Höhere Analysis I: Lineare Funktionalanalysis
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Exercise Sheet 3

Exercise 12 (Separability) - oral

(a) Show that the Banach spaces $(c_0, \|\cdot\|_\infty)$ and $(\ell_1, \|\cdot\|_1)$ are separable.

(b) Show that the Banach space $\mathrm{BC}^0((0,1)) = \{\, f \in \mathrm{C}^0((0,1)) \mid f \text{ bounded}\,\}$ equipped with the norm $\|\cdot\|_\infty$ is not separable. (Hint: Find functions which correspond to sequences in $\{0,1\}^{\mathbb{N}}$.)

(c) Show that every closed subspace V of a separable Banach space $(X, \|\cdot\|)$ is in turn a separable Banach space. Moreover, find a non-separable Banach space Y which has a closed, infinite dimensional and separable subspace U.

(Partial) Solution

(b) For $k \in \mathbb{N}$, let $f_k : (0,1) \to [0,1]$ be a continuous function such that

$$f_k(x) \begin{cases} \in (0,1) & \text{for } x \in (\frac{1}{2^{k+1}}, \frac{1}{2^{k-1}}) \\ = 1 & \text{for } x = \frac{1}{2^k} \\ = 0 & \text{otherwise} \end{cases}$$

(e.g. linear on the intervals $(\frac{1}{2^{k+1}}, \frac{1}{2^k})$ and $(\frac{1}{2^k}, \frac{1}{2^{k-1}})$).

To a sequence $a = (a_n)_n \in l^\infty$, we now associate the function

$$f_a : (0,1) \to \mathbb{R}\,, \qquad x \mapsto \sum_{k=1}^{\infty} f_k(x) \cdot a_k$$

Let $y \in (0,1)$. Find $l \in \mathbb{N}$ such that $y \in I_y := (\frac{1}{2^{l+1}}, \frac{1}{2^{l-1}})$. Then, upon restriction to I_y, the sum over $k \in \mathbb{N}$ in the definition of f_a is finite (in fact, there are at most three summands $k \in \{l-1, l, l+1\}$). Therefore, f_a is continuous. Moreover, it is easily seen to preserve norms and as such is bounded:

$$F : l^\infty \to BC^0((0,1))\,, \qquad a \mapsto f_a\,, \qquad \|f_a\|_\infty = \|a\|_{l^\infty}$$

F is injective: Whenever $f_a = f_b$, then $f_a(\frac{1}{2^k}) = f_b(\frac{1}{2^k})$ and thus $a_k = b_k$. Therefore, l^∞ can be identified via F with a subspace of $BC^0((0,1))$. If $BC^0((0,1))$ was separable, this would imply separability of l^∞, which is a contradiction (see next part).

(please turn over)

(c) We prove: Let X be a separable metric space and $V \subseteq X$ be a subset. Then V is separable. (this concludes the proof of (b) and solves most of (c)).

Let $A \subseteq X$ be countable and dense. Let $\varepsilon > 0$. Let $B_{\varepsilon/2}(a)$ denote the open ball with radius $\varepsilon/2$ around $a \in A$. Consider the set

$$A_\varepsilon = \{a \in A \,\|\, B_{\varepsilon/2}(a) \cap V \neq \emptyset\}$$

By hypothesis, this set is countable. Now replace every $a \in A_\varepsilon$ by some $a' \in B_{\varepsilon/2}(a) \cap V$ and call the resulting set A'_ε. By construction, A'_ε is a countable subset of V, and for every $v \in V$, we find an $a' \in A'_\varepsilon$ with $d(v, a') < \varepsilon$.

In the second step, consider a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to 0$ (e.g. $\varepsilon_n := \frac{1}{n}$). We define the set

$$A' := \bigcup_{n\in\mathbb{N}} A'_{\varepsilon_n} \subseteq V$$

It is countable (triangular enumeration...). Moreover, for every $v \in V$ and $\varepsilon > 0$ we find an element $a' \in A'$ such that $d(v, a') < \varepsilon$, and thus A' is dense in V.