Exercise Sheet 8

Exercise 29 (Orthogonal Complements) - written

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. For any subset $M \subset H$ we define

$$\begin{aligned} M^\perp &:= \{\, x \in H \mid \forall\, y \in M :\ \langle x, y\rangle = 0 \,\} \quad \text{(orthogonal complement)} \\ [M] &:= \mathrm{Span}(M) = \{\, \textstyle\sum_{j=1}^N \lambda_j x_j \mid N \in \mathbb{N},\ \lambda_j \in \mathbb{K},\ x_j \in M \,\} \\ \overline{M} &:= \mathrm{Closure}(M) = \mathrm{cl}(M) = \{\, \lim_{n\to\infty} x_n \mid x_n \in M \,\}. \end{aligned}$$

Furthermore, we set $M^{\perp\perp} = (M^\perp)^\perp$. Show the following assertions.

(a) $M^\perp$ is a closed linear subspace of H.

(b) $M_1 \subset M_2 \subset H \quad \Longrightarrow \quad M_2^\perp \subset M_1^\perp$.

(c) $M^\perp = [M]^\perp = \left(\overline{[M]}\right)^\perp$.

(d) $[M] \subset M^{\perp\perp}$, $H = \overline{[M]} \oplus M^\perp$ and $\overline{[M]} = M^{\perp\perp}$.

(e) For the orthogonal projection P onto $\overline{[M]}$, we have $\langle Px, y\rangle = \langle x, Py\rangle$ for all $x, y \in H$.

Exercise 30 (Four Principles in Functional Analysis) - oral

Let $X = \mathrm{L}^2((1,\infty))$ be equipped with the usual norm $\|\cdot\|_2$.

(a) Let $\mathcal{D} = \{\, f \in X \mid \int_1^\infty x^2 |f(x)|^2 \,\mathrm{d}x < \infty \,\}$ and $T : \mathcal{D} \to X$ with

$$(Tf)(x) = x f(x) \text{ for almost all } x \in (1,\infty).$$

We consider $\mathcal{D}$ as a subset of X and equip it with the same norm. Show that $T : \mathcal{D} \to X$ is linear but not continuous. Is the graph of T closed in $\mathcal{D}\times X$?

(b) Consider

$$\mathcal{C} = \mathrm{C}^0_\mathrm{c}((1,\infty)) = \{\, f \in \mathrm{C}^0((1,\infty)) \mid \exists\, \delta > 0 : \mathrm{supp}(f) \subset [1+\delta, 1/\delta] \Subset (1,\infty) \,\}$$

as a subset of X with the same norm. For $n \in \mathbb{N}$, we define the operators $T_n : \mathcal{C} \to \mathcal{C}$ by

$$(T_n f)(x) = \max\big\{0, \min\{x, 2n-x\}\big\} f(x) \text{ for } x > 1.$$

Is the sequence $(T_n)_{n\in\mathbb{N}}$ pointwisely or uniformly bounded?

(c) Find an example for X, T and a sequence $(T_n)_{n\in N}$ in $\mathcal{L}(X,X)$ such that, for all $x \in X$, convergence $T_n x \to Tx$ in X holds but not $T_n \to T$ in $\mathcal{L}(X,X)$.

(please turn over)

Exercise 31 (Translation Operators) - oral

For $p \in [1, \infty)$ and $h \in \mathbb{R}^d$, we define the translation operator

$$\mathcal{T}_h : \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d); \; (\mathcal{T}_h f)(x) = f(x-h) \text{ for almost all } x \in \mathbb{R}^d.$$

Show the following assertions.

(a) $\mathcal{T}_h \in \mathcal{L}(\mathrm{L}^p(\mathbb{R}^d), \mathrm{L}^p(\mathbb{R}^d))$. Moreover, calculate the operator norm.

(b) For all $f \in \mathrm{C}^0_\mathrm{c}(\mathbb{R}^d)$, $\mathcal{T}_h f$ converges in $\mathrm{L}^p(\mathbb{R}^d)$ to f as $h \to 0$.

(c) For $f \in \mathrm{L}^p(\mathbb{R}^d)$, we have convergence $\|\mathcal{T}_h f - f\|_p \to 0$ as $h \to 0$.