Exercise Sheet 7

Exercise 25 (Banach Space of Bounded Linear Operators) - written

For normed vector spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, let $\mathcal{L}(X, Y)$ denote the set of bounded linear operators $X \to Y$.

(a) Show that the operator norm $\|A\|_{X\to Y} = \sup\{\, \|Ax\|_Y \mid \|x\|_X \leq 1 \,\}$ is indeed a norm.

(b) Prove that the normed vector space $(\mathcal{L}(X, Y), \|\cdot\|_{X\to Y})$ is complete if and only if $(Y, \|\cdot\|_Y)$ is a Banach space.
Hint: You may use that for every Banach space X there exists a non-trivial bounded linear map in $\mathcal{L}(X, \mathbb{K})$ (Theorem of Hahn–Banach).

(c) Let X and Y be Hilbert spaces. Is this sufficient for $(\mathcal{L}(X, Y), \|\cdot\|_{X\to Y})$ being a Hilbert-Raum?

Exercise 26 (Hamel Basis) - oral

A subset $B \subseteq X$ of a vector space X is called Hamel basis if every $x \in X$ can be written as a unique finite linear combination of elements of B.

(a) Let X be a normed vector space and $U \subseteq X$ be a finite dimensional subspace. Show that U is complete and closed.

(b) Let X be a Banach space with a countable Hamel basis. Prove that X is in fact finite-dimensional.
Hint: Use (a) and Baire's category theorem.

Exercise 27 (A Characterisation of Continuity) - oral

Let X and Y be topological vector spaces with systems $\{p^X_\alpha \mid \alpha \in I^X\}$ and $\{p^Y_\beta \mid \beta \in I^Y\}$ of semi-norms which are Hausdorff (i.e. for every $x \neq 0$ there exists $\alpha \in I$ such that $p_\alpha(x) > 0$). Prove the following characterisation of continuity for a linear map $T : X \to Y$.

T is continuous if and only if for every $\beta \in I^Y$ there exist $N \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_N \in I^X$ and $C > 0$ such that for all $x \in X$ the following estimate holds.

$$p^Y_\beta(Tx) \leq C \cdot \sum_{j=1}^{N} p^X_{\alpha_j}(x)$$

(please turn over)

Exercise 28 (SCHWARTZ **Functions) - oral**

Let $\mathrm{S}(\mathbb{R}^d)$ denote the set of "rapidly decreasing" or SCHWARTZ functions, defined as follows.

$$\mathrm{S}(\mathbb{R}^d) = \{ f \in \mathrm{C}^\infty(\mathbb{R}^d; \mathbb{R}) \mid \forall\, k, l \in \mathbb{N}_0 : \ p_{k,l}(f) < \infty \}$$

where $p_{k,l}(f) = \sup\{ (1 + |x|)^k |\mathrm{D}^\alpha f(x)| \mid x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}_0^d,\ |\alpha|_1 = l \}$.

(a) Show that $\mathrm{S}(\mathbb{R}^d)$ equipped with the system $SN = \{ p_{k,l} \mid k, l \in \mathbb{N}_0 \}$ of semi-norms is a complete topological vector space.

(b) Show that $\mathrm{C}_c^\infty(\mathbb{R}^d; \mathbb{R})$ is dense in $\mathrm{S}(\mathbb{R}^d)$.

(c) The SCHRÖDINGER operator in $\mathbb{R}^d$ is defined as

$$\big(H_{\text{Schröd}} f\big)(x) = -(\Delta f)(x) + |x|^2 f(x) \quad \text{with } \Delta f = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} f$$

Is $H_{\text{Schröd}}$, considered as a map $\mathrm{S}(\mathbb{R}^d) \to \mathrm{S}(\mathbb{R}^d)$, continuous?

<u>Remark:</u> We have already encountered $H_{\text{Schröd}}$ in exercise 20.