(t) - \tilde{u}(t)|^2 \, dt = \|\tilde{u}_n - \tilde{u}\|_{L^2((-b, b))}^2 \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Because of  $g_j = \sqrt{2} \cdot f_{-j}|_{(0, b)}$ , we get

$$\begin{aligned} \tilde{u}_n|_{(0, b)} &= \sum_{j=1}^n \langle \tilde{u}, f_{-j} \rangle_{L^2((-b, b))} \cdot f_{-j}|_{(0, b)} = \sum_{j=1}^n 2 \cdot \langle u, f_{-j}|_{(0, b)} \rangle_{L^2((0, b))} \cdot f_{-j}|_{(0, b)} \\ &= \sum_{j=1}^n \left\langle u, \sqrt{2} f_{-j}|_{(0, b)} \right\rangle_{L^2((0, b))} \cdot \sqrt{2} f_{-j}|_{(0, b)} = \sum_{j=1}^n \langle u, g_j \rangle_{L^2((0, b))} \cdot g_j. \end{aligned}$$

In summary, we have shown that for every  $u \in L^2((0, b))$  the sequence of the functions  $\sum_{j=1}^n \langle u, g_j \rangle_{L^2((0, b))} \cdot g_j \in L^2((0, b))$  converges to  $u$  in the normed space  $(L^2((0, b)), \|\cdot\|_{L^2((0, b))})$ . By exercise 9, it thus remains to prove that the  $g_j$  form an ONS to conclude that they in fact form a complete ONS. But this is clear from

$$\begin{aligned} \delta_{jk} &= \langle f_{-j}, f_{-k} \rangle_{L^2((-b, b))} = \int_{-b}^b f_{-j}(t) f_{-k}(t) \, dt = 2 \cdot \int_0^b f_{-j}(t) f_{-k}(t) \, dt \\ &= \left\langle \sqrt{2} f_{-j}|_{(0, b)}, \sqrt{2} f_{-k}|_{(0, b)} \right\rangle_{L^2((0, b))} = \langle g_j, g_k \rangle_{L^2((0, b))}. \end{aligned}$$

for all  $j, k > 0$  ( $f_{-j}$  and  $f_{-k}$  are odd and so their product is again even). ■

**Exercise 15 (A Differential Operator) - oral**

On the interval  $\Omega = (0, 1)$ , we define the differential operator

$$L : Y \rightarrow X, \quad u \mapsto u'' \quad \text{with} \quad Y \subseteq X = C^0(\overline{\Omega})$$

where  $Y := \{u \in C^2(\overline{\Omega}) \mid u(0) = 0, \quad u'(1) = 0\}$ .

- (a) Determine all eigen-pairs  $(\lambda_k, u_k) \in \mathbb{R} \times Y$  (i.e. pairs such that  $Lu_k = \lambda_k u_k$ ).
- (b) Show that for all  $u, v \in Y$ ,  $\langle Lu, v \rangle = \langle u, Lv \rangle$  holds, where  $\langle \cdot, \cdot \rangle$  denotes the standard  $L^2$ -scalar product.
- (c) Conclude that there are no eigen-pairs  $(\lambda, u)$  with  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , and that eigenfunctions with respect to different eigenvalues are orthogonal on each other.
- (d) Show that the normed eigenfunctions form a complete orthonormal system.

**Solution for (d):** It is not mentioned for which space the normed eigenfunctions should form a complete ONS. However, we will see that they do so for all of the three spaces  $L^2((0, 1))$ ,  $X$  and  $Y$ .

The normed eigenfunctions  $u_k$  (to the eigenvalues  $-\frac{(2k+1)^2\pi^2}{4}$ ) are given by

$$u_k(t) = \sqrt{2} \cdot \sin\left(\frac{(2k+1)\pi}{2} \cdot t\right)$$

for  $k \geq 0$ . These are pairwise orthogonal by (c) (or also by the following considerations). By the previous Lemma (for  $b = 2$ ), the functions  $g_j \in L^2((0, 2))$  for  $j \in \mathbb{N}$  defined by  $g_j(t) = \sin\left(\frac{\pi}{2}jt\right)$  form a complete ONS of  $L^2((0, 2))$ . Now we use just the same method as before to get rid of those  $g_j$  with  $j$  even. For an arbitrary  $h \in L^2((0, 1))$  we define  $\tilde{h} \in L^2((0, 2))$  by

$$\tilde{h}(t) := \begin{cases} h(t) & \text{for } t \in (0, 1) \\ 0 & \text{for } t = 1 \\ h(2-t) & \text{for } t \in (1, 2). \end{cases}$$

The key property of  $\tilde{h}$  is the fact that it is axially symmetric with respect to the axis  $\{(1, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$ , i.e. „ $t = 1$ “ (in other words: the function  $t \mapsto \tilde{h}(t+1)$  from  $(-1, 1)$  to  $\mathbb{R}$  is even). Once again, the sequence  $(\tilde{h}_n)_{n \in \mathbb{N}}$  of the functions

$$\tilde{h}_n = \sum_{j=1}^n \langle \tilde{h}, g_j \rangle_{L^2((0,2))} \cdot g_j$$

converges to  $\tilde{h}$  in  $L^2((0, 2))$ . If  $j = 2k$  is even, then  $g_j = g_{2k}$  is given by  $g_{2k}(t) = \sin(k\pi t)$  and this function is then point symmetric to the point  $(1, 0) \in \mathbb{R}^2$ , because of

$$g_{2k}(2-t) = \sin(2k\pi - k\pi t) = \sin(-k\pi t) = -\sin(k\pi t) = -g_{2k}(t)$$

for all  $t \in (0, 2)$  (in other words: the function  $t \mapsto g_{2k}(t+1)$  from  $(-1, 1)$  to  $\mathbb{R}$  is odd). Hence,

$$\langle \tilde{h}, g_{2k} \rangle_{L^2((0,2))} = \int_0^2 \tilde{h}(t)g_{2k}(t) dt = 0.$$

On the other hand, if  $j = 2k + 1$  is odd, then for all  $t \in (0, 2)$  we have

$$\begin{aligned} g_{2k+1}(2-t) &= \sin\left(\frac{\pi}{2}(2k+1)(2-t)\right) = \sin\left((2k+1)\pi - \frac{\pi}{2}(2k+1)t\right) \\ &= \sin\left(\pi - \frac{\pi}{2}(2k+1)t\right) = \sin\left(\frac{\pi}{2}(2k+1)t\right) = g_{2k+1}(t), \end{aligned}$$

i.e.  $g_{2k+1}$  is then axially symmetric with respect to the line „ $t = 2$ “ as well. If now a function  $v \in L^2((0, 2))$  also has this property (e.g.  $v = \tilde{h}$ ), so does the product  $vg_{2k+1}$ , hence

$$\begin{aligned} \langle v, g_{2k+1} \rangle_{L^2((0,2))} &= \int_0^2 v(t)g_{2k+1}(t) dt = 2 \cdot \int_0^1 v(t)g_{2k+1}(t) dt \\ &= 2 \cdot \langle v|_{(0,1)}, g_{2k+1}|_{(0,1)} \rangle_{L^2((0,1))}. \end{aligned}$$

In particular, it holds

$$\begin{aligned} \delta_{jk} &= \langle g_{2l+1}, g_{2k+1} \rangle_{L^2((0,2))} = 2 \cdot \langle g_{2l+1}|_{(0,1)}, g_{2k+1}|_{(0,1)} \rangle_{L^2((0,1))} \\ &= \left\langle \sqrt{2}g_{2l+1}|_{(0,1)}, \sqrt{2}g_{2k+1}|_{(0,1)} \right\rangle_{L^2((0,1))} = \langle u_l, u_k \rangle_{L^2((0,1))}. \end{aligned}$$

for all  $j, k \geq 0$ . Thus, the  $u_k$  form in fact an ONS. Moreover,  $\tilde{h}_n \rightarrow \tilde{h}$  in  $L^2((0, 2))$  particularly implies  $\tilde{h}_{2n+1}|_{(0,1)} \rightarrow \tilde{h}|_{(0,1)} = h$  in  $L^2((0, 1))$  for  $n \rightarrow \infty$ , where

$$\begin{aligned} \tilde{h}_{2n+1}|_{(0,1)} &= \sum_{j=1}^{2n+1} \left\langle \tilde{h}, g_j \right\rangle_{L^2((0,2))} \cdot g_j|_{(0,1)} = \sum_{k=0}^n 2 \cdot \left\langle \tilde{h}|_{(0,1)}, g_{2k+1}|_{(0,1)} \right\rangle_{L^2((0,2))} \cdot g_{2k+1}|_{(0,1)} \\ &= \sum_{k=0}^n \left\langle h, \sqrt{2}g_{2k+1}|_{(0,1)} \right\rangle_{L^2((0,2))} \cdot \sqrt{2}g_{2k+1}|_{(0,1)} = \sum_{k=0}^n \langle h, u_k \rangle_{L^2((0,2))} \cdot u_k. \end{aligned}$$

In conclusion,  $\{u_k \mid k \in \mathbb{N}_0\}$  is in fact a complete ONS in  $L^2((0, 1))$  (and therefore also in the pre-Hilbert spaces  $X$  and  $Y$  endowed with the  $L^2$ -scalar product).

For part (c) of exercise 15, we will use the following Proposition (see basic analysis lectures or books):

**Proposition**

Let  $a < b$  be real numbers, let  $(f_n)_{n \in \mathbb{N}}$  be a series of functions  $f_n : [a, b] \rightarrow \mathbb{R}$  (e.g.  $f_n = \sum_{j=1}^n u_j$ ) and let the following conditions be fulfilled:

- (a) Every  $f_n$  is differentiable on  $[a, b]$  and the sequence  $(f'_n)_{n \in \mathbb{N}}$  converges uniformly on  $[a, b]$  (to some function  $g$  on  $[a, b]$ ).
- (b) There is a  $x_0 \in [a, b]$  such that  $(f_n(x_0))_{n \in \mathbb{N}}$  is a convergent sequence of real numbers (i.e. the sequence  $(f_n)_{n \in \mathbb{N}}$  converges „pointwise in at least one point“).

Then the sequence  $(f_n)_{n \in \mathbb{N}}$  is uniformly converging to a function  $f : [a, b] \rightarrow \mathbb{R}$ , the limit function  $f$  is differentiable and we have  $f = g$ , i.e.  $(f'_n)_{n \in \mathbb{N}}$  converges (uniformly) to  $f'$ .

**Exercise 16 (Heat Conduction) - oral**

Let  $\kappa > 0$ ,  $\Omega := (0, b)$  und  $u_0 : \Omega \rightarrow \mathbb{R}$  be given. We are looking for a function  $u : [0, \infty) \times [0, b] \rightarrow \mathbb{R}$  such that

$$(HE) \quad \frac{\partial u}{\partial t} = \kappa \cdot \frac{\partial^2}{\partial x^2} \quad \text{in } (0, \infty) \times \Omega,$$

$$(IC) \quad u(0, x) = u_0(x) \quad \text{for } x \in \Omega, \quad (BC) \quad u(t, 0) = u(t, b) = 0 \quad \text{for } t > 0.$$

- (a) Show that there are solutions of (HE) and (BC) in the form  $u(t, x) = a(t) \sin(\mu x)$  with  $a$  and  $\mu$  yet to be determined.
- (b) Choose a suitable orthonormal system  $\{e_j \mid j \in \mathbb{N}\}$  in  $L^2((0, b))$  such that there are solutions of (HE) and (BC) in the form  $u(t, x) = \sum_{j=1}^n a_j(t) e_j(x)$ .
- (c) How can we find, for arbitrary  $u_0 \in L^2((0, b))$ , a solution of (HE) which also satisfies (BC) and (IC)? (Additional question: Why is  $u$  thus constructed for  $(t, x) \in (0, \infty) \times (0, b)$  suitably often differentiable?)

**Solution**

(a)  $u \equiv 0$  is the trivial solution. One gets nontrivial solutions of the form  $u(t, x) = a_0 \cdot e^{-\kappa \frac{j^2 \pi^2}{b^2} t} \cdot \sin\left(\frac{\pi}{b} j x\right)$  with  $a_0 \in \mathbb{R}$  and  $j \in \mathbb{N}$  arbitrary (we skip the calculation).

(b) Just choose  $e_j(x) := \sqrt{\frac{2}{b}} \cdot \sin\left(\frac{\pi}{b} j x\right)$  for  $j \in \mathbb{N}$ . This is a complete (!) ONS by the Lemma above. We could then, for example, take  $a_j(t) := e^{-\kappa \frac{j^2 \pi^2}{b^2} t}$  to obtain solutions  $u_j(t, x) := a_j(t) e_j(x)$  like in (a). Since all the conditions ((HE) and (IC)) are linear in  $u$ , finite sums of such solutions will again yield solutions of (HE) and (IC).

(c) Aside from (HE), (IC) and (BC) there are no other conditions posed to the function  $u$  (except for the fact that from (HE) we just see that  $u$  should apparently be two times

(partially) differentiable on  $(0, \infty) \times \Omega$ ). So, if  $\tilde{u}$  is a solution of (HE) and (IC) (e.g. one of the functions in (b)), then  $u$  defined by

$$u(t, x) := \begin{cases} u_0(x) & \text{for } (t, x) \in \{0\} \times (0, b) \\ \tilde{u}(t, x) & \text{otherwise} \end{cases},$$

clearly satisfies (HE), (IC) and (BC) (and is a  $C^\infty$ -function on  $(0, \infty) \times (0, b)$  if  $\tilde{u}$  is just a function as in (b)). Although this is a correct solution of the exercise (because, for example, there was not given any specification of a function space  $u$  should belong to), this was of course not the intention of the exercise. So, let's take it a bit more serious:

Since the functions  $e_j$  defined in (b) form a complete orthonormal system, we have  $u_0 = \sum_{j=1}^{\infty} \langle u_0, e_j \rangle e_j$  (this equation means that the series on the right hand side converges to  $u_0$  in  $L^2((0, b))$ ). We would now like to define

$$u(t, x) := \sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot a_j(t) e_j(x) = \sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot u_j(t, x)$$

with  $a_j(t) = e^{-\kappa \frac{j^2 \pi^2}{b^2} t}$  as in (b). So we have to show the convergence of this series for all  $(t, x) \in [0, \infty) \times [0, b]$ . For  $t = 0$  this is not always possible, because the Fourier series of  $u_0$  need not converge pointwise, but at least it converges pointwise almost everywhere on  $(0, b)$  to  $u_0$ . Of course, (IC) is then just satisfied almost everywhere on  $\Omega$ , but that's already the best possible result for this part. (So we should actually change the condition (IC) to „for almost all  $x \in \Omega$ “).

Now we concentrate on  $t > 0$ . To show the pointwise convergence of the series on  $(0, \infty) \times [0, b]$ , it suffices to show that this series is uniformly converging on  $[a, \infty) \times [0, b]$  for all  $a > 0$ . To cover the additional question, too, we consider a more general case: Let  $p : \mathbb{R} \rightarrow \mathbb{R}$  be a polynomial. We want to show that the series  $\sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot p(j) \cdot u_j$  (i.e.

the sequence  $(S_n)_{n \in \mathbb{N}}$  defined by  $S_n := \sum_{j=1}^n \langle u_0, e_j \rangle \cdot p(j) \cdot u_j$ ) converges uniformly on  $[a, \infty) \times [0, b]$ . Firstly, it holds

$$|u_j(t, x)| = e^{-\kappa \frac{j^2 \pi^2}{b^2} t} \cdot \left| \sqrt{\frac{2}{b}} \cdot \sin\left(\frac{\pi}{b} j x\right) \right| \leq \sqrt{\frac{2}{b}} \cdot e^{-\kappa \frac{j^2 \pi^2}{b^2} a}$$

for all  $(t, x) \in [a, \infty) \times [0, b]$ . For simplification we put  $c := \kappa \frac{\pi^2}{b^2} a > 0$ , so that we have shown  $\|u_j|_{[a, \infty) \times [0, b]}\|_{\infty} \leq \sqrt{\frac{2}{b}} \cdot e^{-c j^2}$ . We thus get

$$\|\langle u_0, e_j \rangle \cdot p(j) \cdot u_j|_{[a, \infty) \times [0, b]}\|_{\infty} \leq |\langle u_0, e_j \rangle| \cdot |p(j)| \cdot \sqrt{\frac{2}{b}} \cdot e^{-c j^2}$$

for all  $j \in \mathbb{N}$ . By Weierstraß' M-test (in german sometimes called „Weierstraßsches Majorantenkriterium“) it therefore suffices to show that the series of real numbers

$$\sqrt{\frac{2}{b}} \cdot \sum_{j=1}^{\infty} |\langle u_0, e_j \rangle| \cdot |p(j)| \cdot e^{-c j^2}$$

converges. This is provided by the root test, because for  $j \in \mathbb{N}$  we have

$$\sqrt[j]{|\langle u_0, e_j \rangle| \cdot |p(j)| \cdot e^{-cj^2}} = \sqrt[j]{|\langle u_0, e_j \rangle|} \cdot \sqrt[j]{|p(j)|} \cdot e^{-cj}.$$

By Bessel's estimate the series  $\sum_{j=1}^{\infty} |\langle u_0, e_j \rangle|^2$  converges, so that  $|\langle u_0, e_j \rangle|$  has limit 0 and therefore  $\sqrt[j]{|\langle u_0, e_j \rangle|}$  is at least bounded (e.g. by 1 for almost all  $j$ ). The second term  $\sqrt[j]{|p(j)|}$  converges to 1 as  $j \rightarrow \infty$  as is known from basic analysis. As the third factor  $e^{-cj}$  converges to 0, so does the whole product. Therefore the series in question is in fact convergent (by the root test).

We have thus shown that  $\sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot u_j$  converges uniformly on  $[a, \infty) \times [0, b]$  for all  $a > 0$  and therefore pointwise on  $(0, \infty) \times [0, b]$ . In conclusion,  $u$  exists and is even continuous on every interval  $[a, \infty) \times [0, b]$  ( $u$  is the uniform limit of the sequence of continuous functions  $S_n$ ) and hence on all of  $(0, \infty) \times [0, b]$ . The general considerations above now yield even more:

Firstly, we can show by induction over  $k \in \mathbb{N}_0$  that all  $k$ -th partial derivatives of  $u$  do exist on  $(0, \infty) \times [0, b]$  and are continuous and that for  $m = (m_1, m_2) \in \mathbb{N}_0^2$  with  $|m| := m_1 + m_2 = k$  it holds (with  $\partial^m = \frac{\partial^{m_1}}{\partial t^{m_1}} \frac{\partial^{m_2}}{\partial x^{m_2}}$  as usual in analysis)

$$\partial^m u = \lim_{n \rightarrow \infty} \partial^m s_n = \sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot \partial^m u_j = \sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot \frac{\partial^{m_1} a_j}{\partial t^{m_1}} \frac{\partial^{m_2} e_j}{\partial x^{m_2}}$$

pointwise on  $(0, \infty) \times [0, b]$  (and even uniformly on  $[a, \infty) \times [0, b]$ ), where we put  $s_n := \sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot u_j$  here. The base case  $k = 0$  is already established. We now assume  $k \geq 0$  and want to show that all  $(k+1)$ -st partial derivatives of  $u$  exist. Let  $m = (m_1, m_2) \in \mathbb{N}_0^2$  be arbitrary with  $|m| = m_1 + m_2 = k$ . By induction hypothesis, the partial derivative  $\partial^m u$  exists, is continuous and is given by the (pointwise converging) series

$$\partial^m u = \lim_{n \rightarrow \infty} \partial^m s_n = \sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot \partial^m u_j = \sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot \frac{\partial^{m_1} a_j}{\partial t^{m_1}} \frac{\partial^{m_2} e_j}{\partial x^{m_2}}.$$

One easily sees that  $\frac{\partial^{m_1} a_j}{\partial t^{m_1}} \frac{\partial^{m_2} e_j}{\partial x^{m_2}}$  is of the form  $p_m(j)u_j = p_m(j) \cdot a_j e_j$  or of the form  $p_m(j)\hat{u}_j = p_m(j) \cdot a_j \hat{e}_j$  (where  $\hat{e}_j(x) := \cos(\frac{\pi}{b} j x)$ ) with a suitable polynomial  $p_m$ . Let  $t \in [a, \infty)$  be fixed. The partial sums

$$\partial^m s_n = \sum_{j=1}^n \langle u_0, e_j \rangle \cdot \partial^m u_j = \sum_{j=1}^n \langle u_0, e_j \rangle \cdot p_m(j) \cdot u_j$$

(or with  $\hat{u}_j$  instead of  $u_j$ ) are obviously differentiable with respect to  $x$  on  $[0, b]$  ( $t$  is still fixed) and moreover, the series  $\sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot \partial_x \partial^m u_j = \sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot p_{(m_1, m_2+1)}(j) \cdot u_j$  (or with  $\hat{u}_j$  instead of  $u_j$ ), i.e. the sequence  $(\partial_x \partial^m s_n)_{n \in \mathbb{N}}$ , which we obtain by derivating partially with respect to  $x$  term by term, is also uniformly convergent on  $[a, \infty) \times [0, b]$  (see

above) and thus in particular on  $\{t\} \times [0, b]$ . Hence, by the Proposition  $\partial^m u$  is partially differentiable with respect to  $x$  on  $\{t\} \times [0, b]$  and the derivation is given by

$$\partial_x \partial^m u = \lim_{n \rightarrow \infty} \partial_x s_n = \sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot \partial_x \partial^m u_j$$

on  $\{t\} \times [0, b]$ . Since  $t$  was arbitrary, this holds on all of  $(0, \infty) \times [0, b]$  (and even uniformly on  $[a, \infty) \times [0, b]$ ). For the partial derivative of  $\partial^m u$  in a point  $(t, x)$  with respect to  $t$ , one argues analogously (let  $x$  be fixed, choose  $a$  and  $d$  with  $0 < a < t < d$  and then use the Proposition on the set  $[a, d] \times \{x\}$ ).

Now we have to check that the constructed function  $u : [0, \infty) \times [0, b] \rightarrow \mathbb{R}$  (which is in fact just defined on  $((0, \infty) \times [0, b]) \cup (\{0\} \times A)$  where  $A \subseteq [0, b]$  is such that  $(0, b) \setminus A$  is a zero set with respect to the Lebesgue measure) which we have just shown to be a  $C^\infty$ -function on  $(0, \infty) \times (0, b)$  actually satisfies the required conditions, i.e. (HE), (IC) and (BC). We have already discussed (IC) above, because

$$u(0, x) = \sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot a_j(0) e_j(x) = \sum_{j=1}^{\infty} \langle u_0, e_j \rangle \cdot e_j(x)$$

is just the Fourier series of  $u$  with respect to the complete ONS  $\{e_j \mid j \in \mathbb{N}\}$ . As  $e_j(0) = e_j(b) = 0$  for all  $j \in \mathbb{N}$ , we obviously have  $u(t, 0) = u(t, b) = 0$  for all  $t > 0$ , which is (BC). Moreover, every  $s_n = \sum_{j=1}^n \langle u_0, e_j \rangle \cdot u_j$  satisfies condition (HE) by (b). As we have just discussed how to derivate  $u$  partially (namely term by term), we get

$$\frac{\partial u}{\partial t} = \partial_t u = \lim_{n \rightarrow \infty} \partial_t s_n = \lim_{n \rightarrow \infty} \kappa \cdot \partial_x^2 s_n = \kappa \cdot \partial_x^2 u = \kappa \cdot \frac{\partial^2 u}{\partial x^2}$$

on  $(0, \infty) \times (0, b)$ . In conclusion,  $u$  has all the desired properties.