

Exercise Sheet 9

Exercise 32 (Quadratic Minimisation Problems) - written

Let H be a real HILBERT space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $C > 0$ and $a : H \times H \rightarrow \mathbb{R}$ be a symmetric bilinear form such that $\frac{1}{C}\|x\|^2 \leq a(x, x) \leq C\|x\|^2$ for all $x \in H$.

(a) Show that $((x, y)) := a(x, y)$ defines a scalar product on H such that the induced norm $\| \cdot \|$ is equivalent to the original norm.

(b) Let $F \in H' = \mathcal{L}(H, \mathbb{R})$ be a continuous linear form on H and

$$J(x) = \frac{1}{2}a(x, x) - F(x).$$

Show that $x_* \in H$ minimises the function J (i.e., $J(x_*) = \mu := \inf\{J(x) \mid x \in H\}$) if and only if for all $y \in H$ we have $a(x_*, y) = F(y)$.

(c) Show that for J there exists a unique minimising element $x_* \in H$.

Exercise 33 (FOURIER Transformation) - oral

For $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$, the FOURIER transform $\mathcal{F}f$ is defined by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) dx$$

For $a > 0$ and $\eta \in \mathbb{C}$, we define $E_{a,\eta} \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ by $E_{a,\eta}(x) = e^{-ax^2 + \eta x}$.

(a) Show the following equality for the complex scalar product $\langle \cdot, \cdot \rangle$ on $L^2(\mathbb{R}, \mathbb{C})$,

$$\langle E_{a,\eta}, E_{b,\mu} \rangle = \sqrt{\frac{\pi}{a+b}} \exp\left(\frac{(\eta + \bar{\mu})^2}{4(a+b)}\right)$$

Hint: Bring the integral to the form $I(\theta) = C \cdot \int_{\mathbb{R}} e^{-t^2 + i\theta t} dt$ with $\theta \in \mathbb{R}$ and $C \in \mathbb{C}$ and then establish a differential equation for $I(\theta)$. You may use $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$.

(b) Show that $(\mathcal{F}E_{a,\eta})(\xi) = \sqrt{\frac{\pi}{a}} e^{\eta^2/(4a)} E_{1/(4a), -i\eta/(2a)}(\xi)$.

(c) Conclude that $\langle \mathcal{F}E_{a,\eta}, \mathcal{F}E_{b,\mu} \rangle = 2\pi \langle E_{a,\eta}, E_{b,\mu} \rangle$ and show $\langle \mathcal{F}f, \mathcal{F}g \rangle = 2\pi \langle f, g \rangle$ for all $f, g \in \text{Span}\{E_{a,\eta} \mid a > 0, \eta \in \mathbb{C}\}$.

(d) Prove that $\|\mathcal{F}f\|_{L^2(\mathbb{R})} = \sqrt{2\pi}\|f\|_{L^2(\mathbb{R})}$ holds for all $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$.

Hint: You may use that $\text{Span}\{E_{a,\eta} \mid a > 0, \eta \in \mathbb{C}\}$ is dense in $\mathcal{S}(\mathbb{R}, \mathbb{C})$

(please turn over)

Exercise 34 (One-dimensional Differential Operators) - oral

Let $\Omega = (0, \ell) \subset \mathbb{R}$ be a bounded interval. We study weak formulations of the ordinary differential equation

$$-\frac{d}{dx}\left(a(x)\frac{d}{dx}u(x)\right) + b(x)\frac{d}{dx}u(x) + c(x)u(x) = f(x) \text{ in } \Omega \text{ and } u(0) = u(\ell) = 0 \text{ (ODE)}$$

Let $a, b, c \in L^\infty(\Omega)$ and $a(x) \geq \alpha$ for some $\alpha > 0$ almost everywhere in Ω .

(a) Let $\rho \in C^1(\bar{\Omega}; (0, \infty))$. Show that

$$\|f\|_\rho = \left(\int_0^\ell |f(x)|^2 \rho(x) dx\right)^{1/2} \text{ and } \|u\|_{1,\rho} = \left(\int_0^\ell [|u(x)|^2 + |u'(x)|^2] \rho(x) dx\right)^{1/2}$$

defines, respectively, a norm on $L^2(\Omega)$ and a norm on $H_0^1(\Omega)$ which, in either case, is equivalent to the respective standard norm.

(b) For $f \in L^2(\Omega)$, write (ODE) in the weak form

$$\forall v \in H_0^1(\Omega) : \mathcal{A}_\rho(u, v) := \int_0^\ell (au'v' + \beta u'v + cuv)\rho dx = \langle f, v \rangle_\rho := \int_0^\ell f(x)v(x)\rho(x) dx.$$

How must $\beta \in L^\infty(\Omega)$ be chosen?

(c) Can ρ be chosen such that the bilinear form \mathcal{A}_ρ in (b) becomes symmetric?

(d) Conclude that, under the assumption $c(x) \geq \gamma$ for some $\gamma > 0$ almost everywhere in Ω , (ODE) is always soluble. In particular, b is not required to be "small".